SCRIPTORES LOGARITHMICI.

SCRIPTORES LOGARITHMICI;

OR,

A COLLECTION

OF

SEVERAL CURIOUS TRACTS

ON THE

NATURE AND CONSTRUCTION

OF

LOGARITHMS,

MENTIONED IN

DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW EDITION OF SHERWIN'S MATHEMATICAL TABLES:

TOGETHER WITH

SOME TRACTS ON THE BINOMIAL THEOREM AND OTHER SUBJECTS CONNECTED WITH THE DOCTRINE OF LOGARITHMS.

VOLUME I.

LONDON:

PRINTED BY J. DAVIS,

AND SOLD BY B. WHITE AND SON, FLEET-STREET.

MDCCXCI.

P R E F A C E.

THE prefent collection of tracts relating to logarithms, and intitled "*Scriptores Logarithmici*," &c, was owing to the publication of Dr. Hutton's very curious hiftorical introduction to his new edition of Sherwin's Mathematical Tables in the year 1785. That introduction (which I have here reprinted at the beginning of this collection, together with Dr. Hutton's preface to that edition of the Mathematical Tables) excited my admiration in a high degree; not only becaufe it afforded a ftriking proof of the unwearied induftry and very great extent of reading, as well as of the uncommon mathematical fkill and judgment, of its learned author, but alfo on account of its great ufefulnefs, in exhibiting, to perfons of an inferior degree of knowledge in thefe fciences, a full and clear view of all the different fteps that had been taken by mathematicians of all countries, both in ancient and modern times, towards the cultivation and improvement of trigonometry and the doctrine of logarithms, which are two of the moft ufeful and important branches of the mathematicks. And, as moft of the tracts that have been written on the latter of thefe fubjects, namely, the doctrine of logarithms, fince the publication of Briggs's *Arithmetica Logarithmica* and *Trigonometria Britannica*, and that are mentioned in Dr. Hutton's introduction as deferving notice, are but fhort,

I thought it would be by no means impracticable to collect them all together into one book; and I did not doubt that such a collection of them, ranged in the same order in which they had been originally published, and in which, for the most part, they had been mentioned in the aforesaid historical introduction, would prove an acceptable present to all the lovers of these sciences. With this view I undertook the present publication, in which I have omitted the two tracts of Briggs above mentioned, namely, the *Arithmetica Logarithmica* and the *Trigonometria Britannica*, on account of their length (they being small folio volumes), and the tract of Baron Napier, called *Canon mirificus Logarithmorum* (though it is but a short one), because it was published before the said two treatises of Briggs. But these omissions are pretty well supplied by the ample accounts which Dr. Hutton has given of the contents of these valuable pieces in his aforesaid historical introduction, which is here reprinted. The present collection begins with the two tracts published on this subject by the famous John Kepler in the years 1624 and 1625; in which (as Dr. Hutton has justly observed in his above-mentioned historical introduction) that very elegant and accurate geometrician has treated of logarithms according to the true and genuine idea of them, as being *measures of ratios, or proportions*, and has delivered his whole doctrine concerning them in a very full and scientifick manner.

When this collection of tracts was first undertaken, I had thought that they might all have been comprised in one volume, quarto. But, as some of the tracts were written in a very obscure style and manner, and seemed much to stand in need of explanation;—and, as they were also founded on a supposition of the truth of the famous Binomial Theorem, both in the case of integral powers and in the case of roots, which theorem but few mathematical authors have attempted to demonstrate;—I resolved to endeavour to supply these defects as well as I was able, partly by adopting

ing and inſerting a demonſtration of the ſaid binomial theorem, both in the caſe of integral powers and in the caſe of roots, that had been publiſhed by the late very learned Mr. John Landen of Walton near Peterborough in Northamptonſhire, in the years 1758 and 1764, and by adopting ſome other hints given by other authors on the ſame ſubject, and partly by notes and tracts written by myſelf with a view to anſwer the ſame purpoſe, and on which (as the reader will eaſily perceive) I have beſtowed no ſmall ſhare of attention and labour. And by theſe additional and explanatory tracts, together with a few other mathematical tracts of my own compoſition, on ſubjects that have for the moſt part a remarkable connection with the binomial theorem, this collection has been ſwelled to ſuch a ſize that it cannot be concluded in leſs than three volumes, quarto; of which the two firſt are now preſented to the publick.

The tracts in theſe two volumes that are of my own compoſition are as follows.

The firſt of them is intitled, "*Remarks on the two infinite Serieſes* $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ *&c and* $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ *&c, which were found by Mr. Nicholas Mercator and Dr. John Wallis in the foregoing Tracts for the Purpoſe of ſquaring the Hyperbolick Spaces* BI*ru and* FI*ru*."

This is a very long tract, extending from page 233 of the firſt volume of this collection to page 344. My deſign in writing it was as follows.

Mr. Mercator, after explaining in the firſt 13 propoſitions of his *Logarithmotechnia*, the methods by which he adviſes the computers of a table of logarithms to proceed in their calculations (which methods are purely arithmetical and derived from the nature of ratios and of

the powers of numbers, without any mention of the hyperbola, or of any other geometrical figure whatſoever), concludes his diſcourſe with ſome curious propoſitions concerning the hyperbola, from which he derives a method of ſquaring an aſymptotick area of a rectangular hyperbola by means of the infinite ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c, in which the letter A is ſuppoſed to denote a portion of one of the aſymptotes of an hyperbola contiguous to the central ſquare of the hyperbola (or the ſquare formed by the two aſymptotes of it, and by two lines drawn from its vertex parallel to the ſaid aſymptotes reſpectively), and to be leſs than the ſide of the ſaid ſquare (which is ſuppoſed to be denoted by 1), and conſequently the ſeveral powers of A, to wit, A, A^2, A^3, A^4, A^5, A^6, &c, will form a ſeries of decreaſing quantities. For he demonſtrates that, upon theſe ſuppoſitions, the ſaid infinite ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c will be equal to the aſymptotick area correſponding to the line A, or of which the line A in the aſymptote forms the baſe. And, as this aſymptotick area is the meaſure of the ratio of $1 + A$ to 1, it is evident that the ſaid ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c muſt conſequently be the meaſure of the ſaid ratio of $1 + A$ to 1, or, if computed in numbers, muſt be its logarithm. And thus, in theſe latter propoſitions of his ſaid tract, Mr. Mercator exhibits a new method of computing logarithms totally different from the former methods, which are the ſubject of the firſt 13 propoſitions of his book, and which he ſeems principally to recommend.

But, though Mr. Mercator himſelf, at the time of publiſhing his *Logarithmotechnia*, ſeems moſt to value his firſt method of computing logarithms, which is contained in the firſt 13 propoſitions of his ſaid tract, the generality of mathematicians have given the preference to his ſecond method of computing them by means of the infinite ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c, which he has given

given us for the quadrature of the hyperbola. And, notwithſtanding the many learned tracts that have been written on the ſubject ſince his time by the moſt eminent mathematicians, I do not know any one method of computing logarithms that deſerves to be preferred to it, though Dr. Wallis's ſeries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c *ad infinitum* (which exhibits the logarithm of the ratio of 1 to 1 — A, and which the Doctor found out in conſequence of the peruſal of Mr. Mercator's *Logarithmotechnia* and after an attentive examination of the aforeſaid ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c *ad infinitum* invented by Mercator) may be conſidered as *equally* convenient for this purpoſe.

But both Mercator's ſeries, $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c *ad infinitum*, and Dr. Wallis's ſeries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c *ad infinitum*, are, by their reſpective inventors, derived from the contemplation of the hyperbola, and are given by them rather as methods of ſquaring the hyperbola than as methods of computing logarithms, though they are alſo capable of being applied to this latter purpoſe with great ſucceſs. This manner of deriving them, Dr. Halley thought, was a defect in the foundation of them, when conſidered as methods of computing logarithms; and therefore he endeavoured to ſupply this defect by giving other inveſtigations of them, which were founded on the mere abſtract principles of arithmetick and the nature of ratios, and particularly on Sir Iſaac Newton's famous theorem for raiſing the powers, either integral or fractional, of a binomial quantity. And he accordingly publiſhed ſuch inveſtigations of them in the Philoſophical Tranſactions of the royal Society of London in the year 1695, in a tract which he intitled "*A moſt compendious and facile Method for conſtructing Logarithms, exemplified and demonſtrated from the Nature of Numbers, without*

any

any Regard to the Hyperbola, with a ſpeedy Method for finding the Number from the Logarithm given."

But this tract by no means anſwers to the title Dr. Halley has given it, of *a compendious and facile method for conſtructing logarithms*. It is indeed *compendious*: but *facile* it moſt certainly is not. For, if ever there was a piece of writing that was peculiarly fit to ſerve as an example of the danger of an author's falling into obſcurity by aiming at too much brevity, which Horace has mentioned in his judicious admonition on the ſubject, *brevis eſſe laboro*; *obſcurus fio*; it is this tract of Dr. Halley: it being ſo uncommonly obſcure, that I ſcarce ever knew a mathematician that had read it (I except not the moſt learned and acute) that did not confeſs that he had found the greateſt difficulty in comprehending it. And, for my part, I cannot underſtand ſome parts of it even now, after having employed ſo much time and pains upon the ſubject. And this was the conſideration that induced me to write the "*Remarks on Mr. Mercator's and Dr. Wallis's Serieſes*," which conſtitute the ſeventh tract of the firſt volume of this collection. I have endeavoured to do that *intelligibly* which Dr. Halley has done in a manner that is *almoſt unintelligible*; that is, I have endeavoured to give ſuch inveſtigations of thoſe two logarithmick ſerieſes, invented by Mercator and Dr. Wallis, as Dr. Halley either has given, or has attempted to give, of them (I know not which), in the aforeſaid tract, namely, inveſtigations founded entirely on arithmetical principles, and on Sir Iſaac Newton's binomial theorem, without any mention of the hyperbola, or any other geometrical figure; and alſo without having recourſe to the doctrine of indiviſibles, or infinitely ſmall quantities, or the doctrine of fluxions, or, in general, to the arithmetick of infinites in any of its modifications. Theſe inveſtigations I have drawn up with great care, and have ſet them forth in different forms and ſtyles, namely, firſt at great length, and afterwards more conciſely, in order to adapt them to different ſorts of readers, and to render them as clear and eaſy as the nature of the ſubject (which

is a very abſtruſe one) will admit: and I have ſubjoined no leſs than nine examples of the computation of logarithms by each of the ſaid two ſerieſes, in order to make the manner of applying them to that purpoſe familiar to the reader, being of opinion (as I before obſerved) that no better method of computing logarithms has yet been found out, or need be ſought for, than the judicious application of theſe two ſerieſes.

From the computations contained in theſe examples I have derived the logarithms of the ratios of the firſt 22 numbers, to wit, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, and 23, to 1, both in Napier's ſyſtem and in Briggs's, to eighteen places of figures; which calculations are, I believe, enough to make the induſtrious reader that ſhall patiently go through them all, become very well acquainted with this method of computing logarithms.

The next tract of my compoſition in this collection is intitled "*An Appendix to the foregoing Remarks on the two Logarithmick Serieſes invented by Mr. Mercator and Dr. Wallis*," and extends from page 345 to page 383, and concludes the firſt volume of theſe tracts.

This tract alſo has a reference to Dr. Halley's *compendious and facile method*, &c, above mentioned. For his inveſtigations of the two anti-logarithmick ſerieſes which he gives us in that tract, to wit, the ſerieſes $1 + L + \frac{L^2}{2} + \frac{L^3}{3} + \frac{L^4}{4} + \frac{L^5}{5} + \frac{L^6}{6} +$ &c *ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{3} + \frac{L^4}{4} - \frac{L^5}{5} + \frac{L^6}{6} -$ &c *ad infinitum*, ſeemed to me almoſt as obſcure as thoſe of the two logarithmick ſerieſes invented by Mercator and Wallis. I therefore reſolved to give new inveſtigations of theſe two ſerieſes derived from the pure principles of Arithmetick and from the Binomial Theorem, in the ſame manner

manner as the inveſtigations given of them by Dr. Halley himſelf in the aforeſaid tract. Theſe inveſtigations are alſo drawn up with great care and pains, and ſet forth at conſiderable length, in order to make them as clear and as eaſy as poſſible. And, after the inveſtigations of theſe two anti-logarithmick ſerieſes in this appendix, I have given ſome examples of the application of the ſaid ſerieſes to the computation of the numbers correſponding to given logarithms, in order to illuſtrate the method of making ſuch computations, and render it familiar to the reader. There is alſo in this tract, amongſt other curious matters, a computation of the terms of the remarkable ratio which Mr. Cotes has called the *ratio modularis*, carried to 24 places of decimal fractions. Theſe terms are 2.718,281,828,459,045,235,360,274, &c and 1, or 1 and 0.367,879,441,171,442,321,595,522, &c, of which numbers all but the three laſt figures may be depended on as exact.

The third tract of my compoſition in this collection is in the ſecond volume, and conſiſts of notes on thoſe parts of Dr. Halley's above-mentioned Diſcourſe on Logarithms, which I had found extremely difficult, but yet had, after repeated peruſals of them, been able to comprehend. By the help of the two tracts above mentioned, to wit, the *Remarks on the two Logarithmick Serieſes invented by Mr. Mercator and Dr. Wallis*, and the *Appendix to thoſe Remarks*, and of theſe explanatory notes on Dr. Halley's tract contained in the ſecond volume of this collection, my readers will, I hope, be enabled to perceive the truth of all the propoſitions advanced in that celebrated diſcourſe. Theſe notes take up 30 pages, to wit, pages 92, 93, 94, &c . . . . . . 122.

The fourth tract of my compoſition in this collection has alſo ſome relation to Dr. Halley's above-mentioned diſcourſe. For it contains a direct method of computing logarithms founded on the principles which Dr. Halley recommends, namely, the pure prin-

ciples of arithmetick, without any reference to the hyperbola, or any other geometrical figure, and likewiſe without having recourſe to the method of indiviſibles, or to the doctrine of fluxions, or to the arithmetick of infinites in any of its forms: and, further, it is grounded (as well as Dr. Halley's inveſtigations in the above-mentioned diſcourſe) on Sir Iſaac Newton's binomial theorem, though by means of a different application of it from that made by Dr. Halley. And for theſe reaſons I have intitled it "*An Appendix to the foregoing Diſcourſe of Dr. Edmund Halley concerning Logarithms.*" This tract, or *Appendix*, &c, takes up pages 123, 124, 125, &c. . . . . 152.

Several of the remaining tracts in the ſecond volume of this collection are allotted to the demonſtration of the Binomial Theorem itſelf; that theorem being ſo very cloſely connected with the ſubject of logarithms as to be the foundation of the beſt methods of computing them, and having been itſelf but imperfectly demonſtrated in moſt of the books of mathematicks that have treated of it. For example, Sir Iſaac Newton (who invented it, or, at leaſt, extended it to the caſe of roots and other fractional powers) has never attempted to demonſtrate it, even in the caſe of integral powers: and Profeſſor Saunderſon, in his Algebra (though written for the uſe of beginners), takes it alſo for granted, or gives no proof of it but from induction: and Dr. Halley (though he has made ſo much uſe of it in the above-mentioned diſcourſe) does not give us the leaſt proof of its truth. And the ſame omiſſion is to be complained of in the works of many other eminent writers on mathematicks. I therefore thought that, in a collection of tracts relating to logarithms (of which that theorem is ſuch an important foundation) it would be proper to ſupply this defect. And this has been the occaſion of my compoſing many of the following tracts in the ſecond volume of this collection, as well as of my inſerting in it Mr. Landen's demonſtration of the ſaid theorem.

The fiſth tract of my compoſition in this collection is a demonſtration of the binomial theorem in the caſe of integral powers, which is nearly the ſame with one given by Mr. John Stewart, of Aberdeen, in the ſixth ſection of his Commentary on Sir Iſaac Newton's learned little tract, called *Analyſis per æquationes numero terminorum infinitas*, publiſhed at the end of his commentary on Sir Iſaac Newton's *Quadrature of Curves*. But the demonſtration of this theorem given in the preſent tract is much fuller than that in Mr. Stewart's book. And in the latter part of the preſent tract the demonſtration is extended from $\overline{a + b}|^{m}$, or the *m*th power of the binomial quantity $a + b$, to $\overline{a - b}|^{m}$, or the *m*th power of the reſidual quantity $a - b$; the letter m denoting any whole number whatſoever in both caſes.

This fifth tract takes up pages 153, 154, 155, &c . . . . . 169.

The next tract in the ſecond volume of this collection is the demonſtration given of the binomial theorem by Mr. John Landen, of Walton near Peterborough in Northamptonſhire, which has been already mentioned. This demonſtration extends to the caſe of the fractional powers of the binomial quantity $1 + x$, as well as to that of its integral powers, being a general demonſtration of the theorem in the caſe of the quantity $\overline{1 + x}|^{\frac{m}{n}}$, when m and n repreſent any whole numbers whatſoever. It is extracted partly from Mr. Landen's *Diſcourſe concerning the Reſidual Analyſis*, publiſhed in the year 1758, and partly from another tract of Mr. Landen, on the ſame ſubject, publiſhed in the year 1764, and intitled, *The Reſidual Analyſis*, to which the former tract, called *A Diſcourſe concerning the Reſidual Analyſis*, had been only an introduction. This demonſtration of Mr. Landen (though I believe it to be juſt) is extremely intricate and perplexed in the algebraick operations that are neceſſary to be performed in it, and is, upon the whole, exceeding difficult to underſtand: for which reaſon I have ſubjoined to it another tract of my own compoſition, in which I have uſed my beſt

endeavours to explain it. Mr. Landen's demonſtration extends through pages 170, 171, 172, 173, 174, 175; and the ſubſequent explanation of it extends through pages 176, 177, 178, &c . . . . . . 193.

The next tract in the ſecond volume of this collection is alſo of my compoſition, and relates to the ſame ſubject of the binomial theorem. It is a very full diſcourſe on the ſaid theorem in the caſe of fractional powers, containing, firſt, various illuſtrations and exemplifications of it, that are calculated to make the reader become familiarly acquainted with it, and ready in the application of it to any propoſed particular caſes; and, afterwards, exhibiting two different methods of demonſtrating it, of which the latter and more compleat was ſuggeſted to me by a very cloſe and attentive peruſal of Mr. Landen's demonſtration above mentioned, to which it bears a conſiderable reſemblance. It does not, however, ſeem to me to be quite the ſame with Mr. Landen's, though I acknowledge it to have been ſuggeſted by it: and it is at leaſt more fully and clearly expreſſed than Mr. Landen's, and vaſtly eaſier to underſtand; and will, I believe, if properly attended to, give the readers of it all the ſatisfaction that can reaſonably be expected on ſo ſubtle and abſtruſe a ſubject. This tract is intitled, "*A Diſcourſe concerning the Binomial Theorem in the caſe of fractional Powers, or Powers of which the Indexes are Fractions.*" It extends from page 194 to page 344.

The next tract in the ſecond volume of this collection is alſo written by myſelf. It is intitled, "*A Diſcourſe concerning Sir Iſaac Newton's Reſidual Theorem, or Theorem for raiſing the Powers of the Reſidual Quantity* $1 - x$, *in the caſe of fractional Powers, or Powers of which the Indexes are Fractions.*"

The object of this tract is to transfer all the concluſions that have been demonſtrated in the foregoing long diſcourſe concerning

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the quantity $\overline{1+x}|^{\frac{m}{n}}$, or the $\frac{m}{n}$th power of the binomial quantity $1+x$, to the quantity $\overline{1-x}|^{\frac{m}{n}}$, or the $\frac{m}{n}$th power of the residual quantity $1-x$; and to do this without repeating over again the long trains of reasoning by which those conclusions had been obtained in the said former discourse. This tract extends from page 345 to page 378.

All the remaining tracts in the second volume of this collection are likewise of my composition. The subjects of them are as follows.

The tract which comes immediately after the last-mentioned Discourse concerning Sir Isaac Newton's Residual Theorem, is a very curious application of the binomial and residual theorems to the finding of the lesser of the two roots of the cubick equation $qy - y^3 = r$ by means of one of the expressions given by Cardan's first rule for the root of the equation $qy + y^3 = r$, or $qx + x^3 = r$. But this application, or extension of Cardan's rule, can take place only when the absolute term r of the equation $qy - y^3 = r$ is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{2}$ is less than $\frac{q^3}{27}$, or when $\frac{rr}{4}$ is less than $\frac{q^3}{54}$.

And, as the root of the cubick equation $x^3 - qx = r$ may always be derived from either of the roots of the cubick equation $qy - y^3 = r$ by means of a quadratick equation, when the absolute term r is of any magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ or when $\frac{rr}{4}$ is of any magnitude less than $\frac{q^3}{27}$, it follows that, when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, the root of the cubick equation $x^3 - qx = r$ may be deduced from Cardan's expression for the root of the cubick equation $x^3 + qx = r$, or $y^3 + qy = r$, or $qy + y^3 = r$, by the mediation

diation of the leſſer root of the cubick equation $qy - y^3 = r$, which may be derived from Cardan's ſaid expreſſion for the root of the equation $qy + y^3 = r$ by the method deſcribed in this tract.

The matters contained in this tract are in their nature extremely ſubtle and difficult; which made it neceſſary for me to be very full and particular in the explanation of them. And this is the cauſe of the length to which this diſcourſe is carried, which is greater than I could at firſt have imagined. It reaches from page 378 to page 440.

The next tract in the ſecond volume of this collection is on the ſame ſubject as the laſt. It is an application of the binomial and reſidual theorems to the purpoſe of extending one of the expreſſions given by Cardan's ſecond rule for the root of the cubick equation $x^3 - qx = r$, in the firſt caſe of that equation, or when the abſolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, to the reſolution of the ſecond caſe of the ſame equation, in which the abſolute term r is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$; provided that r (though leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) be greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{rr}{4}$ (though leſs than $\frac{q^3}{27}$) be greater than $\frac{q^3}{54}$.

This tract is full of abſtruſe and ſubtle matter, as well as the foregoing tract, and has been therefore treated in a very full and explanatory manner, which is the cauſe of its extending through no leſs than 135 pages, or from page 440 to page 575. It had formerly been publiſhed in the Philoſophical Tranſactions for the year 1778. But the ſubject is here explained in a much fuller and better manner than before.

By means of the expreſſions obtained in theſe two tracts we may always

always find the root of the cubick equation $x^3 - qx = r$ in the second case of it, or that in which the absolute term r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$ (which has obtained the name of *the irreducible case* amongst many of the writers on Algebra) by the help of one or other of Cardan's two rules for the resolution of the equations $x^3 + qx = r$ and $x^3 - qx = r$, to wit, by the help of the former of these rules, if r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, and by the help of the latter of them if r is greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, though less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{54}$, though less than $\frac{q^3}{27}$.

These two tracts enable us to do what is equivalent to what is called by Algebräists *finding the cube-roots of two impossible binomial quantities.* But they enable us to do this by means of reasonings conducted in an intelligible manner, and without any mention, or supposition, of either impossible or negative quantities; which, I hope, will be a recommendation of them to many of my readers, and be considered as an excuse for their extraordinary length and difficulty.

The next tract in the second volume of this collection is a conjecture concerning the method by which Cardan's two rules for resolving the cubick equations $x^3 + qx = r$ and $x^3 - qx = r$ might have been found out by *Scipio Ferreus of Bononia*, and *Nicholas Tartalea*, or whoever else were the first inventors of them.

This tract has no connection either with logarithms or the binomial theorem. But, as the two celebrated rules of Cardan to which it relates, had been so much the object of the reader's attention in the two preceeding tracts, I thought it might be agreeable to him to see a probable conjecture concerning the invention of them, more especially as it takes up only eight pages, to wit, pages 579, ... 586.

This

This tract was formerly printed in the Philoſophical Tranſactions for the year 1780. It is now reprinted without any additions.

The next tract in the ſecond volume of this collection, and with which the volume concludes, contains only five pages, to wit, page 587, 588, 589, 590, and 591. It is an improvement upon a hint given by Profeſſor Saunderſon in his Algebra, relating to the old ſubject of the Binomial Theorem, and ſhews how a mathematician, that had been in purſuit of a general rule for deriving the numeral co-efficients of the terms of any integral power of a binomial quantity, $a + b$, one from another in terms of the index m (as Dr. Wallis tells us he himſelf had been, but without ſucceſs) might have been led to ſuſpect that they might be generated by the continual multiplication of 1 into the following ſeries of fractions, to wit, $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, &c; after which ſuſpicion he would naturally have tried theſe fractions in a variety of inſtances, or with a variety of different values of the index m; and, having found that they always produced the true numeral co-efficients of the ſecond, third, fourth, and other following terms of the mth power of $a + b$, he would then, of courſe, have concluded, by induction from the inſtances in which he had found this method of generating the ſaid co-efficients to be true, that it muſt be true in all other caſes whatſoever, or when the index m of the power to which the binomial quantity $a + b$ was to be raiſed, was equal to any other whole number than thoſe in which he had tried it.

And, when this concluſion had thus been formed by an induction grounded on a great number of trials, the mathematician might have confirmed the truth of it by a general demonſtration extending it from the caſes in which it had been tried and found to be true, to the caſes of all higher powers whatſoever, in the manner

ner defcribed in the former tract contained in the fecond volume of this collection, in pages 153, 154, 155, &c. . . . . . 169.

I think therefore that thefe two tracts may be confidered as forming a compleat and fatisfactory inveftigation and demonftration of the Binomial and Refidual Theorems in the cafe of integral powers: after which the reader may proceed to the two tracts above defcribed, which extend from page 194 to page 378, in which he will find pretty full, and, I hope, fatisfactory demonftrations of the truth of thefe important theorems; firft, in the cafe of roots, and fecondly, in the cafe of the powers of roots, and confequently in the cafes of all fractional powers whatfoever.

INNER-TEMPLE, *Aug.* 16, 1791.

FRANCIS MASERES.

A

T A B L E

OF THE

C O N T E N T S

OF THE

F I R S T V O L U M E;

EXCLUSIVE OF DR. HUTTON'S HISTORICAL INTRODUCTION.

VII.

VIII.

ERRATA IN VOL. I.

In page 202, in the figure to the firſt lemma, inſtead of the letter B at the end of the right hand line of the figure, inſert the letter E.

In the ſame page, the firſt line of the ſecond lemma, inſtead of "*dico*, $AC^2 \times CB^3$," read "*dico, ſi* $AC^2 \times CB^3$."

In page 208, line 6 from the bottom, inſtead of the words, "*ducto in altero ſegmentum A*," read "*ducto in alterum ſegmentum A*."

In page 209, line 10, inſtead of "*dignitatem*," read "*dignitatum*."

In page 210, line 19, inſtead of "*ſecet E*," read "*ſecet in E*," and inſtead of "*productum*," read "*productam*."

In page 224, line 6, inſtead of "*accomodari*," read "*accommodari*."

In page 229, line 14th from the bottom, inſtead of "L 99," read "L 49."

In page 231, in line 17th and in line 14th from the bottom, inſtead of "*infinitiſſimâ*," read "*infi-nitesſimâ*."

In page 307, line 10, inſtead of "*will be the ſeries*," read "*will be to the ſeries*."

DR. HUTTON'S PREFACE

TO HIS

MATHEMATICAL TABLES.

THE very ample introduction, prefixed to the following collection of Mathematical Tables, has superseded the necessity of using many words here by way of preface, and has left me little more to mention than the necessity and occasion of this work, with some account of the contents and mode of execution.

The undertaking was occasioned by the extreme incorrectness of the 5th, or last, edition of Sherwin's once very useful book of tables. Finding, as well from the report of others, as from my own experience, that that edition (to say nothing of the very improper alteration in the form of the table of sines, tangents, and secants) was so very incorrectly printed, the errors being multiplied beyond all tolerable bounds, and no dependence to be placed on it for any thing of real practice, I was led to undertake the painful office of preparing a correct edition of another similar work. And I was lucky enough to meet with a bookseller of sufficient spirit to be at the great expence of printing the book, as well as to allow me what I demanded for my trouble in preparing it; which demand, however, was nothing adequate to the great labour attending it, as I was well aware that the profits of the book would not afford him the means of rewarding my pains.

I have, in the first place, therefore, used all the means in my power to render the work correct. I began by collating the 3d, or best edition of Sherwin's tables, with some others of the most perfect works of

the ſame kind, as Briggs's, Vlacq's, Gardiner's quarto book, &c. by which means I detected many errors in each of them, which had not before been diſcovered; and of theſe between twenty and thirty were in the two editions of Gardiner's work, printed at London in 1742, and at Avignon in 1770; the errata of which two books are here printed at the end of our tables. But beſides detecting many unknown errors in my copy of the ſaid 3d Edition of Sherwin, which was no more than what I expected, I diſcovered, with no ſmall ſurprize, that the laſt figures in the table of logarithms were not uniformly true to the neareſt unit, except in a very few pages at the beginning and end of the table; although Mr. Gardiner, the editor of that edition, had made the table correct in that reſpect in his own quarto work before-mentioned, which was alſo printed in the ſame year 1742 with the ſaid third edition of Sherwin! The errors from this cauſe amounted to ſeveral thouſands; and they have continued to run through all the editions of Sherwin ever ſince that time! But I have here corrected them. Nor have I employed leſs attention in correcting the preſs, than in previouſly correcting the copy; every proof having been ſeveral times read over, and compared with the beſt of the books before-mentioned.

But in giving this edition to the world, I was not ſatisfied with barely making it correct. I was aware that the materials themſelves might be much improved; and I have accordingly enlarged, or otherwiſe greatly amended them in various reſpects. Among the improvements of the old materials, may be reckoned the following: namely, in the large table of logarithms, the proportional parts, near the beginning, are more conveniently arranged, being now all placed in the ſame opening of the book where their correſponding differences occur: The logarithms to ſixty-one figures are brought to their proper place in the book, and more conveniently diſpoſed all in one page: The large table of ſines, tangents and ſecants, is more commodiouſly arranged, and rendered more diſtinct and convenient for uſe; the natural ſines, tangents, ſecants and verſed ſines, being all ſeparated from the others, and placed all together on the left-hand pages, and the logarithmic ones facing them

them on the right-hand pages; the common differences, in both, ſet between the two columns to which each of them anſwers; and the verſed ſines are introduced into their proper place in the ſame pages with the ſines, tangents and ſecants. Beſides theſe, there are ſome other alterations in the new tables here given, and the reader will find a number of very important improvements in the deſcription and uſe of the whole; eſpecially in the arithmetic of logarithms, and in the reſolution of plane and ſpherical triangles, according to the preſent improved methods of calculation uſed by the Aſtronomer Royal, and other perſons the moſt experienced in theſe matters.

The improvements in the tables by the introduction of new matter, are both great and numerous. The tables numbered 2, 3, and 4 are here added, being an entire new ſet, with their differences, for finding numbers and logarithms to twenty places. The columns of common differences in the pages of natural ſines, &c, are now firſt introduced: As are alſo the tables of hyperbolic and logiſtic logarithms; the logarithmic ſines and tangents for every ſecond, in the firſt two degrees of the quadrant; together with a table of the lengths of arcs; a table to change common and hyperbolic logarithms from the one to the other; &c: the uſes and exemplifications of the whole being very amply detailed.

But the greateſt alteration of all, is the very extenſive and new introduction here given, inſtead of the former inadequate and heterogeneous one, conſiſting of about 180 pages of new matter, on a methodical plan, containing the hiſtorical account and deſcription of all trigonometrical writings, and the tables relating to that ſubject, both natural and logarithmic; beſides the complete uſe of our own tables. Inventions are here aſcribed to the proper authors, and their methods and improvements deſcribed and compared. This hiſtorical deſcription will evidently appear to be the reſult of immenſe labour and reading. And indeed I have painfully gone over all the books which are here ſo minutely deſcribed; and that deſcription with a detail in ſome degree adequate to their great merits; eſpecially the works of Napier, Briggs,

 Kepler,

Kepler, &c; which was the more neceſſary, as the writings and methods of thoſe great maſters had not been any where properly deſcribed and diſcriminated, although they are in themſelves highly curious and important.

Theſe readings and commentaries have been carried on to an extent far beyond what was at firſt intended. But the tables having been in the preſs for the ſpace of ſeven or eight years, I had thereby an opportunity of collecting and examining a ſtill greater number of books; ſo that I was gradually led on, and my views and plans rendered ſtill more extenſive and compleat. This delay, therefore, though in many reſpects it proved very inconvenient and diſagreeable, has at length given the occaſion of rendering theſe commentaries more perfect and ſatisfactory.

Beſides what immediately relates to trigonometrical ſubjects, the reader will here find many other curious and uncommon articles, relating to the ſeveral authors and their diſcoveries, which have occurred in the courſe of my reading, and which appeared of too much conſequence to be paſſed over unnoticed, in the analyſis of their ſeveral compoſitions. Among theſe is the diſcovery of the firſt author of the binomial theorem, and the differential method, which are due to Mr. Henry Briggs, whoſe writings are replete with ingenious and original matter, and are well deſerving to be more generally known and ſtudied than they have been for ſome time paſt.

This long courſe of examination and deſcription, however, having been carried on for ſo many years, at different intervals, and interrupted by various avocations, and by buſineſs of different kinds, it will be no wonder if this circumſtance may have occaſioned ſome inequalities in the ſtyle and compoſition of this hiſtory; and for which therefore, ſhould any ſuch appear, it is hoped the occaſion will plead an apology.

INTRO-

INTRODUCTION.

I. OF TRIGONOMETRICAL TABLES, &c.

NECESSITY, the fruitful mother of moſt uſeful inventions, gave birth to the various numerical tables which compoſe the following work. Aſtronomy has been cultivated from the earlieſt ages. The progreſs of that ſcience, requiring numerous arithmetical computations of the ſides and angles of triangles, both plane and ſpherical, gave riſe to trigonometry; for thoſe frequent calculations ſuggeſted the neceſſity of performing them by the property of ſimilar triangles; and for the ready application of this property, it was neceſſary that certain lines deſcribed in and about circles, to a determinate radius, ſhould be computed and diſpoſed in tables. Navigation, and the continually improving accuracy of aſtronomy, have alſo occaſioned as perpetual an increaſe in the accuracy and extent of thoſe tables. And this, it is evident, muſt ever be the caſe, the improvement of trigonometry uniformly following the improvement of thoſe other uſeful ſciences, for the ſake of which it is more eſpecially cultivated.

The ancients performed their trigonometry by means of the chords of arcs, which with the chords of their ſupplemental arcs, and the conſtant diameter, formed all ſpecies of right-angled triangles. Beginning with the radius, and the arc whoſe chord is equal to the radius, they divided them both into 60 equal parts, and eſtimated all other arcs and chords by thoſe parts, namely all arcs by 60ths of that arc, and all chords by 60ths of its chord or the radius: At leaſt this method is as old as the writings of Ptolemy, who uſed the ſexagenary arithmetic for this diviſion of chords and arcs, and for aſtronomical purpoſes.—And this, by the bye, ſhews the reaſon why the whole circumference is divided into 360, or 6 times 60, equal parts or degrees, the whole circumference being equal to 6 times the firſt arc whoſe chord is equal to the radius: Unleſs perhaps we are to ſeek for the diviſion of the circle in the number of days in the year; for thus, the ancient year conſiſting of 360 days, the ſun or earth in each day deſcribed the 360th part of the orbit; and thence might ariſe the method of dividing every circle into 360 parts; and the radius being equal to the chord of 60 of thoſe parts, the ſexageſimal diviſion both of the radius and of the parts might thence ariſe. Trigonometry however muſt have been cultivated long before the time of Ptolemy; and indeed Theon, in his commentary on Ptolemy's Almageſt,

Almageſt, l. 1. ch. 9, mentions a work of the philoſopher Hipparchus, written about a century and a half before Chriſt, conſiſting of 12 books on the chords of circular arcs; which muſt have been a treatiſe on trigonometry. And Menelaus alſo, in the firſt century of Chriſt, wrote 6 books concerning ſubtenſes or chords of arcs. He uſed the word *nadir* (of an arc), which he defined to be the right line ſubtending the double of the arc; ſo that his nadir of an arc, was the double of our ſine of the ſame arc; and therefore whatever he proves of the former, may be applied to the latter, ſubſtituting the double ſine for the nadir.

The radius has ſince been decimally divided; but the ſexageſimal diviſions of the arc have continued in uſe to this day. Indeed our countrymen Briggs and Gellibrand, having a general diſlike to all ſexageſimal diviſions, made an attempt at ſome reformation of this cuſtom, by dividing the degrees of the arcs, in their tables, into centeſms or hundredth parts, inſtead of minutes or 60th parts. The ſame was alſo recommended by Vieta and others; and a decimal diviſion of the whole quadrant might perhaps ſoon have followed, had it not been for the tables of Vlacq which came out a little after, to every ten ſeconds, or 6th part of a minute.—But the compleat reformation would be, to expreſs all arcs by their real lengths, namely in equal parts of the radius decimally divided: of which more in its proper place.

It is not to be doubted that many of the ancients wrote on the ſubject of trigonometry, as being a neceſſary part of aſtronomy; although few of their labours on that branch have come to our knowledge, and ſtill fewer of the writings themſelves have been handed down to us.

We are in poſſeſſion of the three books of Menelaus on ſpherical trigonometry; but the 6 books are loſt which he wrote on chords, being probably a treatiſe on the conſtruction of trigonometrical tables.

The trigonometry of Menelaus was much improved by Ptolemy (Claudius Ptolemæus) the celebrated philoſopher and mathematician. He was born at Peluſium, taught aſtronomy at Alexandria in Egypt, and died in the year of Chriſt 147, being the 78th year of his age. In the firſt book of his Almageſt, Ptolemy delivers a table of arcs and chords, with the method of conſtruction. This table contains 3 columns: in the 1ſt are the arcs to every half degree or 30 minutes; in the ſecond are their chords, expreſſed in degrees, minutes and ſeconds, of which degrees the radius contains 60; and in the 3d column are the differences of the chords anſwering to 1 minute of the arcs, or the 30th part of the differences between the chords in the 2d column. In the conſtruction of this table, among others, Ptolemy ſhews, for the firſt time that we know of, this property of any quadrilateral figure inſcribed in a circle, namely that the rectangle under the two diagonals, is equal to the ſum of the two rectangles under the oppoſite ſides.

This method of computation by the chords, continued in uſe till about the middle centuries after Chriſt; when it was changed for that of the ſines, which were about that time introduced into trigonometry by the Arabians, who in other reſpects much improved this ſcience, which they received from the Greeks, introducing among other things the three or four theorems, or axioms, which we uſe at preſent as the foundation of our modern trigonometry.

 The

The other great improvements that have been made in this branch, are due to the Europeans. Theſe improvements they have gradually introduced ſince they received this ſcience from the Arabians. And, although theſe latter people had long uſed the Indian, or decimal, ſcale of arithmetic, it does not appear that they varied from the Greek, or ſexageſimal, diviſion of the radius, by which the chords and ſines were expreſſed.

This alteration is ſaid to have been firſt made by George Purbach, who was ſo called from his being a native of a place of that name between Auſtria and Bavaria. He was born in 1423, ſtudied mathematics and aſtronomy at the univerſity of Vienna, where he was afterwards profeſſor of thoſe ſciences, though but for a ſhort time, the learned world quickly ſuffering a great loſs by his immature death, which happened in 1462, at the age of 39 years only. Purbach, beſides enriching trigonometry and aſtronomy with ſeveral new tables, theorems, and obſervations, ſuppoſed the radius to be divided into 600,000 equal parts, and computed the ſines of the arcs, for every ten minutes, in ſuch equal parts of the radius, by the decimal notation.

This project of Purbach was completed by his diſciple, companion, and ſucceſſor John Muller, or Regiomontanus, who was ſo called from the place of his nativity, the little town of Mons Regius, or Koningſberg in Franconia, where he was born in the year 1436. Regiomontanus not only extended the ſines to every minute, the radius being 600,000, as deſigned by Purbach, but afterwards diſliking that ſcheme, as evidently imperfect, he computed them likewiſe to the radius of 1,000,000, for every minute of the quadrant. He alſo introduced the tangents into trigonometry, the canon of which he called *fœcundus*, becauſe of the many and great advantages ariſing from them. Beſides theſe he enriched trigonometry with many theorems and precepts. Through the benefit of all theſe improvements, except for the uſe of logarithms, the trigonometry of Regiomontanus is but little inferior to that of our own time. His treatiſe on both plane and ſpherical trigonometry is in 5 books; it was written about the year 1464, and printed in folio at Nuremberg in 1533. And in the fifth book are various problems concerning rectilinear triangles, ſome of which are reſolved by means of algebra: a proof that this ſcience was not wholly unknown in Europe before the treatiſe of Lucas de Burgo. Regiomontanus died in 1476 at the age of 40 years only, being then at Rome, whither he had been invited by the pope, to aſſiſt in the reformation of the calendar, and was ſuſpected to have been poiſoned there by the ſons of George Trebizonde, in revenge for the death of their father, which was ſaid to have been cauſed by the grief he felt on account of the criticiſms made by Regiomontanus on his tranſlation of Ptolemy's Almageſt.

Soon after this, ſeveral other mathematicians contributed to the improvement of trigonometry, by extending and enlarging the tables, though few of their works have been printed; and particularly John Werner of Nuremburg, who was born in 1468 and died in 1528, and who is ſaid to have written five books on triangles.

About the year 1500, Nicholas Copernicus, the famous modern reſtorer of the true ſolar ſyſtem, wrote a brief treatiſe on trigonometry both plane and ſpherical, with the deſcription and conſtruction of the canon of chords, or their

halves,

halves, nearly in the manner of Ptolemy; to which is ſubjoined a canon of ſines, with their differences, for every ten minutes of the quadrant, to the radius 100,000. This tract is inſerted in the firſt book of his *Revolutiones Orbium Cœleſtium*, firſt printed in folio at Nuremburg 1543. It is remarkable that he does not call theſe lines *ſines*, but *ſemiſſes ſubtenſarum*, namely of the double arcs.—Copernicus was born at Thorn in 1473, and died in 1543.

In 1553 was publiſhed the *Canon Fœcundus*, or table of tangents, of Eraſmus Reinhold, profeſſor of mathematics in the academy of Wurtemburg. He was born at Salfieldt in Upper Saxony, in the year 1511, and died in 1553.

To Franciſcus Maurolycus, abbot of Meſſina in Sicily, we owe the introduction of the *Tabula Benefica*, or canon of ſecants, which came out about the ſame time, or little before. But Lanſbergius erroneouſly aſcribes this to Rheticus. And the tangents and ſecants are both aſcribed to Reinhold, by Briggs, in his *Mathematica ab antiquis minùs cognita* (pa. 30. Appendix to Ward's Lives of the Profeſſors of Greſham College).

Francis Vieta was born in 1540 at Fontenai, or Fontenai-le-Comte, in Lower Poitou, a province of France. He was maſter of requeſts at Paris, where he died in 1603, being the 63d of his age. Among other branches of learning in which he excelled, he was one of the moſt reſpectable mathematicians of the 16th century, or indeed of any age. His writings abound with marks of great originality, and the fineſt genius, as well as intenſe application. Among them are ſeveral pieces relating to trigonometry, which may be found in the collection of his works publiſhed at Leyden in 1646, by Francis Schooten, beſides another large and ſeparate volume in folio, publiſhed in the author's life-time at Paris in 1579, containing trigonometrical tables with their conſtruction and uſe; very elegantly printed by the king's mathematical printer, with beautiful types and rules, the differences of the ſines, tangents and ſecants, and ſome other parts, being printed with red ink for the better diſtinction; but inaccurately executed, as he himſelf teſtifies in pa. 323 of his other works above-mentioned. The firſt part of this curious volume is intitled *Canon Mathematicus, ſeu ad Triangula, cum Appendicibus*, and contains a great variety of tables uſeful in trigonometry. The firſt of theſe is what he more peculiarly calls *Canon Mathematicus, ſeu ad Triangula*, which contains all the ſines, tangents, and ſecants for every minute of the quadrant, to the radius 100,000 with all their differences; and towards the end of the quadrant the tangents and ſecants are extended to 8 or 9 places of figures. They are arranged like our tables at preſent, increaſing on the left-hand ſide to 45 degrees, and then returning upwards by the right-hand ſide to 90 degrees; ſo that each number and its complement ſtand together on the ſame line. But here the canon of what we now call tangents is denominated *fœcundus*, and that of the ſecants *fœcundiſſimus*. For the general idea prevailing in the form of theſe tables, is not that the lines repreſented by the numbers are thoſe which are drawn in and about a circle, as ſines, tangents and ſecants, but the three ſides of right-angled triangles; this being the way in which thoſe lines had always been conſidered, and which ſtill continued for ſome time longer. And therefore he conſiders the canon as a ſeries of plane right-angled triangles, one ſide being conſtantly 100,000; or rather as three ſeries of ſuch triangles, for he

makes

makes a diſtinct ſeries for each of the three varieties, namely, according as the hypotenuſe or the baſe or the perpendicular is repreſented by the conſtant number 100,000, which is ſimilar to the radius. Making each ſide conſtantly 100,000, the other two ſides are computed to every magnitude of the acute angle at the baſe, from one minute up to 90 degrees or the whole quadrant. Each of the three ſeries therefore conſiſts of two parts, as repreſenting the two variable ſides of the triangle. When the hypotenuſe is made the conſtant number 100,000, the two variable ſides of the triangle are the perpendicular and baſe, or our ſine and coſine; when the baſe is 100,000, the perpendicular and hypotenuſe are the variable parts, forming the *canon fœcundus et fœcundiſſimus*, or our tangent and ſecant; and when the perpendicular is made the conſtant 100,000, the ſeries contains the variable baſe and hypotenuſe, or alſo *canon fœcundus et fœcundiſſimus*, or our cotangent and coſecant. Of courſe therefore the table conſiſts of ſix columns, 2 for each of the three ſeries, beſides the two columns on the right and left for minutes, from 0 to 60 in each degree.

The ſecond of theſe tables is ſimilar to the firſt, but all in rational numbers, conſiſting, like *it*, of 3 ſeries of 2 columns each, the radius, or conſtant ſide of the triangle, in each ſeries, being 100,000, as before, and the other two ſides *accurately* expreſſed in integers and rational vulgar fractions. So that we have here the canon of *accurate* ſines, tangents and ſecants, or a ſeries of about 4300 rational right-angled triangles. But then the ſeveral correſponding arcs of the quadrant, or angles of thoſe triangles, are not expreſſed. Inſtead of them are inſerted, in the firſt column next the margin, a ſeries of numbers decreaſing from the beginning to the end of the quadrant, which are called *numeri primi baſeos*. It is from theſe numbers that Vieta conſtructs the ſides of the three ſeries of right-angled triangles, one ſide in each ſeries being the conſtant number 100,000, as before. The theorems by which theſe ſeries of rational triangles are computed from the *numeri primi baſeos*, or marginal numbers, are inſerted all in one page at the end of this ſecond table, and in the modern notation they may be briefly expreſſed thus. Let p be the primary or marginal number on any line, and r the conſtant radius or number 100,000; then if r denote the hypotenuſe of the right-angled triangle, the perpendicular and baſe, or the ſine and coſine, will be reſpectively

$$\frac{pr}{\frac{1}{4}p^2+1} \text{ and } r - \frac{2r}{\frac{1}{4}p^2+1}, \text{ which laſt we may reduce to } \frac{\frac{1}{4}p^2-1}{\frac{1}{4}p^2+1}r);$$

when r denotes the baſe of the right-angled triangle, the perpendicular and hypotenuſe, or the tangent and ſecant, are expreſſed by

$$\frac{pr}{\frac{1}{4}p^2-1} \text{ and } r + \frac{2r}{\frac{1}{4}p^2-1}, \text{ which laſt we may reduce to } \frac{\frac{1}{4}p^2+1}{\frac{1}{4}p^2-1}r);$$

and when r denotes the perpendicular of the right-angled triangle, the baſe and hypotenuſe, or the cotangent and coſecant, are then expreſſed by

$$\tfrac{1}{4}pr - \frac{r}{p} \text{ (or } \frac{\frac{1}{4}p^2-1}{p}r) \text{ and } \tfrac{1}{4}pr + \frac{r}{p} \text{ (or } \frac{\frac{1}{4}p^2+1}{p}r).$$

So that Vieta's general values will be as we have here collected them together in the following expreſſions immediately under the words ſine, coſine, &c; and

juſt below Vieta's forms I have here placed the others to which they reduce and are equivalent, which are more contracted, but not ſo well adapted to the expeditious computation as Vieta's forms.

Sine	Coſine	Tangent	Secant	Cotangent	Coſecant
$\frac{pr}{\frac{1}{4}p^2+1}$	$r - \frac{2r}{\frac{1}{4}p^2+1}$	$\frac{pr}{\frac{1}{4}p^2-1}$	$r + \frac{2r}{\frac{1}{4}p^2-1}$	$\frac{1}{4}pr - \frac{r}{p}$	$\frac{1}{4}pr + \frac{r}{p}$
$\frac{p}{\frac{1}{4}p^2+1}r$	$\frac{\frac{1}{4}p^2-1}{\frac{1}{4}p^2+1}$	$\frac{p}{\frac{1}{4}p^2-1}r$	$\frac{\frac{1}{4}p^2+1}{\frac{1}{4}p^2-1}r$	$\frac{\frac{1}{4}p^2-1}{p}r$	$\frac{\frac{1}{4}p^2+1}{p}r$

All theſe expreſſions it is evident are rational; and by aſſuming p of different values, from the firſt theorems Vieta computed the correſponding ſides of the triangles, and ſo expreſſed them all in integers and rational fractions.

To the foregoing principal tables are ſubjoined ſeveral other ſmaller tables, or ſhort ſpecimens of large ones: as, a table of the ſines, tangents and ſecants for every ſingle degree of the quadrant, with the correſponding lengths of the arcs, the radius being 100,000,000; another table of the ſines, tangents and ſecants, for each degree alſo, expreſſed in ſexageſimal parts of the radius as far as the 3d order of parts; alſo two other tables for the multiplication and reduction of ſexageſimal quantities.

The ſecond part of this volume is intituled *Univerſalium Inſpectionum ad Canonem Mathematicum Liber ſingularis.* It contains the conſtruction of the tables, a compendious treatiſe on plane and ſpherical trigonometry, with the application of them to a great variety of curious ſubjects in geometry and menſuration, treated in a very learned manner; as alſo many curious obſervations concerning the quadrature of the circle, the duplication of the cube, &c. Computations are here given of the ratio of the diameter of a circle to the circumference, and of the length of the ſine of 1 minute, both to many places of figures; by which he found that the ſine of 1 minute is between 2,908,881,959
2,908,882,056; alſo that, the diameter of a circle being 100,000,000,00, the perimeter of the inſcribed and circumſcribed polygon of 393216 ſides, will be as follows,

perim. of the inſcrib. polygon 314,159,265,35
perim. of the circum. polygon 314,159,265,37

and that therefore the circumference of the circle lies between thoſe two numbers.

Although no author's name appears to the volume I have been deſcribing, there can be no doubt of its being the performance of Vieta; for, beſides bearing evident marks of his maſterly hand, it is mentioned by himſelf in ſeveral parts of his other works collected by Schooten, and in the preface to thoſe works by Elzevir, the printer of them; as alſo in M. Montucla's *Hiſtoire des Mathematiques*; which are the only notices I have ever ſeen or heard of concerning this book, the copies of which are ſo rare, that I never ſaw one beſides that which is in my own poſſeſſion, nor ever met with any other perſon at all acquainted with ſuch a book.

In the other works of Vieta, publiſhed at Leyden in 1646 by Schooten, as mentioned above, there are ſeveral other pieces relating to trigonometry; ſome of

of which, on account of their originality and importance, are very deſerving of particular notice in this place. And firſt, the very excellent theorems, here firſt of all given by our author, relating to angular ſections, the geometrical demonſtrations of which are ſupplied by that ingenious geometrician Alexander Anderſon, a native of Aberdeen. We find here theorems for the chords (and conſequently ſines) of the ſums and differences of arcs; and for the chords of arcs that are in arithmetical progreſſion, namely, that the firſt or leaſt chord is to the 2d, as any one after the firſt is to the ſum of the two next leſs and greater, for example as the 2d to the ſum of the 1ſt and 3d, and as the 3d to the ſum of and 2d and 4th, and as the 4th to the ſum of the 3d and 5th, &c; ſo that the 1ſt and 2d being given, all the reſt are found from them by one ſubtraction and one proportion for each, in which the 1ſt and 2d terms are conſtantly the ſame: next are given theorems for the chords of any multiples of a given arc or angle, as alſo the chords of their ſupplements to a ſemicircle, which are ſimilar to the ſines and coſines of the multiples of given angles; and the concluſions from them are expreſſed in this manner: 1ſt, that if c be the chord of the ſupplement of a given arc a, to the radius 1, then the chords of the ſupplements of the multiple arcs, will be as in the annexed table: where the author obſerves that the ſigns are alternately + and —; that the vertical columns of numeral co-efficients to the terms of the chords, are the ſeveral orders of figurate numbers, which he calls triangular, pyramidal, triangulo-triangular, triangulo-pyramidal, &c. *generated in the ordinary way by continual additions; not indeed from unity,* AS IN THE GENERATION OF POWERS, but beginning with the number 2; and that the powers obſerve always the ſame progreſſion: ſecondly, that if the chord of an arc a be called 1, and d the chord of the double arc $2a$, then the chords of the ſeries of multiple arcs will be as in this table: where the author remarks as before on the law of the powers, ſigns, and co-efficients, theſe being the orders of figurate numbers, raiſed from unity by continual additions, *after the manner of the geneſis of powers,* which generation in that way he ſpeaks of as a thing generally known, but without giving any hint how the co-efficients of the terms of any power may be found from one another only, and independent of thoſe of any other power, as it was afterwards, and firſt of all, I believe, done by Henry Briggs, about the year 1600: and 3dly, that if C be the chord of any arc a, to the radius 1, then the ſeries of the chords

Arcs	Chords of the Supplements.
$1a$	c
$2a$	$c^2 - 2$
$3a$	$c^3 - 3c$
$4a$	$c^4 - 4c^2 + 2$
$5a$	$c^5 - 5c^3 + 5c$
$6a$	$c^6 - 6c^4 + 9c^2 - 2$
$7a$	$c^7 - 7c^5 + 14c^3 - 7c$ &c.
&c.	

Arcs	Chords.
$1a$	1
$2a$	d
$3a$	$d^2 - 1$
$4a$	$d^3 - 2d$
$5a$	$d^4 - 3d^2 + 1$
$6a$	$d^5 - 4d^3 + 3d$
$7a$	$d^6 - 5d^4 + 6d^2 - 1$
$8a$	$d^7 - 6d^5 + 10d^3 - 4d$
&c.	&c.

and ſupplemental chords of the multiple arcs, will be thus; where the values are alternately chords and chords of the ſupplements of the arcs on the ſame line, and the law of the powers and co-efficients as before, but every alternate couplet of lines having their ſigns changed.

Arcs	Chords and Chords of Sup.
$1a$	Chord $= + C$
$2a$	Sup. ch. $= - C^2 + 2$
$3a$	Chord $= - C^3 + 3C$
$4a$	Sup. ch. $= + C^4 + 4C^2 + 2$
$5a$	Chord $= + C^5 - 5C^3 + 5C$
$6a$	Sup. ch. $= - C^6 + 6C^4 - 9C^2 + 2$
$7a$	Chord $= - C^7 + 7C^5 - 14C^3 + 7C$
&c.	&c.

Another curious theorem is added to the above, for finding the ſum of all theſe chords drawn in a ſemicircle, from one end of the diameter to every point in the circumference, thoſe points dividing the circumference into any number of equal parts; namely, as the leaſt chord is to the diameter, ſo is the ſum of the ſaid leaſt chord and diameter and greateſt chord, to double the ſum of all the chords including the diameter as one of them.

As the above theorems are chiefly adapted for the chords of multiple angles, a few problems and remarks are then added (whether by Vieta or Anderſon does not clearly appear, but I think by the latter) concerning the application of them, to the ſection of angles into ſubmultiples, and thence to the computation of the chords or ſines, or a canon of triangles. The general precept for the angular ſections is this; ſelect one of the above equations adapted to the proper number of the ſection, in which will be concerned the powers of the unknown or required quantity, as high as the index of the ſection; and from this equation find that quantity by the known methods for the reſolution of equations. Examples are given of three different ſections, namely, for 3, 5, and 7 equal parts, the forms for which are reſpectively theſe

$$3C - C^3 \ldots\ldots\ldots = g$$
$$5C - 5C^3 + C^5 \ldots\ldots = g$$
$$7C - 14C^3 + C^5 - C^7 = g$$

where g is the chord of the given arc or angle, and C the required chord of the 3d, 5th, or 7th part of it. And it is ſhewn geometrically that the firſt of theſe equations has two real poſitive roots, the ſecond 3, and the laſt 4; alſo from the ſame principles the relations of theſe roots are pointed out.

The method then annexed for conſtructing the canon of ſines from the foregoing theorems, is thus: By dividing the radius in extreme-and-mean ratio, is obtained the ſine of 18 degrees; this quinquiſected gives the ſine of 3° 36′: Again by triſecting the arc of 60°, there is obtained the ſine of 20°; this again triſected gives that of 6° 40′; and this biſected gives that of 3° 20′: Then, by the theorem for the difference of two arcs, there will be found the ſine of 16′, the difference between 3° 36′ and 3° 20′: Laſtly, by four ſucceſſive biſections, will at length be found the ſines of 8′, 4′, 2′ and 1′. This laſt being found, the ſines of its multiples, and again of the multiples of theſe multiples, &c, throughout the quadrant, are to be taken by the proper theorems before laid down.

And the ſame ſubject is ſtill farther purſued and explained in the tract containing the anſwer given by Vieta to the problem propoſed to the whole world by Adrianus Romanus.

In the ſame collection of Vieta's works, from page 400 to 432, is given a complete

plete treatife on practical trigonometry, containing rules for refolving all the cafes of plane and fpherical triangles, by the *Canon Mathematicus*, or table of fines, tangents and fecants.

The next authors whofe labours in this way have been printed, are Rheticus, Otho, and Pitifcus: to all of whom we owe very great improvements in trigonometry.

George Joachim Rheticus, profeffor of mathematics in the univerfity of Wurtemburg, and fometime pupil to Copernicus, died in 1576, in the 60th year of his age. He conceived and executed the great defign of computing the triangular canon for every ten feconds of the quadrant, to the radius (1,000,000,000,000,000) confifting of 1 followed by 15 cyphers. The feries of fines which Rheticus computed to this radius, for every ten feconds, and for every fingle fecond in the firft and laft degree of the quadrant, was publifhed in folio at Francfort 1613 by Pitifcus, who himfelf added a few of the firft fines computed to the radius 10,000,000,000,000,000,000,000.

But the large work, or whole trigonometrical canon, computed by Rheticus, was publifhed in 1596 by Valentine Otho, mathematician to the Electoral Prince Palatine. This vaft work contains all the three feries for the whole canon of right-angled triangles (being fimilar to the fines, tangents and fecants, by which names I fhall call them) with all the differences of the numbers, to the radius 10,000,000,000.

Prefixed to thefe tables are feveral books on their conftruction and ufe in plane and fpherical trigonometry, &c. Of thefe, the firft three are by Rheticus himfelf; namely, book the 1ft containing the demonftrations of 9 lemmas concerning the properties of certain lines drawn in and about circles: the 2d book contains 10 propofitions relative to the fines and cofines of arcs, together with thofe of their fums and differences, their halves and doubles, &c. The 3d book teaches, in 13 propofitions, the conftruction of the canon to the radius 1,000,000,000,000,000. By fome of the common properties of geometry having determined the fines of a few principal arcs, as 30°, 36°, &c, in the firft propofition by continual bifections, he finds the fines of various other arcs, down to 45 minutes. Then in the fecond propofition, by the theorems for the fums and differences of arcs, he finds all the fines and cofines, up to 90 degrees, in a feries of arcs differing by 1° 30′. And, in the third propofition, by the continual addition of 45′, he obtains all the fines and cofines in the feries whofe common difference is 45′. In the 4th propofition, beginning with 45′, and continually bifecting, he finds the fines and cofines of the feries of half arcs till he arrives at the arc of 14^{viii} 19^{ix}, the fine of which is found to be 1, and its cofine 999,999,999,999,999. In the fifth propofition are computed the fine and cofine of 30 feconds or half a minute. In the 6th and 7th propofitions are computed the fines and cofines for every minute, from 1′ to 45′, as well as of many larger arcs. The 8th propofition extends the computation for fingle minutes much farther. In propofition 9 and 10 are computed the tangents and fecants for all arcs in the feries whofe common difference is 45′; and thefe are deduced from the fines of the fame arcs by one proportion for each. In the remaining three propofitions, 11, 12, 13, are computed the tangents and fecants for feveral fmall angles. And from

from all these primary ſines, tangents, and ſecants, the whole canon is deduced and compleated.

The remaining books in this work are by the editor Otho; namely, a treatiſe, in one book, on right-angled plane triangles, the caſes of which are reſolved by the tables; then right-angled ſpherical trigonometry in four books; next oblique ſpherical trigonometry in five books; and laſtly ſeveral other books, containing various ſpherical problems.

Next after the above are placed the tables themſelves, containing, for every 10 ſeconds, the ſines, tangents and ſecants, with all the differences annexed to each, in a ſmaller character. The numbers, however, are not called ſines, tangents, and ſecants, but, like Vieta's before deſcribed, they are conſidered as repreſenting the ſides of right-angled triangles, and titled accordingly. They are alſo in like manner divided into three ſeries, namely, according as the radius, or conſtant ſide of the triangle, is made the hypotenuſe, or the greater leg, or the leſſer leg of the triangle. When the hypotenuſe is made the conſtant radius 10,000,000,000, the two columns of this caſe or ſeries are called the perpendicular and baſe, which are our ſine and coſine; when the greater leg is the conſtant radius, the two columns of this ſeries are titled hypotenuſe and perpendicular, which are our ſecant and tangent; and when the leſſer leg is conſtant, the two columns in this caſe are called hypotenuſe and baſe; which are our coſecant and cotangent. After this large canon is printed another ſmaller table, which is ſaid to be the two columns of the third ſeries, or coſecants and cotangents, with their differences, but to 3 places of figures leſs, or to the radius 10,000,000. But I cannot diſcover the reaſon for adding this leſſer table, even if it were correct, which is very far from being the caſe, the numbers being uniformly erroneous, and different from the former through the greateſt part of the table.

Towards the cloſe of the 16th century many perſons wrote on the ſubject of trigonometry, and the conſtruction of the triangular canon. But, their writings being ſeldom printed till many years afterwards, it is not eaſy to aſſign their order in reſpect of time. I ſhall therefore mention but a few of the principal authors, and that without pretending to any great preciſion on the ſcore of chronological precedence.

In 1591 Philip Lanſbergius firſt publiſhed his *Geometria Triangulorum* in four books, with the canon of ſines, tangents, and ſecants; a brief but very elegant work; the whole being clearly explained: and it is perhaps the firſt ſet of tables titled with thoſe words. The ſines, tangents and ſecants of the arcs to 45 degrees, with thoſe of their complements, are each placed in adjacent columns, in a very commodious manner, continued forwards and downwards to 45 degrees, and then returning backwards and upwards to 90 degrees: the radius is 10,000,000, and a ſpecimen of the firſt page of the table is as follows.

0 Sines

0	Sines		Tangents		Secants		
0	0	10,000,000	0	Infinitum.	10,000,000	Infinitum.	60
1	2,909	9,999,999	2,909	34,377,466,738	10,000,000	34,377,468,193	59
2	5,818	9,999,998	5,818	17,188,731,915	10,000,002	17,188,734,824	58
3	8,727	9,999,996	8,727	11,459,152,994	10,000,004	11,459,157,357	57
4	11,636	9,999,993	11,636	8,594,363,048	10,000,007	8,594,368,866	56
5	14,544	9,999,989	14,544	6,875,488,693	10,000,011	6,875,495,966	55
&c							&c
							89

Of this work, the firſt book treats of the magnitude and relations of ſuch lines as are conſidered in and about the circle, as the chords, ſines, tangents, and ſecants. In the ſecond book is delivered the conſtruction of the trigonometrical canon, by means of the properties laid down in the firſt book: After which follows the canon itſelf. And in the third and fourth books is ſhewn the application of the table, in the reſolution of plane and ſpherical triangles.—Lanſberg, who was born in Zealand 1561, was many years a miniſter of the goſpel, and died at Middleburg in 1632.

The trigonometry of Bartholomew Pitiſcus was firſt publiſhed at Francfort in the year 1599. This is a very compleat work; containing, beſides the triangular canon, with its conſtruction and uſe in reſolving triangles, the application of trigonometry to problems of ſurveying, altimetry, architecture, geography, dialling, and aſtronomy. The conſtruction of the canon is very clearly deſcribed: And in the third edition of the book in the year 1612, he boaſts to have added, in this part, arithmetical rules for finding the chords of the 3d, 5th, and other uneven parts of an arc, from the chord of that arc being given; ſaying that it had been heretofore thought impoſſible to give ſuch rules: But, after all, thoſe boaſted methods are only the application of the double rule of Falſe-Poſition to the then known rules for finding the chords of multiple arcs; namely, making the ſuppoſition of ſome number for the required chord of a ſubmultiple of any given arc, then from this aſſumed number computing what will be the chord of its multiple arc, and which is to be compared with that of the given arc; then the ſame operation is performed with another ſuppoſition; and ſo on as in the double rule of poſition. The canon contains the ſine, tangent, and ſecant for every minute of the quadrant, in ſome parts to 7 places of figures, in others to 8; as alſo the differences for every 10 ſeconds. The ſines, tangents, and ſecants are alſo given for every 10 ſeconds in the firſt and laſt degree of the quadrant, for every 2 ſeconds in the firſt and laſt 10 minutes, and for every ſingle ſecond in the firſt and laſt minute. In this table the ſines, tangents and ſecants are continued downwards on the left-hand pages as far as to 45 degrees, and then returned upwards on the right-hand pages, ſo that the complements are always on the ſame line in the oppoſite or facing pages.

The mathematical works of Chriſtopher Clavius (a German jeſuit, who was born at Bamberg in 1537) in five large folio volumes, were printed at Moguntia, or Mentz, in 1612, the year in which the author died, at the age of 75. In the firſt volume we find a very ample and circumſtantial treatiſe on trigonometry, with

with Regiomontanus's canon of ſines for every minute, as alſo canons of tangents and ſecants, each in a ſeparate table, to the radius 10000000, and in a form continued forwards all the way up to 90 degrees. The explanation of the conſtruction of the tables is very compleat, and is chiefly extracted from Ptolemy, Purbach, and Regiomontanus. The ſines have the differences ſet down for each ſecond, that is, the quotients ariſing from the differences of the ſines divided by 60. About the year 1600 Ludolph Van Collen or à Ceulen, a reſpectable Dutch Mathematician, wrote his book *de circulo & adſcriptis*, in which he treats fully and ably of the properties of lines drawn in and about the circle, and eſpecially of chords or ſubtenſes, with the conſtruction of the canon of ſines. The geometrical properties from which theſe lines are computed, are the ſame as thoſe uſed by former writers: but his mode of computing and expreſſing them, is different from theirs; for they actually extracted all the roots, &c, at every ſtep or ſingle operation, in decimal numbers; but he retained the radical expreſſions to the laſt, making them however always as ſimple as poſſible: thus, for inſtance, he determines the ſides of the polygons of 4, 8, 16, 32, &c, ſides inſcribed in the circle whoſe radius is 1, to be as in the table annexed: where the point before any figure ($\surd .2$) ſignifies the root of all that follows it; ſo the laſt line is in our notation the ſame as $\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}$. And as the perfect management of ſuch ſurds was then not generally known, he added a very neat tract on that ſubject, to facilitate the computations. Theſe, together with other diſſertations on ſimilar geometrical matters, were tranſlated from the Dutch language into Latin by Willebrord Snell, and publiſhed at (Lugd. Batav.) Leyden in 1619. It was in this work that Ludolph determined the ratio of the diameter to the circumference of the circle to 36 figures, ſhewing that, if the diameter be 1, the circumference will be

No. of ſides	Length of each ſide.
4	$\surd 2$
8	$\surd .2 - \surd 2$
16	$\surd .2 - \surd .2 + \surd 2$
32	$\surd .2 - \surd .2 + \surd .2 - \surd 2$
&c	&c.

greater than 3·14159,26535,89793,23846,26433,83279,50288,
but leſs than 3·14159,26535,89793,23846,26433,83279,50289;

which ratio was by his order, in imitation of Archimedes, engraven on his tombſtone, as is witneſſed by the ſaid Snell, pa. 54, 55, *Cyclometricus*, publiſhed at Leyden two years after, in which he treats the ſame ſubject in a ſimilar manner, recomputing and verifying Ludolph's numbers. And in the ſame book he alſo gives a variety of geometrical approximations, or mechanical ſolutions, to determine very nearly the lengths of arcs, and the areas of ſectors and ſegments of circles.

Beſides the *Cyclometricus*, and another geometrical work *(Apollonius Batavus)* publiſhed in 1608, the ſame Snellius wrote alſo four others intitled *doctrinæ triangulorum canonicæ*, in which are contained the canon of ſecants, and in which the conſtruction of ſines, tangents and ſecants, together with the dimenſion or calculation of triangles, both plane and ſpherical, are briefly and clearly treated. After the author's death this work was publiſhed in 8vo, at Leyden 1627, by

Martinus Hortenſius, who added to it a tract on ſurveying and ſpherical problems. Willebrord Snell was born in 1591 at Royen, and died in 1627, being only 35 years of age. He was profeſſor of mathematics in the univerſity of Leyden, as was alſo his father Rodolph Snell.

Alſo in 1627, Francis van Schooten publiſhed at Amſterdam, in a ſmall neat form, tables of ſines, tangents and ſecants for every minute of the quadrant, to 7 places of figures, the radius being 10,000,000; together with their uſe in the trigonometry of plane triangles. Theſe tables have a great character for their accuracy, being declared by the author to be without one ſingle error. This, however, muſt not be underſtood of the laſt figure of the numbers, which I find to be very often erroneous, ſometimes in exceſs and ſometimes in defect, by not being always ſet down to the neareſt unit. Schooten died in 1659, while the ſecond volume of his ſecond edition of Deſcartes' geometry was in the preſs. He was alſo author of ſeveral other valuable works in geometry and other branches of the mathematics.

The foregoing are the principal writers on the tables of ſines, tangents and ſecants, before the invention of logarithms, which happened about this time, namely, ſoon after the year 1600. Tables of the natural numbers were now all compleated, and the methods of computing them nearly perfected: And therefore, before entering on the diſcovery and conſtruction of logarithms, I ſhall ſtop here awhile to give a ſummary of the manner in which the ſaid natural ſines, tangents and ſecants were actually computed, after having been gradually improved from Hipparchus, Menelaus, and Ptolemy, who uſed only the chords, down to the beginning of the 17th century, when ſines, tangents, ſecants and verſed ſines were in uſe, and when the method hitherto employed had received its utmoſt improvement.

In this explanation I ſhall here firſt enumerate the theorems by which the calculations were made, and then deſcribe the application of them to the computation itſelf.

Theorem 1. The ſquare of the diameter of a circle is equal to the ſum of the ſquares of the chord of an arc and of the chord of its ſupplement to a ſemicircle.

2. The rectangle under the two diagonals of any quadrilateral figure inſcribed in a circle, is equal to the ſum of the two rectangles under the oppoſite ſides.

3. The ſum of the ſquares of the ſine and coſine (hitherto called the ſine of the complement), is equal to the ſquare of the radius.

4. The difference between the ſines of two arcs that are equally diſtant from 60 degrees, or $\frac{1}{6}$ of the whole circumference, the one as much greater as the other is leſs, is equal to the ſine of half the difference of thoſe arcs, or of the difference between either arc and the ſaid arc of 60 degrees.

5. The ſum of the coſine and verſed ſine, is equal to the radius.

6. The ſum of the ſquares of the ſine and verſed ſine, is equal to the ſquare of the chord, or to the ſquare of double the ſine of half the arc.

7. The ſine is a mean proportional between half the radius and the verſed ſine of double the arc.

8. A mean proportional between the verſed ſine and half the radius, is equal to the ſine of half the arc.

9. As radius is to the ſine, ſo is twice the coſine to the ſine of twice the arc.

10. As the chord of an arc is to the ſum of the chords of the ſingle and double arc, ſo is the difference of thoſe chords to the chord of thrice the arc.

11. As the chord of an arc is to the ſum of the chords of twice and thrice the arc, ſo is the difference of thoſe chords to the chord of five times the arcs.

12. And in general, as the chord of an arc is to the ſum of the chords of n times and $n + 1$ times the arc, ſo is the difference of thoſe chords to the chord of $2n + 1$ times the arc.

13. The ſine of the ſum of two arcs, is equal to the ſum of the products of the ſine of each multiplied by the coſine of the other and divided by the radius.

14. The ſine of the difference of two arcs, is equal to the difference of the ſaid two products divided by the radius.

15. The coſine of the ſum of two arcs, is equal to the difference between the products of their ſines and of their coſines divided by the radius.

16. The coſine of the difference of two arcs, is equal to the ſum of the ſaid products divided by the radius.

17. A ſmall arc is equal to its chord or ſine, nearly.

18. As coſine is to ſine, ſo is radius to tangent.

19. The radius is a mean proportional between the tangent and cotangent.

20. Half the difference between the tangent and cotangent of an arc, is equal to the tangent of the difference between the arc and its complement. Or, the ſum ariſing from the addition of double the tangent of an arc with the tangent of half its complement, is equal to the tangent of the ſum of that arc and the ſaid half complement.

21. The ſquare of the ſecant of an arc, is equal to the ſum of the ſquares of the radius and tangent.

22. The radius is a mean proportional between the ſecant and coſine. Or, as coſine is to radius, ſo is radius to ſecant.

23. The radius is a mean proportional between the ſine and coſecant.

24. The ſecant of an arc is equal to the ſum of its tangent and the tangent of half its complement. Or, the ſecant of the difference between an arc and its complement, is equal to the tangent of the ſaid difference added to the tangent of the leſſer arc.

25. The ſecant of an arc is equal to the difference between the tangent of that arc and the tangent of the arc added to half its complement. Or, the ſecant of the difference between an arc and its complement, is equal to the difference between the tangent of the ſaid difference and the tangent of the greater arc.

From ſome of theſe 25 theorems, extracted from the writers before mentioned, and a few propoſitions of Euclid's Elements, they compiled the whole table of ſines, tangents, and ſecants, nearly in the following manner.

By the elements were computed the ſides of a few of the regular figures inſcribed in a circle, which were the chords of ſuch parts of the whole circumference as are expreſſed by the number of ſides, and therefore the halves of thoſe chords the ſines of the halves of the arcs. So, if the radius be 10,000,000, the ſides of the following figures will give the annexed chords and ſines.

The

The figure	Arcs ſub-tended	Its chord, or ſide	Half arc	Its ſines or $\frac{1}{2}$ chord
Triangle	120°	17,320,508	60°	8,660,254
Square	90	14,142,136	45	7,071,068
Pentagon	72	11,755,705	36	5,877,853
Hexagon	60	10,000,000	30	5,000,000
Decagon	36	6,180,340	18	3,090,170
Quindecagon	24	4,158,234	12	2,079,117

Of ſome, or all of theſe, the ſines of the halves were continually taken, by theorem the 6th, 7th, or 8th, and of their complements, by the ſame theorems ; and ſo on alternately of the halves and complements, till we arrive at an arc which is nearly equal to its ſine. Thus, beginning with the above arc of 12 degrees, and its ſine, we obtain the halves as follows :

The halves		Their ſines
6°	′	1,045,285
3		523,360
1	30	261,769
	45	130,896
The comp. of theſe		
84		9,945,218
87		9,986,295
88	30	9,996,573
89	15	9,999,143
The halves of theſe		
42		6,691,306
21		3,583,679
10	30	1,822,355
5	15	915,016
43	30	6,883,545
21	45	3,705,574
44	15	6,977,905

The comp. of theſe		Sines
48°	′	7,431,448
69		9,335,804
79	30	9,832,549
84	45	9,958,049
46	30	7,253,744
68	15	9,288,095
45	45	7,163,019
The halves of theſe		
24		4,067,366
34	30	5,664,062
17	15	2,965,416
39	45	6,394,390
23	15	3,947,439
The comp.		
66		9,135,455
55	30	8,241,262
72	45	9,550,199
50	15	7,688,418
66	45	9,187,912

The halves		Sines.
33°	′	5,446,390
16	30	2,840,153
8	15	1,439,426
27	45	4,656,145
Comps.		
57		8,386,706
73	30	9,588,197
81	45	9,896,514
62	15	8,849,876
Halves		
28	30	4,771,588
14	15	2,461,533
36	45	5,983,246
Comps.		
61	30	8,788,171
75	40	9,692,309
53	15	8,012,538
Half		
30	45	5,112,931
Comp.		
59	15	8,594,064

The ſines of ſmall arcs are then deduced in this manner. From the ſine of 45′ above determined, are found the halves, which will be thus :

45′	0″	130,896
22	30	65,449,4
11	15	32,724,8

Now

Now theſe laſt two ſines being evidently in the ſame ratio as their arcs, the ſines of all the leſs ſingle minutes will be found by ſingle proportion. So the 45th part of the ſine of 45′, gives 2909 for the ſine of 1′; which may be doubled, tripled, &c, for the ſines of 2′, 3′, &c, up to 45′.

Then, from all the foregoing primary ſines, by the theorems for halving, doubling, or tripling, and by thoſe for the ſums and differences, the reſt of the ſines are deduced, to compleat the quadrant.

But having thus determined the ſines and coſines of the firſt 30° of the quadrant, that is the ſines of the firſt and laſt 30°, thoſe of the intermediate 30° are, by theor. 4, found by one ſingle ſubſtraction for each ſine.

The ſines of the whole quadrant being thus compleated, the tangents are found by theor. 18, 19, 20, namely for one half of the quadrant by the 18th and 19th, and the other half, by one ſingle addition or ſubſtraction for each, by the 20th theorem.

And laſtly, by theor. 24 and 25, the ſecants are deduced from the tangents by addition and ſubſtraction only.

Among the various means uſed for conſtructing the canon of ſines, tangents and ſecants, the writers above enumerated ſeem not to have been poſſeſſed of the method of differences, ſo profitably uſed ſince, and firſt of all, I believe, by Briggs, in computing his trigonometrical canon and his logarithms, as we ſhall ſee hereafter when we come to deſcribe thoſe works. They took, however, the ſucceſſive differences of the numbers after they were computed, to verify or prove the truth of them; and, if found erroneous, by any irregularity in the laſt differences, from thence they had a method of correcting the original numbers themſelves. At leaſt this method is uſed by Pitiſcus, *trig. lib.* 2, where the differences are extended to the third order.——In pa. 44 of the ſame book alſo is deſcribed, for the firſt time that I know of, the common notation of decimal fractions as now uſed. And this ſame notation was afterwards deſcribed and uſed by baron Neper in *poſitio* 4 and 5 of his poſthumous work on the conſtruction of logarithms, publiſhed by his ſon in the year 1619. But the decimal fractions themſelves may be conſidered as having been introduced by Regiomontanus, by his decimal diviſion of the radius &c. of the circle; and from that time gradually brought into uſe; but continued long to be denoted after the manner of vulgar fractions, by a line drawn between the numerator and denominator, which laſt, however, was ſoon omitted, and only the numerator ſet down with the line below it; thus it was firſt $31\,\frac{35}{100}$, then $31\underline{^{35}}$; afterwards, omitting the line, it became 31^{35}, and laſtly 31_{35} or 31.35 or 31·35: As may be traced in the works of Vieta, and others ſince his time, gradually into the preſent century.

Having often heard it remarked that the word *ſine*, or in Latin and French *ſinus*, is of doubtful origin; and as the various accounts which I have ſeen of its derivation, are very different from one another, it may not be amiſs here to employ a few lines on this matter. Some authors ſay this is an Arabic word; others that it is the ſingle latin word *ſinus*; and in Montucla's *Hiſtoire des Mathematiques*, it is conjectured to be an abbreviation of two Latin words. The conjecture is thus expreſſed by the ingenious and learned author of that excellent hiſtory, at pa. xxxiii among the additions and corrections of the firſt volume: "A l'occaſion

l'occaſion des ſinus dont on parle dans cette page, comme d'une invention des Arabes, voici une étymologie de ce nom, tout-à-fait heureuſe et vraiſemblable. Je la dois à M. Godin, de l'Académie Royale des Sciences, Directeur de l'Ecole de Marine de Cadix. Les ſinus ſont, comme l'on ſçait, des moitiés de cordes; et les cordes en Latin ſe nomment *inſcriptæ*. Les ſinus ſont donc *ſemiſſes inſcriptarum*, ce que probablement on écrivit ainſi pour abréger, S. Ins. Delà enſuite s'eſt fait par abus le mot de ſinus." Now, ingenious as this conjecture is, there appears to be little or no probability for the truth of it. For, in the firſt place, it is not in the leaſt ſupported by quotations from any of the more early books to ſhew that it ever was the practice to write or print the words thus *S. Ins.* upon which the conjecture is founded. Again, it is ſaid the chords are called in Latin *inſcriptæ*; and it is true that they ſometimes are ſo; but I think they are more frequently called *ſubtenſæ*, and the ſines *ſemiſſes ſubtenſarum* of the double arcs, which will not abbreviate into the word *ſinus*. But it may be ſaid, what reaſon have we to ſuppoſe this word to be either a Latin word, or the abbreviation of any Latin words whatever? that it ſeems but proper to ſeek for the etymology of *words* in the language of the inventors of the *things*. For which reaſon it is, that we find the two other words, *tangens* and *ſecans*, are Latin, as they were invented and uſed by authors who wrote in that language. But the ſines are acknowledged to have been invented and introduced by the Arabians, and thence by analogy it would ſeem probable that this is a word of *their* language, and from them adopted, together with the uſe of it, by the Europeans. And indeed Lanſbergius, in the 2d pa. of his trigonometry above-mentioned, expreſsly ſays that it *is* Arabic: His words are, *Vox* ſinus *Arabica eſt, et proinde barbara; ſed, cum longo uſu approbata ſit, & commodior non ſuppetat, nequaquam repudianda eſt: faciles enim in verbis nos eſſe opportet, cum de rebus convenit.* And Vieta ſays ſomething to the ſame purport in pa. 9 of his *Univerſalium Inſpectionum ad Canonem Mathematicum Liber:* His words are, *Breve* ſinus *vocabulum, cum ſit artis, Saracenis præſertim quàm familiare, non eſt ab artificibus explodendum, ad laterum ſemiſſium inſcriptorum denotationem, &c.*

Guarinus alſo is of the ſame opinion: in his *Euclides Adauctus, &c. tract.* xx. *pa.* 307. he ſays, SINUS *vero eſt nomen Arabicum uſurpatum in hanc ſignificationem à mathematicis*; although he was aware that a Latin origin was aſcribed to it by Vitalis, for he immediately adds, *Licet Vitalis in ſuo Lexico Mathematico ex eo velit ſinum appellatum, quòd claudat curvitatem arcûs.*

Long before I either ſaw or heard of any conjecture or obſervation concerning the etymology of the word *ſinus*, I remember that I *imagined* it to be taken from the ſame Latin word, ſignifying breaſt or boſom, and that our ſine was ſo called allegorically. I had obſerved that ſeveral of the terms in trigonometry were derived from a bow to ſhoot with, and its appendages; as *arcus* the bow, *chorda* the ſtring, and *ſagitta* the arrow, by which name the verſed ſine which repreſents it was ſometimes called; alſo that the *tangens* was ſo called from its office, being a line *touching* the circle, and the *ſecans* from its *cutting* the ſame; I therefore imagined that the *ſinus* was ſo called, either from its reſemblance to the breaſt or boſom, or from its being a line drawn within the boſom (*ſinus*) of the arc, or from its being that part of the ſtring (*chorda*) of a bow (*arcus*) which is drawn near the breaſt

(*ſinus*)

(*ſinus*) in the act of ſhooting. And perhaps Vitalis's definition above quoted has ſome alluſion to the ſame ſimilitude.

Alſo Vieta ſeems to allude to the ſame thing in calling *ſinus* an allegorical word, in pa. 417 of his works as publiſhed by Schooten, where, with his uſual judgment and preciſion, he treats of the propriety of the terms uſed in trigonometry for certain lines drawn in and about the circle, of which, as it very well deſerves, I ſhall here extract the principal part, to ſhew the opinion and arguments of ſo great a man on thoſe names. "Arabes autem ſemiſſes inſcriptas duplo, numeris præſertìm æſtimatas, vocaverunt allegoricè SINUS, atque ideò ipſam ſemidiametrum, quæ maxima eſt ſemiſſium inſcriptarum, SINUM TOTUM. Et de iis ſuâ methodo canones exaraverunt qui circumferuntur, ſupputante præſertìm Regiomontano benè juſtè & accuratè, in iis etiam particulis qualium ſemidiameter adſumitur 10,000,000,

"Ex canonibus deinde ſinuum derivaverunt recentiores canonem ſemiſſium circumſcriptarum, quem dixêre *Fœcundum*; & canonem eductarum è centro, quem dixêre *Fœcundiſſimum* et *Beneficum*, hypotenuſis addictum. Atque adeò ſemiſſes circumſcriptas, numeris præſertìm æſtimatas, vocaverunt *Fœcundos*, Sinus numeróſve videlicet; quanquam nihil vetat *Fœcundi* nomen ſubſtantivè accipi. Hypotenuſas autem *Beneficas*, vel etiam ſimpliciter Hypotenuſas: quoniam hypotenuſa in primâ ſerie ſinûs totius nomen retinet. Itaque ne novitate verborum res adumbretur, & alioqui ſua artificibus, eo nomine debita, præripiatur gloria, præpoſita in Canone Mathematico canonicis numeris inſcriptio candidè admonet primam ſeriem eſſe Canonem Sinuum. In ſecundâ verò, partem canonis fœcundi, partem canonis fœcundiſſimi, contineri. In tertiâ, reliquam.

"Sanè præter inſcriptas & circumſcriptas, circulum etiàm adficiunt aliæ lineæ rectæ, velut Incidentes, Tangentes, & Secantes. Verùm illæ voces ſubſtantivæ ſunt, non peripheriarum relativæ. Ac ſecare quidem circulum linea recta tunc intelligitur, cum in duobus punctis ſecat. Itaque non loquuntur benè geometricè, qui eductas è centro ad metas circumſcriptarum vocant ſecantes impropriè, cum ſecantes & tangentes ad certos angulos vel peripherias referunt. Immò verò artem confundunt, cum his vocibus neceſſe habeat uti geometra abs relatione.

"Quare ſi quibus arrideat Arabum metaphora; quæ quidem aut omninò retinenda videtur, aut omninò explodenda; ut ſemiſſes inſcriptas, Arabes vocant ſinus; ſic ſemiſſes circumſcriptæ, vocentur Proſinus Amſinúſve; et eductæ è centro, Tranſſinuoſæ. Sin allegoria diſpliceat, geometrica ſanè inſcriptarum et circumſcriptarum nomina retineantur. Et cum eductæ è centro ad metas circumſcriptarum, non habeant hactenus nomen certum neque elegans, vocentur ſanè *proſemidiametri*, quaſi protenſæ ſemidiametri, ſe habentes ad ſuas circumſcriptas, ſicut ſemidiametri ad inſcriptas."

Againſt the Arabic origin, however, of this word (*ſinus*) may be urged its being varied according to the fourth declenſion of Latin nouns, like *manus*; and that if it were an Arabic word latinized, it would have been ranked under either the firſt, ſecond, or third declenſion, as is uſual in ſuch adopted words.

So that, upon the whole, it will perhaps rather ſeem probable, that the term

ſinus

ſinus is the Latin word anſwering to the name by which the Saracens called that line, and not their word itſelf. And this conjecture ſeems to be rendered ſtill more probable by ſome expreſſions in pa. 4 and 5 of Otho's preface to Rheticus's Canon, where it is not only ſaid that the Saracens called the half-chord of double the arc *ſinus*, but alſo that they called the part of the radius lying between the ſine and the arc *ſinus verſus, vel ſagitta*, which are evidently Latin words, and ſeem to be intended for the Latin tranſlations of the names by which the Arabians called theſe lines, or the numbers expreſſing the lengths of them.

OF

OF LOGARITHMS.

THE trigonometrical canon of natural fines, tangents and fecants, being now brought to a confiderable degree of perfection; the great length and accuracy of the numbers, together with the increafing delicacy and number of aftronomical problems and fpherical triangles, to the refolution of which the canon was applied, urged many perfons, converfant in thofe matters, to endeavour to difcover fome means of diminifhing the great labour and time, requifite for fo many multiplications and divifions, in fuch large numbers as the tables then confifted of. And their chief aim was, to reduce the multiplications and divifions to additions and fubftractions, as much as poffible.

For this purpofe, Nicolas Raymer Urfus Dithmarfus invented an ingenious method, which ferves for one cafe in the fines, namely, when radius is the firft term in the proportion, and the fines of two arcs are the fecond and third terms; for he fhewed that the fourth term or fine, would be found by only taking half the fum or difference of the fines of two other arcs, which fhould be the fum and difference of the lefs of the two former given arcs and the complement of the greater. This is no more in effect than the following well-known theorem in trigonometry: As half the radius is to the fine of one arc, fo is the fine of another arc to the cofine of the difference *minus* the cofine of the fum of the faid arcs. The author publifhed this ingenious device in 1588, in his *Fundamentum Aftronomiæ*. And three or four years afterwards it was greatly improved by Clavius, who adapted it to all proportions in the refolution of fpherical triangles, both for fines, tangents, fecants, verfed fines, &c; and that, whether radius be in the proportion or not. All which he explains very fully in *lem.* 53 *lib.* 1. of his treatife on the *Aftrolabe*. This method, although ingenious, depends not on any abftract property of numbers, but only on the relations of certain lines drawn in and about the circle; and it was therefore rather limited, and fometimes attended with trouble in the application.

After perhaps various other contrivances, inceffant endeavours at length produced the happy invention of logarithms, which are of direct and univerfal application to all numbers abftractedly confidered, being derived from a property inherent in themfelves. This property may be confidered, either as the relation between a geometrical feries of terms and a correfponding arithmetical one, or as the relation between ratios and the meafures of ratios, which comes to much the fame thing, they having been conceived in one of thefe ways by fome of the writers on this fubject, and in the other by the reft of them, as well as in both ways at different times by the fame writer. A fummary idea of this property, and of the probable reflections made on it by the firft writers on logarithms, may be to the following effect.

The learned calculators, about the clofe of the 16th, and beginning of the 17th

century, finding the operations of multiplication and divifion by very long numbers of 7 or 8 places of figures, which they had frequently occafion to perform in folving problems relating to geography and aftronomy, to be exceedingly troublefome, fet themfelves to confider whether it was not poffible to find fome method of leffening this labour, by fubftituting other eafier operations in their ftead. In purfuit of this object they reflected that, fince in every multiplication by a whole number, the ratio, or proportion, of the product to the multiplicand, is the fame as the ratio of the multiplier to unity, it will follow that the ratio of the product to unity (which, according to Euclid's definition of compound ratios, is compounded of the ratios of the faid product to the multiplicand and of the multiplicand to unity) muft be equal to the fum of the two ratios of the multiplier to unity, and of the multiplicand to unity. Confequently, if they could find a fet of artificial numbers that fhould be the reprefentatives of, or fhould be proportional to, the ratios of all forts of numbers to unity, the addition of the two artificial numbers that fhould reprefent the ratios of any multiplier and multiplicand to unity, would anfwer to the multiplication of the faid multiplicand by the faid multiplier; or the fum arifing from the addition of the faid reprefentative numbers, would be the reprefentative number of the ratio of the product to unity; and confequently the natural number to which it fhould be found, in the table of the faid artificial or reprefentative numbers, that the faid fum belonged, would be the product of the faid multiplicand and multiplier. Having fettled this principle, as the foundation of their wifhed-for method of abridging the labour of calculations, they refolved to compofe a table of fuch artificial numbers, or numbers that fhould be reprefentatives of, or proportional to, the ratios of all the common or natural numbers to unity.

The firft obfervation that naturally occurred to them, in the purfuit of this fcheme, was that, whatever artificial numbers fhould be chofen to reprefent the ratios of other whole numbers to unity, the ratio of equality, or of unity to unity, muft be reprefented by 0; becaufe *that* ratio has properly no magnitude, fince, when it is added to or fubtracted from any other ratio, it neither increafes nor diminifhes it.

The fecond obfervation that occurred to them was, that any number whatever might be chofen at pleafure for the reprefentative of the ratio of any given natural number to unity: but that, when once fuch choice was made, all the other reprefentative numbers would be thereby determined; becaufe they muft be greater or lefs than that firft reprefentative number, in the fame proportions in which the ratios reprefented by them, or the ratios of the correfponding natural numbers to unity, were greater or lefs than the ratio of the faid given natural number to unity. Thus, either 1, or 2, or 3, &c, might be chofen for the reprefentative of the ratio of 10 to 1. But if 1 be chofen for it, the reprefentatives of the ratios of 100 to 1 and 1000 to 1, which are double and triple of the ratio of 10 to 1, muft be 2 and 3, and cannot be any other numbers; and, if 2 be chofen for it, the reprefentatives of the ratios of 100 to 1 and 1000 to 1 will be 4 and 6, and cannot be any other numbers; and, if 3 be chofen for it, the reprefentatives of the ratios of 100 to 1 and 1000 to 1 will be 6 and 9, and cannot be any other numbers; and fo on.

The third obſervation that occurred to them was, that, as theſe artificial numbers were repreſentatives of, or proportional to, the ratios of the natural numbers to unity, they muſt be expreſſions of the numbers of ſome ſmaller equal ratios that are contained in the ſaid ratios. Thus, if 1 be taken for the repreſentative of the ratio of 10 to 1, then 3, which is the repreſentative of the ratio of 1000 to 1, will expreſs the number of ratios of 10 to 1 that are contained in the ratio of 1000 to 1. And if, inſtead of 1, we make 10,000,000, or ten millions, the repreſentative of the ratio of 10 to 1 (in which caſe 1 will be the repreſentative of a very ſmall ratio, or *ratiuncula*, which is only the ten-millionth part of the ratio of 10 to 1, or will be the repreſentative of the ratio of the 10,000,000th root of 10, or of the firſt, or ſmalleſt, of 9,999,999 mean proportionals interpoſed between 1 and 10, to 1), the repreſentative of the ratio of 1000 to 1, which will in this caſe be 30,000,000, will expreſs the number of thoſe *ratiunculæ*, or ſmall ratios of the 10,000,000th root of 10 to 1, which are contained in the ſaid ratio of 1000 to 1: and the like may be ſhewn of the repreſentative of the ratio of any other number to unity. And therefore they thought theſe artificial numbers, which thus repreſent, or are proportional to, the magnitudes of the ratios of the natural numbers to unity, might not improperly be called the LOGARITHMS of thoſe ratios, ſince they expreſs the numbers of ſmaller ratios of which they are compoſed. And then, for the ſake of brevity, they called them the *logarithms of the ſaid natural numbers themſelves*, which are the antecedents of the ſaid ratios to unity, of which they are in truth the repreſentatives.

The foregoing method of conſidering this property, leads to much the ſame concluſions as the other way, in which the relations between a geometrical ſeries of terms, and their exponents, or the terms of an arithmetical ſeries, are contemplated. In this latter way, it readily occurred that the addition of the terms of the arithmetical ſeries correſponded to the multiplication of the terms of the geometrical ſeries; and that the arithmeticals would therefore form a ſet of artificial numbers, which, when arranged in tables with their geometricals, would anſwer the purpoſes deſired, as has been explained above.

From this property, by aſſuming four quantities, two of them as two terms in a geometrical ſeries, and the others as the two correſponding terms of the arithmeticals, or artificials, or logarithms, it is evident that all the other terms of both the two ſerieſes may thence be generated: and therefore there may be as many ſets or ſcales of logarithms as we pleaſe, ſince they depend entirely on the arbitrary aſſumption of the firſt two arithmeticals. And all poſſible natural numbers may be ſuppoſed to coincide with ſome of the terms of any geometrical progreſſion whatever, the logarithms or arithmeticals determining which of the terms in that progreſſion they are.

It was proper, however, that the arithmetical ſeries ſhould be ſo aſſumed, as that the term 0 in it might anſwer to the term 1 in the geometricals; otherwiſe the ſum of the logarithms of any two numbers would be always to be diminiſhed by the logarithm of 1, to give the logarithm of the product of thoſe numbers: for which reaſon, making 0 the logarithm of 1, and aſſuming any quantity whatever for the value of the logarithm of any one number, the logarithms of all other numbers were thence to be derived. And hence, like as the multiplication of two numbers is effected by barely adding their logarithms, ſo diviſion is per-

formed by ſubtracting the logarithm of the one from that of the other, raiſing of powers by multiplying the logarithm of the given number by the index of the power, and extraction of roots by dividing the logarithm by the index of the root. It is alſo evident that, in all ſcales or ſyſtems of logarithms, the logarithm of o will be infinite; namely, infinitely negative if the logarithms increaſe with the natural numbers, but infinitely poſitive if the contrary: becauſe that, while the geometrical ſeries muſt decreaſe through infinite diviſions by the ratio of the progreſſion, before the quotient come to o or nothing; the logarithms, or arithmeticals, will in like manner undergo the correſponding infinite ſubtractions or additions of the common equal difference; which equal increaſe or decreaſe, thus indefinitely continued, muſt needs tend to an infinite reſult.

This, however, was no newly-diſcovered property of numbers, but what was always well known to all mathematicians, being treated of in the writings of Euclid, as alſo by Archimedes, who made great uſe of it in his *Arenarius*, or treatiſe on the number of the ſands, namely, in aſſigning the rank or place of thoſe terms, of a geometrical ſeries, produced from the multiplication together of any of the foregoing terms, by the addition of the correſponding terms of the arithmetical ſeries, which ſerved as the indices or exponents of the former: and the reaſon why tables of theſe numbers were not ſooner compoſed, was, that the accuracy and trouble of trigonometrical computations had not ſooner rendered them neceſſary. It is therefore not to be doubted that, about the cloſe of the ſixteenth and beginning of the ſeventeenth century, many perſons had thoughts of ſuch a table of numbers, beſides the few who are ſaid to have attempted it.

Longomontanus has, by ſome, been ſaid to have invented logarithms: but this cannot well be ſuppoſed to have been much more than in idea, ſince he never publiſhed any thing of the kind, nor ever laid claim to the invention, though he lived thirty-three years after they were firſt publiſhed by baron Neper, as he died only in 1647, when they had been long known and received all over Europe. Some circumſtances of this matter are indeed related by Wood in his *Athenæ Oxonienſes*, under the article Briggs, on the authority of Oughtred and Wingate, viz. "That one Dr. Craig, a Scotchman, coming out of Denmark into his own country, called upon John Neper, baron of Marcheſton near Edenburgh, and told him, among other diſcourſes, of a new invention in Denmark (by Longomontanus, as 'tis ſaid) to ſave the tedious multiplication and diviſion in aſtronomical calculations. Neper being ſolicitous to know farther of him concerning this matter, he could give no other account of it, than that it was by proportionable numbers; which hint Neper taking, he deſired him at his return to call upon him again. Craig, after ſome weeks had paſſed, did ſo; and Neper then ſhewed him a rude draught of that he called *Canon mirabilis Logarithmorum*. Which draught, with ſome alterations, he printing in 1614, it came forthwith into the hands of our author Briggs, and into thoſe of Will. Oughtred, from whom the relation of this matter came."

Kepler alſo ſays that one Juſte Byrge, aſſiſtant-aſtronomer to the Landgrave of Heſſe, invented, or projected, logarithms long before Neper did, but that they had never come abroad on account of the great reſervedneſs of their author with regard to his own compoſitions. Byrge is alſo ſaid to have computed a table of natural ſines for every two ſeconds of the quadrant.

But whatever may have been said or conjectured concerning any thing that may have been done by others, it is certain that the world is indebted, for the first publication of logarithms, to John Napier or Nepair *, or in Latin Neper, baron of Merchiston, or Markinston, in Scotland, who died the 3d of April 1618, at the age of 67 years. Baron Napier added considerable improvements to trigonometry; and the frequent numerical computations he performed in this branch, gave occasion to his invention of logarithms, in order to save part of the trouble attending those calculations; and for this reason he adapted his tables peculiarly to trigonometrical uses.

This discovery he published in 1614, in his book intitled *Mirifici logarithmorum canonis descriptio*, reserving the construction of the numbers till the sense of the learned concerning his invention should be known: and, excepting the construction, this is a perfect work on this kind of logarithms, containing in effect the logarithms of all numbers, and the logarithmic sines, tangents, and secants for every minute of the quadrant, together with the description and uses of the tables, as also his definition and idea of logarithms.

Napier explains his notion of logarithms by lines described or generated by the motion of points, in this manner: He first conceives a line to be generated by the equable motion of a point which passes over equal portions of it in equal small moments or portions of time: he then considers another line as generated by the unequal motion of a point, in such manner, that, in the aforesaid equal moments or portions of time, there may be described or cut off, from a given line,

* The origin of which name Crawfurd informs us was from a (less) peer*less* action of one of his ancestors, viz. Donald, second son of the earl of Lenox, in the time of David the Second. " Some English writers, mistaking the import of the term *baron*, have called this celebrated person lord Napier, a Scotch nobleman. He was not indeed a peer of Scotland; but the Peerage of Scotland informs us, that he was of a very ancient, honourable, and illustrious family; that his ancestors, for many generations, had been possessed of sundry baronies; and, amongst others, of the barony of Merchistoun, which descended to him by the death of his father in 1608. Mr. Briggs, therefore, very properly stiles him *Baro Merchistonii*. Now according to Skene, *de verborum significatione*, 'In this realm (of Scotland) he is called an Barrone, quha haldis his landes immediatelie in chiefe of the king, and hes power of Pit and Gallows; *Fossa et Furca*; quhilk was first institute and granted be king Malcolme, quha gave power to the Barrones to ave ane Pit, quhairin wemen condemned for thieft suld be drowned, and ane Gallows, whereupon men thieves and trespassowres suld be hanged, conforme to the doome given in the Barron Court thereanent.' That is, in our modern English, " In this kingdom (of Scotland) a man is called a *Baron* who holds his lands immediately in chief of the King, and has power of Pit and Gallows, *Fossa et Furca*; which was first instituted and granted by king *Malcolm*, who gave power to the Barons to have a Pit, wherein women condemned for theft should be drowned, and a Gallows, whereon men-thieves and offenders (guilty of capital offences) should be hanged, conformably to the judgments given in the court of the Baron concerning the said crimes." So that a Scotch baron, though no peer, was nevertheless a very considerable personage, both in dignity and power. *Reid's Essay on Logarithms*.—The name of the illustrious inventor of logarithms, and his family, has been variously written at different times, and on different occasions. In his own Latin works, and in (perhaps) all other books in Latin, it is *Neper*, or *Neperus Baro Merchistonii:* By Briggs, in a letter to Archbishop Usher, he is called *Naper, lord of Markinston:* In Wright's translation of the logarithms, which was revised by the author himself, and published in 1616, he is called *Nepair, baron of Marchiston*; and the same by Crawfurd and some others. But M'Kenzy and others write it *Napier, baron of Merchiston*; which, being also the orthography now used by the family, I shall adopt in this work. I observe also that the Scotch Compendium of Honour says he was only Sir John Napier; and that his son and heir, Archibald, was the first lord, being raised to that dignity in 1626. Be this however as it may, I shall conform to the common modes of expression, and call him indifferently *baron Napier* or *lord Napier*.

parts

parts which ſhall be continually in the ſame proportion with the reſpective remainders of that line which had before been left: then are the ſeveral lengths of the firſt line the logarithms of the correſponding parts of the latter. Which deſcription of them is ſimilar to this, that the logarithms are a ſeries of quantities or numbers in arithmetical progreſſion, adapted to another ſeries in geometrical progreſſion. The firſt or whole length of the line, which is diminiſhed in geometrical progreſſion, he makes the radius of a circle, and its logarithm o or nothing, repreſenting the beginning of the firſt or arithmetical line; and the ſeveral proportional remainders of the geometrical line, are the natural ſines of all the other parts of the quadrant decreaſing down to nothing, while the ſucceſſive increaſing values of the arithmetical line, are the correſponding logarithms of thoſe decreaſing ſines; ſo that, while the natural ſines decreaſe from radius to nothing, their logarithms increaſe from nothing to infinite. Napier made the logarithm of radius to be o, that he might ſave the trouble of adding and ſubtracting it in trigonometrical proportions, in which it ſo frequently occurred; and he made the logarithms of the ſines, from the entire quadrant down to o, to increaſe, that they might be poſitive, and ſo in his opinion the eaſier to manage, the ſines being of more frequent uſe than the tangents and ſecants, of which the whole of the latter, and half the former, would in his way be of a different affection from the ſines: for it is evident that the logarithms of all the ſecants of the quadrant, and of all the tangents above 45°, or the half-quadrant, would be negative, being the logarithms of numbers greater than the radius whoſe logarithm is made equal to o or nothing.

As to the contents of Napier's table, it conſiſts of the natural ſines and their logarithms, for every minute of the quadrant. Like moſt other tables, the arcs are continued to 45 degrees from top to bottom, on the left-hand ſide of the pages; and then returned backwards from bottom to top, on the right-hand ſide of the pages: ſo that the arcs and their complements, with the ſines, natural and logarithmic, ſtand on the ſame line of the page, in ſix columns; and in another column, in the middle of the page, are placed the differences between the logarithmic ſines and coſines, on the ſame lines, and in the adjacent columns on the right and left; thus making in all ſeven columns in each page. Of theſe columns, the firſt and ſeventh contain the arc and its complement, in degrees and minutes; the ſecond and ſixth, the natural ſine and coſine of each arc; the third and fifth, the logarithmic ſine and coſine; and the fourth, or middle column, the difference between the logarithmic ſine and coſine which are in the third and fifth columns.

To elucidate the deſcription, the firſt page of the table is here inſerted.

Gr. 0		+ | −				
min.	Sinus	Logarithmi	Differentia	Logarithmi	Sinus	
0	0	Infinitum	Infinitum	0	10000000	60
1	2909	81425681	81425680	1	10000000	59
2	5818	74494213	74494211	2	9999998	58
3	8727	70439560	70439560	4	9999996	57
4	11636	67562746	67562739	7	9999993	56
5	14544	65331315	65331304	11	9999989	55
6	17453	63508099	63508083	16	9999984	54
7	20362	61966595	61966573	22	9999980	53
8	23271	60631284	60631256	28	9999974	52
9	26180	59453453	59453418	35	9999967	51
10	29088	58399857	58399814	43	9999959	50
11	31997	57446759	57446707	52	9999950	49
12	34906	56576646	56576584	62	9999940	48
13	37815	55776222	55776149	73	9999928	47
14	40724	55035148	55035064	84	9999917	46
15	43632	54345225	54345129	96	9999905	45
16	46541	53699843	53699734	109	9999892	44
17	49450	53093600	53093577	123	9999878	43
18	52359	52522019	52521881	138	9999863	42
19	55268	51981356	51981202	154	9999847	41
20	58177	51468431	51468361	170	9999831	40
21	61086	50980537	50980450	187	9999813	39
22	63995	50515342	50515137	205	9999795	38
23	66904	50070827	50070603	224	9999776	37
24	69813	49645239	49644995	244	9999756	36
25	72721	49237030	49236765	265	9999736	35
26	75630	48844826	48844539	287	9999714	34
27	78539	48467431	48467122	309	9999692	33
28	81448	48103763	48103431	332	9999668	32
29	84357	47752859	47752503	356	9999644	31
30	87265	47413852	47413471	381	9999619	30
						min.
						Gr. 0

Besides the columns which are actually contained in this table, as above exhibited and described, namely, the natural and logarithmic sines, and the differences of these, the same table is made to serve also for the logarithmic tangents and secants of the whole quadrant, and for the logarithms of common numbers. For the fourth or middle column contains the logarithmic tangents, being equal to the differences between the logarithmic sines and cosines when the logarithm of radius is 0, because cosine : sine : : radius : tangent, that is, in logarithms, tangent = sine − cosine. Also the logarithmic sines made negative become the

the logarithmic cofecants, and the logarithmic cofines made negative are the logarithmic fecants; becaufe fine : radius : : radius : cofecant, and cofine : radius : : radius : fecant; that is, in logarithms, cofecant = o — fine = — fine, and fecant = o — cofine = — cofine. And to make it anfwer the purpofe of a table of logarithms of common numbers, the author directs to proceed thus : A number being given, find that number in any table of natural fines, or tangents, or fecants, and note the degrees and minutes in its arc ; then in his table find the correfponding logarithmic fine, or tangent, or fecant, to the fame number of degrees and minutes ; and it will be the required logarithm of the given number.

After his definitions and defcription of logarithms, Napier explains his table, and illuftrates the precepts with examples, fhewing how to take out the logarithms of fines, tangents, fecants, and of common numbers ; as alfo how to add and fubtract logarithms. He then proceeds to teach the ufes of thofe numbers ; and firft, in finding any of the terms of three or four proportionals, fhewing how to multiply and divide, and to find powers and roots, by logarithms : 2dly, in trigonometry, both plane and fpherical, but efpecially the latter, in which he is very explicit, turning all the theorems for every cafe into logarithms, computing examples to each in numbers, and then enumerating a fet of aftronomical problems of the fphere which properly belong to each cafe. Napier here teaches alfo fome new theorems in fpherical trigonometry ; particularly that the tangent of half the bafe : tang. $\frac{1}{2}$ fum legs : : tang $\frac{1}{2}$ dif. legs : tang. $\frac{1}{2}$ the alternate bafe ; and the general theorem for what are called his five circular parts, by which he condenfes into one rule, in two parts, the theorems for all the cafes of right-angled fpherical triangles, which had been feparately demonftrated by Pitifcus, Lanfbergius, Copernicus, Regiomontanus, and others.

The defcription and ufe of Napier's Canon being in the Latin language, they were tranflated into Englifh by Mr. Edward Wright, an ingenious mathematician, and inventor of the principles of what has commonly, though erroneoufly, been called Mercator's Sailing. He fent the tranflation to the author, at Edinburgh, to be revifed by him before publication ; who having carefully perufed it, returned it with his approbation, and a few lines introduced befides into the tranflation. But, Mr. Wright dying foon after he received it back, it was after his death publifhed, together with the tables, but each number to one figure lefs, in the year 1616, accompanied with a dedication, by his fon Samuel Wright, to the Eaft India Company ; as alfo a preface by Henry Briggs, of whom we fhall prefently have occafion to fpeak more at large, on account of the great fhare he bore in perfecting the logarithms. In this tranflation Mr. Briggs gave alfo the defcription and draught of a fcale that had been invented by Mr. Wright, and feveral other methods of his own, for finding the proportional parts to intermediate numbers, the logarithms having been only printed for fuch numbers as were the natural fines of each minute. And the note which baron Napier inferted in this Englifh edition, and which was not in the original, was as follows : " But becaufe the addition and fubtraction of thefe former numbers may " feem fomewhat painful, I intend (if it fhall pleafe God) in a fecond edition " to fet out fuch logarithms as fhall make thofe numbers above written to fall upon

" decimal

"decimal numbers, ſuch as 100000000, 200000000, 300000000, &c. which are "eaſie to be added or abated to or from any other number." This note had reference to the alteration of the ſcale of logarithms in ſuch manner, that 1 ſhould become the logarithm of the ratio of 10 to 1, inſtead of the number 2·3025851, which Napier had made that logarithm in his table; and which alteration had before been recommended to him by Briggs, as we ſhall ſee preſently. Napier alſo inſerted a ſimilar remark in his *Rabdologia*, which he printed at Edinburgh in 1617.

The following is the preface to Wright's book, which, as far as where it mentions the change from the Latin into Engliſh, is a literal tranſlation of the preface to Napier's original; but what follows that, is added by Napier himſelf: and I willingly inſert it here, as it contains a declaration of the motives which led to this diſcovery, and as the book itſelf is very ſcarce. "Seeing there is nothing (right well-beloved ſtudents in the mathematics) that is ſo troubleſome to mathematicall practiſe, nor that doth more moleſt and hinder calculators, than the multiplications, diviſions, ſquare and cubical extractions of great numbers, which, beſides the tedious expence of time, are for the moſt part ſubject to many ſlippery errors; I began therefore to conſider in my minde, by what certaine and ready art I might remove thoſe hindrances. And having thought upon many things to this purpoſe, I found at length ſome excellent briefe rules to be treated of (perhaps) hereafter. But amongſt all, none more profitable than this, which together with the hard and tedious multiplications, diviſions, and extractions of rootes, doth alſo caſt away from the worke it ſelfe, even the very numbers themſelves that are to be multiplied, divided, and reſolved into rootes, and putteth other numbers in their place, which performe as much as they can do, onely by addition and ſubtraction, diviſion by two, or diviſion by three; which ſecret invention, being (as all other good things are) ſo much the better as it ſhall be the more common, I thought good heretofore to ſet forth in Latine, for the publique uſe of mathematicians. But now ſome of our countrymen in this Iſland, well affected to theſe ſtudies, and the more publique good, procured a moſt learned mathematician to tranſlate the ſame into our vulgar Engliſh tongue; who, after he had finiſhed it, ſent the coppy of it to me, to bee ſeene and conſidered on by myſelf. I having moſt willingly and gladly done the ſame, finde it to bee moſt exact and preciſely conformable to my minde and the originall. Therefore it may pleaſe you, who are inclined to theſe ſtudies, to receive it from me and the Tranſlator, with as much good will as we recommend it unto you. Fare yee well."

There are alſo extant copies of Wright's * tranſlation, with the date 1618 in the title:

* Of this ingenious man I ſhall here inſert in a note the following memoirs, as they have been tranſlated from a Latin piece taken out of the annals of Gonvile and Caius College in Cambridge, viz. "This year (1615) died at London Edward Wright of Garveſton in Norfolk, formerly a fellow of this college; a man reſpected by all for the integrity and ſimplicity of his manners, and alſo famous for his ſkill in the mathematical ſciences: inſomuch that he was deſervedly ſtiled a moſt excellent mathematician by Richard Hackluyt, the author of an original treatiſe of our Engliſh navigations. What knowledge he had acquired in the ſcience of mechanics, and how uſefully he employed that knowledge to the public as well as private advantage, abundantly appear both from the writings he publiſhed, and from the many mechanical operations ſtill extant, which are ſtanding monuments of his

title; but this is not properly a new edition, but only the old work with a new title page adapted to it (the old one being cancelled), together with the addition of fixteen pages of new matter, called "An Appendix to the Logarithms, ſhewing the practice of the calculation of triangles, and alſo a new and ready way for the exact finding out of ſuch lines and logarithms as are not preciſely to be found in the canons." But we are not told by what author; probably it was by Briggs.

Beſides the trouble attending Napier's canon, in finding the proportional parts, when uſed as a table of the logarithms of common numbers, and which was in part remedied by the fore-mentioned contrivances of Wright and Briggs, it was alſo accompanied with another inconvenience, which aroſe from the logarithms being ſometimes + or additive, and ſometimes — or negative, and which required therefore the knowledge of algebraic addition and ſubtraction. And this inconvenience was occaſioned partly by making the logarithm of radius to be o, and the ſines to decreaſe; and partly by the compendious manner in which the author had formed the table, making the three columns of ſines, coſines, and tangents, to ſerve alſo for the other three of coſecants, ſecants, and cotangents.

But this latter inconvenience was well remedied by John Speidell in his *New Logarithms*, firſt publiſhed in 1619, which contained all the ſix columns, and in

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his great induſtry and ingenuity. He was the firſt undertaker of that difficult but uſeful work, by which a little river is brought from the town of Ware, in a new canal, to ſupply the city of London with water; but by the tricks of others he was hindered from completing the work he had begun. He was excellent both in contrivance and execution; nor was he inferior to the moſt ingenious mechanic in the making of inſtruments, either of braſs or any other matter. To his invention is owing whatever advantage Hondius's geographical charts have above others; for it was our Wright that taught Jodocus Hondius the method of conſtructing them, which was till then unknown: but the ungrateful Hondius concealed the name of the true author, and arrogated the glory of the invention to himſelf. Of this fraudulent practice the good man could not help complaining, and juſtly enough, in the preface to his treatiſe of the Correction of Errors in the Art of Navigation; which he compoſed with excellent judgment, and after long experience, to the great advancement of naval affairs. For the improvement of this art he was appointed Mathematical Lecturer by the Eaſt India Company, and read lectures in the houſe of that worthy knight Sir Thomas Smith, for which he had a yearly ſalary of 50 pounds. This office he diſcharged with great reputation, and much to the ſatisfaction of his hearers. He publiſhed in Engliſh a book on the doctrine of the ſphere, and another concerning the conſtruction of ſun-dials. He alſo prefixed an ingenious preface to the learned Gilbert's book on the load-ſtone. By theſe, and other his writings, he has tranſmitted his fame to lateſt poſterity. While he was yet a fellow of this college, he could not be concealed in his private ſtudy, but was called forth to the public buſineſs of the kingdom, by the queen's majeſty, about the year 1593. He was ordered to attend the earl of Cumberland in ſome maritime expeditions. One of theſe he has given a faithful account of, in the way of a journal or ephemeris, to which he has prefixed an elegant hydrographical chart of his own contrivance. A little before his death he employed himſelf about an Engliſh tranſlation of the Book of Logarithms, then lately found out by the honourable baron Napier, a Scotchman, who had a great affection for him. This poſthumous work of his was publiſhed ſoon after, by his only ſon Samuel Wright, who was alſo a ſcholar of this college. He had formed many other uſeful deſigns, but was hindered by death from bringing them to perfection. Of him it may be truly ſaid, that he ſtudied more to ſerve the public than himſelf; and though he was rich in fame, and in the promiſes of the great, yet he died poor, to the ſcandal of an ungrateful age."

Other anecdotes of him, as well as of many other mathematical authors, may be found in the curious hiſtory of navigation, by Dr. James Wilſon, prefixed to Mr. Robertſon's excellent treatiſe on that ſubject.

this order—ſines, coſines, tangents, cotangents, ſecants, coſecants: and they were beſides made all poſitive, by being taken the arithmetical complements of Napier's, that is, they were the remainders left by ſubtracting each of theſe latter from 10,000,000. And the former inconvenience was more effectually removed, by the ſaid Speidell, in an additional table, given in the ſixth impreſſion of the former work, in the year 1624. This was a table of Napier's logarithms for the round or integer numbers 1, 2, 3, 4, 5, &c, to 1000, together with their differences and arithmetical complements; as alſo the halves of the ſaid logarithms, with their differences and arithmetical complements; which halves conſequently were the logarithms of the ſquare roots of the ſaid numbers. Theſe logarithms are however a little varied in their form from Napier's, namely, ſo as to increaſe *from* 1, whoſe logarithm is 0, inſtead of decreaſing *to* 1, or radius, whoſe logarithm Napier made 0 likewiſe; that is, Speidell's logarithm of any number n, is equal to Napier's logarithm of its reciprocal $\frac{1}{n}$: ſo that in this laſt table of Speidell, the logarithm of 1 being 0, the logarithm of 10 is 2,302,584; the logarithm of 100 is twice as much, or 4,605,168; and that of 1000 thrice as much, or 6,907,753.

This table is now commonly called *hyberbolic* logarithms, becauſe the numbers expreſs the areas between the aſymptote and curve of the hyperbola, thoſe areas being limited by ordinates parallel to the other aſymptote, the ordinates decreaſing in geometrical progreſſion. But this is an improper method of denominating them, as ſuch areas may be made to denote any ſyſtem of logarithms whatever, as we ſhall ſhew more at large in the proper place.

In the year 1619, Robert Napier, ſon of the inventor of logarithms, publiſhed a new edition of his late father's *Logarithmorum Canonis Deſcriptio*, together with the promiſed *Logarithmorum Canonis Conſtructio*, and other miſcellaneous pieces written by his father and Mr. Briggs.—Alſo one Bartholomew Vincent, a bookſeller at Lugdunum, or Lyons, in France, printed there an exact copy of the ſame two works in one volume, in the year 1620; which was four years before the logarithms were carried to France by Wingate, wh owas therefore erroneouſly ſaid to have firſt introduced them into that country. But I ſhall treat more particularly of the contents of this work after I have enumerated the other writers on this ſort of logarithms.

In 1618 or 1619, Benjamin Urſinus, mathematician to the Elector of Brandenburg, publiſhed, at Cologn, his *Curſus Mathematicus*, in which is contained a copy of Napier's logarithms, with the addition of ſome tables of proportional parts. And in 1624 he printed, at the ſame place, his *Trigonometria*, with a table of natural ſines and their logarithms, of the Napierian kind and form, to every ten ſeconds in the quadrant; which he had been at much pains in computing.

In the ſame year, 1624, logarithms of nearly the ſame kind were alſo publiſhed at Marpurg, by the famous John Kepler, mathematician to the Emperor Ferdinand the Second, under the title of *Chilias Logarithmorum ad totidem Numeros Rotundos; præmiſſâ Demonſtratione legitimâ Ortûs Logarithmorum, eorúmque Uſûs, &c.* and, the year following, a ſupplement to the ſame; being applied to round or integer numbers, and to ſuch natural ſines as nearly coincide with them.

them. Theſe are exactly the ſame ſort of logarithms as Napier's, being the ſame logarithms of the natural ſines of arcs beginning from the quadrant, whoſe ſine or radius is 10,000,000, the logarithm of which is made 0; and from thence the ſines decreaſing by equal differences down to 0, or the beginning of the quadrant, whilſt their logarithms increaſe to infinity: ſo that the difference between this table and Napier's conſiſts only in this, namely, that in Napier's table the *arc* of the quadrant is divided into equal parts, differing by one minute each, and conſequently their ſines, to which the logarithms are adapted, are irrational or interminate numbers, and only expreſſed by approximate decimals; whereas, in Kepler's table, the *radius* is divided into equal parts, which are conſidered as perfect and terminate ſines, having equal differences, and to which terminate ſines the logarithms are here adapted. By this means indeed the proportions for intermediate numbers and logarithms are eaſier made; but then the correſponding arcs are not terminate, but irrational, and only ſet down to an approximate degree: ſo that Kepler's table is more convenient as a table of the logarithms of common numbers, and Napier's as the logarithmic ſines of the arcs of the quadrant. In both tables the logarithm of the ratio of 10 to 1 is the ſame quantity, namely, 23,025,852; and as the radius, or greateſt ſine, is 10,000,000, whoſe logarithm is made 0, the logarithms of the decuple parts of it will be found by adding 23,025,852 continually, or multiplying this logarithm by 2, 3, 4, &c. and hence the logarithm of 1, the firſt number, or ſmalleſt ſine, in the table, is 161,180,959, or 7 times 2302, &c.

Beſides the two columns of the natural ſines and their logarithms, with the differences of the logarithms, this table of Kepler conſiſts alſo of three other columns; the firſt of which contains the neareſt arcs belonging to thoſe ſines, expreſſed in degrees, minutes, and ſeconds; and the other two expreſs what parts of the radius each ſine is equal to, namely, the one of them in 24th parts of the radius, and minutes and ſeconds of them; and the other in 60th parts of the radius, and minutes of them. As a ſpecimen I have here extracted the laſt page of the table, printed exactly as in the work.

Arcus Circuli, cum differentiis.	Sinus, ſeu numeri abſoluti.	Partes viceſimæ-quartæ.	Logarithmi cum differentiis	Partes ſexagenariæ
19. 34			101.58	
80. 3. 46	98500.00	23. 38. 24	1511.36 +	59. 6
20. 12			101.47	
80. 23. 58	98600.00	23. 39. 50	1409.89 +	59. 10
20. 53			101.37	
80. 44. 51	98700.00	23. 41. 17	1308.52 +	59. 13
21. 42			101.26	
81. 6. 33	98800.00	23. 42. 43	1207.26	59. 17
22. 53			101.17	
81. 29. 26	98900.00	23. 44. 10	1106.09 +	59. 20
24. 6			101.06	
81. 53. 32	99000.00	23. 45. 36	1005.03 +	59. 24
25. 6			100.96	
82. 18. 38	99100.00	23. 47. 2	904.07 +	59. 28
26. 28			100.85	
82. 45. 6	99200.00	23. 48. 29	803.22 +	59. 31
27. 54			100.76	
83. 13. 0	99300.00	23. 49. 55	702.46	59. 35
30. 20			100.65	
83. 43. 20	99400.00	23. 51. 22	601.81	59. 38
32. 40			100.56	
84. 16. 0	99500.00	23. 52. 48	501.25 +	59. 42
36. 30			100.45	
84. 52. 30	99600.00	23. 54. 14	400.80	59. 46
41. 9			100.35	
85. 33. 39	99700.00	23. 55. 41	300.45	59. 49
48. 54			100.25	
86. 22. 33	99800.00	23. 57. 7	200.20	59. 53
1. 3. 42			100.15	
87. 26. 15	99900.00	23. 58. 34	100.05	59. 56
2. 33. 45			100.05	
90. 0. 0	100000.00	24. 0. 0	000000.00	60. 0

To the table Kepler prefixes a pretty conſiderable tract, containing the conſtruction of the logarithms, and a demonſtration of their properties and ſtructure, in which he conſiders logarithms, in the true and legitimate way, as the meaſures of ratios; as ſhall be ſhewn more particularly hereafter, in the next part, where I ſhall treat of the conſtruction of logarithms.

Kepler alſo introduced the logarithmic calculus into his Rudolphine tables, publiſhed in 1627, and inſerted in that work ſeveral logarithmic tables; as, firſt, a table ſimilar to that above deſcribed, except that the ſecond, or column of ſines, or of abſolute numbers, is omitted, and inſtead of it another column is added, ſhewing what part of the quadrant each arc is equal to, namely the quotient, expreſſed in integers and ſexageſimal parts, ariſing from the dividing the whole quadrant by each given arc; 2dly, Napier's table of logarithmic ſines to every minute of the quadrant; alſo two other ſmaller tables adapted for the purpoſes

purpoſes of eclipſes and the latitudes of the planets. In this work alſo Kepler gives a ſummary account of logarithms, with the deſcription and uſe of thoſe that are contained in theſe tables: and here it is that he mentions Juſtus Byrgius, as having had logarithms before Napier publiſhed them.

Beſides the above, ſome few others publiſhed logarithms of the ſame ſort about this time.—But let us now return to treat of the hiſtory of the common or Briggs's logarithms, ſo called becauſe he firſt computed them, and firſt mentioned them, and recommended them to Napier, inſtead of the firſt ſort by him invented.

Mr. Henry Briggs, not leſs eſteemed for his great probity and other eminent virtues than for his excellent ſkill in mathematics, was, at the time of the publication of Napier's logarithms, in 1614, profeſſor of geometry in Greſham college in London, having been appointed the firſt profeſſor after its inſtitution; which appointment he held till January 1620, when he was choſen alſo the firſt Savilian profeſſor of geometry at Oxford, where he died January the 26th, 163$\frac{0}{1}$, aged about 74 years.

On the publication of Napier's logarithms, Briggs immediately applied himſelf to the ſtudy and improvement of them. In a letter to Mr. (afterwards archbiſhop) Uſher, dated the 10th of March 1615, he writes "that he was wholly taken up and employed about the noble invention of logarithms lately diſcovered:" and again, "Napier, lord of Markinſton, hath ſet my head and hands at work with his new and admirable logarithms; I hope to ſee him this ſummer, if it pleaſe God; for I never ſaw a book which pleaſed me better, and made me more wonder." Thus we find that Briggs began very early to compute logarithms: but theſe were not of the ſame kind with Napier's, in which the logarithm of the ratio of 10 to 1 was 2·302,585,1, &c; for in Briggs's firſt attempt he made 1 the logarithm of that ratio; and, from the evidence we have, he appears to be the firſt perſon who formed the idea of this change in the ſcale, which he preſently and generouſly communicated both to the public in his lectures, and to lord Napier himſelf, who afterwards ſaid that he alſo had thought of the ſame thing; as appears by the following extract, tranſlated from the preface to Briggs's *Arithmetica Logarithmica:* "Wonder not (ſays he) that theſe logarithms are different from thoſe which the excellent baron of *Marchiſton* publiſhed in his Admirable Canon. For when I explained the doctrine of them to my auditors at Greſham College in London, I remarked that it would be much more convenient, the logarithm of the ſine total or radius being 0 (as in the *Canon Mirificus*), if the logarithm of the 10th part of the ſaid radius, namely of $5^\circ\ 44'\ 21''$, were 100000 &c. and concerning this I preſently wrote to the author; alſo as ſoon as the ſeaſon of the year, and my public teaching, would permit, I went to Edinburgh, where being kindly received by him, I ſtaid a whole month. But when we began to converſe about the alteration of them, he ſaid that he had formerly thought of it, and wiſhed it; but that he choſe to publiſh thoſe that were already done, till ſuch time as his leiſure and health would permit him to make others more convenient. And as to the nature of the change, he thought it more expedient that 0 ſhould be made the logarithm of 1, and 100000, &c, the logarithm of radius, which I could

I could not but acknowledge was much better. Therefore rejecting thofe which I had before prepared, I proceeded at his exhortation to calculate thefe; and the next fummer I went again to Edinburgh, to fhew him the principal of them; and fhould have been glad to do the fame the third fummer, if it had pleafed God to fpare him fo long."

So that it is plain that Briggs was the inventor of the prefent fcale of logarithms, in which 1 is the logarithm of the ratio of 10 to 1, and 2 that of 100 to 1, &c. and that the fhare which Napier had in them was only advifing Briggs to begin at the loweft number 1, and make the logarithms, or artificial numbers, as Napier had alfo called them, to *increafe* with the natural numbers inftead of *decreafing*; which made no alteration in the figures that expreffed Briggs's logarithms, but only in their affection or figns, changing them from negative to pofitive; fo that Briggs's firft logarithms to the numbers in the fecond column of the annexed tablet, would have been as in the firft column; but, after they were changed, as they are here in the third column; which is a change of no effential difference, as the logarithm of the ratio of 10 to 1, the radix of the natural fyftem of numbers, continues the fame, a change in the logarithm of that ratio being the only circumftance that can effentially alter the fyftem of logarithms, the logarithm of 1 being 0. And the reafon why Briggs, after that interview, rejected what he had before done, and began anew, was probably becaufe he had adapted his new logarithms to the approximate fines of arcs, inftead of the round or integer numbers; and not from their being logarithms of another fyftem, as were thofe of Napier.

B	Num.	N
n	10^n	$-n$
3	·001	-3
2	·01	-2
1	·1	-1
0	1	0
-1	10	1
-2	100	2
-3	1000	3
$-n$	10^n	n

On Briggs's return from Edinburgh to London the fecond time, namely in 1617, he printed the firft thoufand logarithms, to eight places of figures befides the index, under the title of *Logarithmorum Chilias Prima*. But thefe feem not to have been publifhed till after the death of Napier, which happened on the third of April 1618, as before faid; for in the preface to them Briggs fays, "why thefe logarithms differ from thofe fet forth by their moft illuftrious inventor, of ever refpectful memory, in his *Canon Mirificus*, IT IS TO BE HOPED his pofthumous work will fhortly make appear." And as Napier, after communication had with Briggs on the fubject of altering the fcale of logarithms, had given notice, both in Wright's tranflation and in his own *Rabdologia*, printed in 1617, of his intention to alter the fcale (though it appears very plainly that he never intended to compute any more), without making any mention of the fhare which Briggs had in the alteration, this gentleman modeftly gave the above hint. But not finding any regard paid to it in the faid pofthumous work, publifhed by lord Napier's fon in 1619, where the alteration is again adverted to, but ftill without any mention of Briggs; this gentleman thought he could not do lefs than ftate the grounds of that alteration himfelf, as they are above extracted from his work publifhed in 1624.

Thus, upon the whole matter, it feems evident that Mr. Briggs, whether he

he had thought of this improvement in the conſtruction of logarithms, of making 1 the logarithm of the ratio of 10 to 1, before lord Napier, or not, (which is a ſecret that could be known only to Napier himſelf), was the firſt perſon who communicated the idea of ſuch an improvement to the world; and that he did this in his lectures to his auditors at Greſham College, in the year 1615, very ſoon after his peruſal of Napier's *Canon Mirificus Logarithmorum* in the year 1614. He alſo mentioned it to Napier, both by letter in the ſame year, and on his firſt viſit to him in Scotland in the ſummer of the year 1616; when Napier approved the idea, and ſaid it had already occurred to himſelf, and that he had determined to adopt it. It would therefore have been more candid in lord Napier to have told the world, in the ſecond edition of this book, that Mr. Briggs had mentioned this improvement to him, and that he had thereby been confirmed in the reſolution he had already taken, before Mr. Briggs's communication with him, to adopt it in that his ſecond edition, as being better fitted to the decimal notation of arithmetic which was in general uſe. Such a declaration would have been but a piece of juſtice to Mr. Briggs; and the not having made it, cannot but incline us to ſuſpect that lord Napier was deſirous that the world ſhould aſcribe to him alone the merit of this very uſeful improvement of the logarithms, as well as that of having originally invented them; though, if the having firſt communicated an invention to the world be ſufficient to entitle a man to the honour of having firſt invented it, Mr. Briggs had the better title to be called the firſt inventor of this happy improvement of logarithms.

In 1620, two years after the *Chilias Prima* of Briggs came out, Mr. Edmund Gunter publiſhed his *Canon of Triangles*, which contains the artificial or logarithmetic ſines and tangents, for every minute, to ſeven places of figures, beſides the index, the logarithm of radius being 10·0 &c. Theſe logarithms are of the kind laſt agreed upon by Napier and Briggs, and they were the firſt tables of logarithmic ſines and tangents that were publiſhed of this ſort. Gunter alſo in 1623 reprinted the ſame in his book *De Sectore et Radio*, together with the *Chilias Prima* of his old colleague Mr. Briggs, he being profeſſor of Aſtronomy at Greſham College when Briggs was profeſſor of Geometry there; Gunter having been elected to that office the 6th of March 1619, and enjoyed it till his death, which happened on the 10th of December 1626, about the forty-fifth year of his age. In 1623 alſo Gunter applied theſe logarithms of numbers, ſines, and tangents, to ſtraight lines drawn on a ruler; with which proportions in common numbers and trigonometry were reſolved by the mere application of a pair of compaſſes; a method founded on this property, that the logarithms of the terms of equal ratios are equidifferent. This inſtrument, in the form of a two-foot ſcale, is now in common uſe for navigation and other purpoſes, and is commonly called the Gunter. He alſo greatly improved the ſector for the ſame uſes. Gunter was the firſt who uſed the word *co-ſine* for the ſine of the complement of an arc. He alſo introduced the uſe of arithmetical complements into the logarithmical arithmetic, as is witneſſed by Briggs, chap. xv. Arith. Log. And it has been ſaid that he ſtarted the idea of the

logarithmic

logarithmic curve, which was ſo called becauſe the ſegments of its axis are the logarithms of the correſponding ordinates.

The logarithmic lines were afterwards drawn in various other ways. In 1627 they were drawn by Wingate, on two ſeparate rulers, ſliding againſt each other, to ſave the uſe of compaſſes in reſolving proportions. They were alſo in 1627 applied to concentric circles, by Oughtred. Then in a ſpiral form by a Mr. Milburne of Yorkſhire, about the year 1650. And laſtly, in 1657, on the preſent ſliding rule, by Seth Partridge.

The diſcoveries relating to logarithms were carried to France by Mr. Edmund Wingate; but not firſt of all, as he erroneouſly ſays in the preface to his book. He publiſhed at Paris, in 1624, two ſmall tracts in the French language; and afterwards at London, in 1626, an Engliſh edition of the ſame, with improvements. In the firſt of theſe he teaches the uſe of Gunter's ruler; and in the other that of Briggs's logarithms, and the artificial ſines and tangents. Here are contained alſo tables of thoſe logarithms, ſines, and tangents, copied from Gunter. The edition of theſe logarithms printed at London in 1635, and the former editions alſo I ſuppoſe, have the units figures diſpoſed along the tops of the columns, and the tens down the margins, like our tables at preſent; but the whole logarithm, which was only to ſix places of figures, in the angle of meeting: which is the firſt inſtance that I have ſeen of this mode of arrangement.

But proceed we now to the larger ſtructure of logarithms.

Briggs had continued, from the beginning, to labour with great induſtry at the computation of thoſe logarithms of which he before publiſhed a ſhort ſpecimen in ſmall numbers: and in 1624 he produced his *Arithmetica Logarithmica*, a ſtupendous work for ſo ſhort a time! containing the logarithms of 30000 natural numbers, to fourteen places of figures beſides the index, namely from 1 to 20000, and from 90000 to 100000; together with the differences of the logarithms. Some writers ſay that there was another *chiliad*, namely from 100000 to 101000; but none of the copies that I have ſeen have more than the 30000 above mentioned, and they were all regularly terminated in the uſual way with the word FINIS. The preface to theſe logarithms contains, among other things, an account of the alteration made in the ſcale by Napier and himſelf, from which we before gave an extract; and an earneſt ſolicitation to others to undertake the computation for the intermediate numbers, offering to give inſtructions, and paper ready ruled for that purpoſe, to any perſons ſo inclined to contribute to the completion of ſo valuable a work. In the introduction he gives alſo an ample treatiſe on the conſtruction and uſes of theſe logarithms, which will be particularly deſcribed hereafter.——By this invitation, and other means, he had hopes of collecting materials for the logarithms of the intermediate 70000 numbers, whilſt he ſhould employ his own labour more immediately upon the canon of logarithmic ſines and tangents, and ſo carry on both works at once; as indeed they were both equally neceſſary, and he himſelf was now pretty far advanced in years.

Soon after this Adrian Vlacq, or Flack, of Gouda in Holland, completed the intermediate ſeventy chiliads, and republiſhed the *Arithmetica Logarithmica*

at

at that place, in 1627 and 1628, with thofe intermediate numbers, making in the whole the logarithms of all numbers to 10000, but only to ten places of figures. To thefe was added a table of artificial fines, tangents, and fecants, to every minute of the quadrant.

Briggs himfelf lived alfo to complete a table of logarithmic fines and tangents for the hundredth part of every degree, to fourteen places of figures befides the index; together with a table of natural fines for the fame parts to fifteen places, and the tangents and fecants for the fame to ten places; with the conftruction of the whole. Thefe tables were printed at *Gouda*, under the care of Adrian Vlacq, and moftly finifhed off before 1631, though not publifhed till 1633. But his death, which then happened, prevented him from completing the application and ufes of them. However, the performing of this office, when dying, he recommended to his friend Henry Gellibrand, who was then Profeffor of Aftronomy in Grefham College, having fucceeded Mr. Gunter in that appointment. Gellibrand accordingly added a preface, and the application of the logarithms to plane and fpherical trigonometry, &c. and the whole was printed at Gouda, by the fame printer, and brought out in the fame year, 1633, as the *Trigonometria Artificialis* of Vlacq, who had the care of the prefs, as abovefaid. This work was called *Trigonometria Britannica*; and befides the arcs in degrees, and centefms of degrees, it has another column containing the minutes and feconds, anfwering to the feveral centefms in the firft column.

In 1633, as mentioned above, Vlacq printed, at Gouda in Holland, his *Trigonometria Artificialis, five Magnus Canon Triangulorum Logarithmicus ad Decadas Secundorum Scrupulorum conftructus*. This work contains the logarithmic fines and tangents to 10 places of figures, with their differences, for every 10 feconds in the quadrant. To them is alfo added Briggs's table of the firft 20000 logarithms, but carried only to 10 places of figures befides the index, with their differences. The whole is preceded by a defcription of the tables, and the application of them to plane and fpherical trigonometry, chiefly extracted from Briggs's *Trigonometria Britannica*, mentioned above.

Gellibrand publifhed alfo, in 1635, *An Inftitution Trigonometricall*, containing the logarithms of the firft 10000 numbers, with the natural fines, tangents, and fecants, and the logarithmic fines and tangents, for degrees and minutes, all to feven places of figures, befides the index; as alfo other tables proper for navigation; with the ufes of the whole. Gellibrand died the 9th of February 1636, in the 40th year of his age, to the great lofs of the mathematical world.

Befides the perfons hitherto mentioned, who were moftly computers of logarithms, many others have alfo publifhed tables of thofe artificial numbers, more or lefs complete, and fometimes improved and varied in the manner and form of them. I fhall here juft advert to a few of the principal.

In 1626, D. Henrion publifhed, at Paris, a treatife concerning Briggs's logarithms of common numbers from 1 to 20000, to eleven places of figures; with the fines and tangents to eight places only.

In 1631 was printed at London, by one George Miller, a book containing Briggs's logarithms, with their differences, to ten places of figures, befides the

index, for all numbers to 100000; as alſo the logarithmic ſines, tangents, and ſecants for every minute of the quadrant; with the explanation and uſes in Engliſh.

The ſame year, 1631, Richard Norwood publiſhed his *Trigonometrie*; in which we find Briggs's logarithms for all numbers to 10000, and for the ſines, tangents, and ſecants to every minute, both to ſeven places, beſides the index.— In the concluſion of the trigonometry, he complains of the unfair practices of printing Vlacq's book in 1627 or 1628, and the book mentioned in the laſt article. His words are, "Now whereas I have here, and in ſundry places in this book, cited Mr. Briggs his *Arithmetica Logarithmica* (leſt I may ſeem to abuſe the reader), you are to underſtand not the book put forth about a month ſince in Engliſh, as a tranſlation of his, and with the ſame title; being nothing like his, nor worthy his name; but the book which himſelf put forth with this title in Latin, being printed at London, anno 1624. And here I have juſt occaſion to blame the ill dealing of theſe men, both in the matter before-mentioned, and in printing a ſecond edition of his *Arithmetica Logarithmica* in Latin, whilſt he lived, againſt his mind and liking; and brought them over to ſell when the firſt were unſold; ſo fruſtrating thoſe additions which Mr. Briggs intended in his ſecond edition, and moreover leaving out ſome things that were in the firſt edition of ſpecial moment. A practice of very ill conſequence, and tending to the great diſparagement of ſuch as take pains in this kind."

Francis Bonaventure Cavalerius publiſhed at Bologna, in 1632, his *Directorium Generale Uranometricum*, in which are tables of Briggs's logarithms of ſines, tangents, ſecants, and verſed ſines, each to eight places, for every ſecond of the firſt five minutes, for every five ſeconds from five to ten minutes, for every ten ſeconds from ten to twenty minutes, for every twenty ſeconds from twenty to thirty minutes, for every thirty ſeconds from 30′ to 1° 30′, and for every minute in the reſt of the quadrant; which is the firſt table of logarithmic verſed ſines that I know of. In this book are contained alſo the logarithms of the firſt ten chiliads of natural numbers, namely from 1 to 10000, diſpoſed in this manner, all the twenties at top, and from 1 to 19 on the ſide, the logarithm of the ſum being in the ſquare of meeting. In this work alſo I think Cavalerius firſt gave the method of finding the area or ſpherical ſurface contained by various arcs deſcribed on the ſurface of a ſphere.

Alſo in the *Trigonometria* of the ſame author, printed in 1643, beſides the logarithms of numbers from 1 to 1000, to eight places, with their differences, we find both natural and logarithmic ſines, tangents, and ſecants, the former to ſeven and the latter to eight places; namely, to every 10″ of the firſt 30 minutes, to every 30″ from 30′ to 1°; and the ſame for their complements, or backwards through the laſt degree of the quadrant; the intermediate 88° being to every minute only.

Mr. Nathaniel Roe, "Paſtor of Benacre in Suffolke," alſo reduced the logarithmic tables to a contracted form, in his *Tabulæ Logarithmicæ*, printed at London in 1633. Here we have Briggs's logarithms of numbers from 1 to 100000, to eight places; the fifties placed at top, and from 1 to 50 on the

4 ſide;

fide; alfo the firft four figures of the logarithms at top, and the other four down the columns. They contain alfo the logarithmic fines and tangents to every 100th part of the feveral fucceffive degrees, to ten places.

Ludovicus Frobenius publifhed at Hamburgh, in 1634, his *Clavis Univerfa Trigonometriæ*, containing tables of Briggs's logarithms of numbers from 1 to 2000; and of fines, tangents, and fecants, for every minute; both to feven places.

But the table of logarithms of common numbers was reduced to its moft convenient form by John Newton, in his *Trigonometria Britannica*, printed at London in 1658, having availed himfelf of both the improvements of Wingate and Roe, namely, uniting Wingate's difpofition of the natural numbers with Roe's contracted arrangement of the logarithms; the numbers being all difpofed as in our beft tables at prefent, namely, the units along the top of the page, and the tens down the left-hand fide; alfo the firft three figures of each logarithm in the firft column, and the remaining five figures in the other columns, the logarithms being to eight places. This work contains alfo the logarithmic fines and tangents, to eight figures befides the index, for every 100th part of a degree, with their differences, and for 1000th parts in the firft three degrees.—In the preface to this work, Newton takes occafion, as Wingate and Norwood had done before, as well as Briggs himfelf, to cenfure the unfair practices of fome other publifhers of logarithms. He fays, "In the fecond part of this inftitution, thou art prefented with Mr. Gellibrand's Trigonometrie, faithfully tranflated from the Latin copy, that which the author himfelf publifhed under the title of *Trigonometria Britannica*, and not that which Vlacq the Dutchman ftyles *Trigonometria Artificialis*, from whofe corrupt and imperfect copy that feems to be tranflated, which is amongft us generally known by the name of *Gellibrand's Trigonometry*; but thofe who either knew him, or have perufed his writings, can teftify that he was no admirer of the old fexagenary way of working; nay, that he did preferre the decimal way before it, as he hath abundantly teftified in all the examples of this his Trigonometry, which differs from that other which Vlacq hath publifhed, and that which hath hitherto borne his name in Englifh, as in the form, fo likewife in the matter of it; for in the two laft-mentioned editions there is fomething left out in the fecond chapter of plain triangles, the third chapter wholly omitted, and a part of the third in the fpherical, but in this edition nothing; fomething we have added to both, by way of explanation and demonftration."

In 1670, John Caramuel publifhed his *Mathefis Nova*, in which are contained 1000 logarithms both of Napier's and Briggs's form, as alfo 1000 of what he calls the Perfect Logarithms, namely, the fame as thofe which Briggs firft thought of; which differ from the laft only in this, that the one increafes while the other decreafes; the radix, or logarithm of the ratio of 10 to 1, being the fame in both.

The books of logarithms have fince become very numerous; but the logarithms are moftly of that fort invented by Briggs, and which are now in common ufe. Of thefe the moft noted for their accuracy or ufefulnefs, befides the works above mentioned, are Vlacq's fmall volume of tables, particularly

 that

that edition printed at Lyons in 1670; alſo tables printed at the ſame place in 1760; but moſt eſpecially the tables of Sherwin and Gardiner. Of theſe, Sherwin's *Mathematical Tables* in 8vo. form the moſt complete collection of any; containing, beſides the logarithms of all numbers to 101,000, the ſines, tangents, ſecants, and verſed ſines, both natural and logarithmic, to every minute of the quadrant. The firſt edition was in 1706; but the third edition, in 1742, which was reviſed by Gardiner, is eſteemed the moſt correct of any: as to the laſt or fifth edition, in 1771, it is ſo erroneouſly printed, that no dependence can be placed in it, and it is the moſt inaccurate book of tables I ever knew; I have a liſt of ſeveral thouſand errors which I have corrected in it.

Gardiner alſo printed at London, in 1742, a quarto volume of "Tables of Logarithms, for all numbers from 1 to 102,100, and for the ſines and tangents to every ten ſeconds of each degree in the quadrant; as alſo, for the ſines of the firſt 72 minutes to every ſingle ſecond: with other uſeful and neceſſary tables;" namely, a table of Logiſtical Logarithms, and three ſmaller tables to be uſed for finding the logarithms of numbers to twenty places of figures. Of theſe tables of Gardiner, only a ſmall number was printed, and that by ſubſcription; and they are now in the higheſt eſtimation of any logarithms for their accuracy and uſefulneſs.

An edition of Gardiner's collection was alſo elegantly printed at Avignon in France, in 1770, with ſome additions, namely, the ſines and tangents for every ſingle ſecond in the firſt four degrees; and a ſmall table of hyperbolic logarithms, copied from a treatiſe on Fluxions by the late ingenious Mr. Thomas Simpſon: but this is not quite ſo correct as Gardiner's own edition. The tables in all theſe books are to ſeven places of figures.

"The logarithmic canon ſerves to find readily the logarithm of any aſſigned number; and we are told by Dr. Wallis, in the ſecond volume of his Mathematical Works, that an antilogarithmic canon, or one to find as readily the number correſponding to every logarithm, was begun he thinks by Mr. Harriot the algebraiſt (who died in 1621), and completed by Mr. Walter Warner, the editor of Harriot's works, before 1640; which ingenious performance it ſeems was loſt, for want of encouragement to publiſh it."

"A ſmall ſpecimen of ſuch numbers was publiſhed in the Philoſophical Tranſactions, for the year 1714, by Mr. Long of Oxford; but it was not till 1742 that a complete antilogarithmic canon was publiſhed, by Mr. James Dodſon, wherein he has computed the numbers correſponding to every logarithm from 1 to 100,000, to 11 places of figures."

Since the preceeding account was written, and whilſt it was in the preſs, there has been printed at Paris, "*Tables Portatives de Logarithmes, publiées à Londres par Gardiner*," &c. This work is moſt beautifully printed in a neat portable 8vo. volume, and contains all the tables in Gardiner's 4to. volume, with ſome additions and improvements; but with what degree of accuracy remains yet to be determined. And on this, as well as ſeveral other occaſions, it is but juſtice to remark the extraordinary ſpirit and elegance with which the learned men and the artiſans of the French nation undertake and execute works of merit.

THE

THE CONSTRUCTION OF LOGARITHMS, &c.

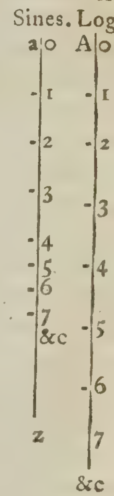
HAVING deſcribed the ſeveral ſorts of logarithms, their riſe and invention, their nature and properties, and given ſome account of the principal early cultivators of them, with the chief collections that have been publiſhed of ſuch tables; I proceed now to deliver a more particular account of the ideas and methods employed by each author, and the peculiar modes of conſtruction which they made uſe of.

And firſt of the great inventor himſelf, lord Napier.

Napier's Conſtruction of Logarithms.

The inventor of logarithms did not adapt them to the ſeries of natural numbers 1, 2, 3, 4, 5, &c, as it was not his principal idea to extend them to all arithmetical operations whatever; but he confined his labours to that circumſtance which firſt ſuggeſted the neceſſity of the invention, and adapted his logarithms to the approximate numbers expreſſing the natural ſines of every minute in the quadrant, as they had been ſet down by former writers on trigonometry.

The ſame reſtricted idea was purſued through his method of conſtructing the logarithms. As the lines of the ſines of all arcs are parts of the radius or ſine of the quadrant, therefore called the *ſinus totus* or whole ſine, he conceived the line of the radius to be deſcribed, or run over, by a point moving along it in ſuch manner, that in equal portions of time it generated, or cut off, parts in a decreaſing geometrical progreſſion, leaving the ſeveral remainders, or ſines, in geometrical progreſſion alſo; whilſt another point, in an indefinite line, deſcribed equal parts of *it* in the ſame equal portions of time; ſo that the reſpective ſums of theſe, or the whole line generated, were always the arithmeticals or logarithms of thoſe ſines. Thus, az is the given radius upon which all the ſines are to be taken, and A &c the indefinite line containing the logarithms; theſe lines being each generated by the motion of a point, beginning at A, a. Now at the end of the 1ſt, 2d, 3d, &c, moments, or equal ſmall portions of time, let the moving points be found at the places marked 1, 2, 3, &c; then za, z1, z2, z3, &c, will be the ſeries of natural ſines, and Ao (or o), A1, A2, A3, &c, will be their logarithms; ſuppoſing the point which generates az to move every where with a velocity decreaſing in proportion to its diſtance from z, namely, its velocity in the points o, 1, 2, 3, &c, to be reſpectively as the diſtances zo, z1, z2, z3, &c; whilſt the velocity of the point generating the logarithmic line A &c remains conſtantly the ſame as at firſt in the point A or o.



Hitherto

Hitherto the author had not fully limited his ſyſtem or ſcale of logarithms, having only ſuppoſed one condition or limitation, namely, that the logarithm of the radius az ſhould be o. But two independent conditions, no matter what they are, were neceſſary to limit the ſcale, or ſyſtem, of logarithms. It did not occur to him, that it was proper to form the other limit by affixing ſome particular logarithm to an aſſigned number, or part of the radius: but as another condition was neceſſary, he aſſumed *this* for it, namely, that the two generating points ſhould begin to move at a, A, with equal velocities; or that the increments a1, A1, deſcribed in the firſt moments, ſhould be equal; as he thought this circumſtance would be attended with ſome little eaſe in the computation: and this is the reaſon that, in his table, the natural ſines and their logarithms, at the complete quadrant, have equal differences; and this is alſo the reaſon why his ſcale of logarithms happens accidentally to agree with what have ſince been called the hyperbolic logarithms, which have numerical differences equal to thoſe of their natural numbers at the beginning; except only that theſe latter increaſe with the natural numbers, and his on the contrary decreaſe; the logarithms of the ratio of 10 to 1 being the ſame in both, namely, 2·302,585,09.

And here by the way it may be obſerved, that Napier's manner of conceiving the generation of the lines of the natural numbers and their logarithms, by the motion of points, is very ſimilar to the manner in which Newton afterwards conſidered the generation of magnitudes in his doctrine of fluxions; and it is alſo remarkable that, in Art. 2. of the *Habitudines Logarithmorum & ſuorum naturalium numerorum invicem*, in the appendix to the *Conſtructio Logarithmorum*, Napier ſpeaks of the velocities of the increments or decrements of the logarithms, in the ſame way as Newton does of his fluxions, namely, where he ſhews that thoſe velocities, or fluxions, are inverſely as the ſines, or natural numbers of the logarithms; which is a neceſſary conſequence of the nature of the generation of thoſe lines, as deſcribed above; with this alteration, however, that now the radius az muſt be conſidered as generated by an equable motion of the point, and the indefinite line A &c by a motion increaſing in the ſame ratio as the other before decreaſed; which is a ſuppoſition that Napier muſt have had in view when he ſtated that relation of the fluxions.

Having thus limited his ſyſtem, Napier proceeds, in the poſthumous work of 1619, to explain his conſtruction of the logarithmic canon: and this he effects in various ways; but chiefly by generating, in a very eaſy manner, a ſeries of proportional numbers and their arithmeticals, or logarithms; and then finding, by proportion, the logarithms to the natural ſines, from thoſe of the neareſt numbers among the original proportionals.

After deſcribing the neceſſary cautions he made uſe of to preſerve a ſufficient degree of accuracy, in ſo long and complex a proceſs of calculation; ſuch as annexing ſeveral cyphers, as decimals ſeparated by a point, to his primitive numbers, and rejecting the decimals thence reſulting after the operations were completed, ſetting the numbers down to the neareſt unit in the laſt figure; and teaching the arithmetical proceſſes of adding, ſubtracting, multiplying, and dividing the limits between which certain unknown numbers muſt lie, ſo as

as to obtain the limits between which the refults muft alfo fall; I fay, after defcribing fuch particulars, in order to clear and fmooth the way, he enters on the great field of calculation itfelf. Beginning at radius 10,000,000, he firft conftructs feveral defcending geometrical feries, but of fuch a nature that they are all quickly formed by an eafy continual fubtraction, and a divifion by 2, or by 10, or 100, &c, which is done by only removing the decimal point fo many places towards the left hand, as there are cyphers in the divifor. He conftructs three tables of fuch feries: The firft of thefe confifts of 100 numbers, in the proportion of radius to radius minus 1, or of 10,000,000 to 9,999,999; all which are found by only fubtracting from each its 10,000,000th part, which part is alfo found by only removing each figure 7 places lower: the laft of thefe 100 proportionals is found to be 9,999,900·000,495,0.

The 2d table contains 50 numbers, which are in the continual proportion of the firft to the laft in the firft table, namely, of 10,000,000.000,000,0 to 9,999,900.000,495,0, or nearly the proportion of 100,000 to 99,999; thefe therefore are found by only removing the figures of each number 5 places lower, and fubtracting them from the fame number: the laft of thefe he finds to be 9,995,001.222,927. And a fpecimen of thefe two tables is here annexed.

No.	Firft Table.	2d Table.
1	10,000,000.000,000,0	10,000,000.000,000
2	9,999,999.000,000,0	9,999,900.000,000
3	9,999,998.000,000,1	9,999,800.001,000
4	9,999,997.000,000,3	9,999,700.003,000
&c	&c till the 100th	&c to the 50th term
50	term, which will be	9,995,001.222,927
100	9,999,900.000,495,0	

The 3d table confifts of 69 columns, and each column of twenty-one numbers or terms; which terms, in every column, are in the continual proportion of 10,000 to 9,995, that is, nearly as the firft is to the laft in the 2d table; and as 10,000 exceeds 9,995 by the 2000th part, the terms in every column will be conftructed by dividing each upper number by 2, removing the figures of the quotient 3 places lower, and then fubtracting them; and in this way it is proper to conftruct only the firft column of 21 numbers, the laft of which will be 9,900,473.578,0: but the 1ft, 2d, 3d, &c, numbers in all the columns, are in the continual proportion of 100 to 99, or nearly the proportion of the firft to the laft in the firft column; and therefore thefe will be found by removing the figures of each preceeding number 2 places lower, and fubtracting them, for the like number in the next column. A fpecimen of this 3d table is as here below.

The 3d Table.					
Terms	1ft Column.	2d Column.	3d Column.	&c. till the	69 Column.
1	10,000,000.000,0	9,900,000.000,0	9,801,000.000,0	&c for the	5,048,858.890,0
2	9,995,000.000,0	9,895,050.000,0	9,796,099.500,0	4th, 5th, 6th,	5,046,334.460,5
3	9,990,002.500,0	9,890,102.475,0	9,791,201.450,3	7th, &c, col.	5,043,811.293,2
4	9,985,007.498,7	9,885,157.423,7	9,786,305.849,5	till the laft	5,041,289.387,9
5	9,980,014.995,0	9,880,214.845,1	9,781,412.696,7	or	5,038,768.743,5
&c	&c till	&c	&c		&c
21	9,900,473.578,0	9,801,468.842,3	9,703,454.153,9		4,998,609.403,4

Thus

Thus he had, in this 3d table, interpofed between the radius and its half, 68 numbers in the continual proportion of 100 to 99; and interpofed between every two of thefe 20 numbers in the proportion of 10,000 to 9,995: and again, in the 2d table, between 10,000,000 and 9,995,000, the two firft of the 3d table, he had 50 numbers in the proportion of 100,000 to 99,999: And laftly, in the 1ft table, between 10,000,000 and 9,999,900, or the 2 firft of the 2d table, 100 numbers in the proportion of 10,000,000 to 9,999,999. That is, in all about 1600 proportionals; all found in the moft fimple manner by little more than eafy fubtractions; which proportionals nearly coincide with all the natural fines from 90° down to 30°.

To obtain the logarithms of all thofe proportionals, he demonftrates feveral properties and relations of the numbers and logarithms, and illuftrates the manner of applying them. The principal of thefe properties are as follows: 1ft, that the logarithm of any fine is greater than the difference between that fine and the radius, but lefs than the faid difference when increafed in the proportion of the fine to radius*; and 2dly, that the difference between the logarithms of two fines, is lefs than the difference of the fines increafed in the proportion of the lefs fine to radius, but greater than the faid difference of the fines increafed in the proportion of the greater fine to radius †.

Hence, by the 1ft theorem, the logarithm of 10,000,000, the radius or firft term in the firft table, being 0, the logarithm of 9,999,999, the 2d term, will be between 1 and 1.000,000,1, and will therefore be equal to 1.000,000,05 very nearly: and this will be alfo the common difference of all the terms or proportionals in the firft table; and therefore by the continual addition of this logarithm, there will be obtained the logarithms of all thefe 100 proportionals: confequently 100 times the faid firft logarithm, or the laft of the above fums, will give 100.000,005, for the logarithm of 9,999,900.000,495,0, the laft of the faid 100 proportionals.

Then, by the 2d theorem, it eafily appears that .000,495,0 is the difference between the logarithms of 9,999,900.000,495,0 and 9,999,900, the laft term of the firft table and the 2d term of the fecond table; this then being added to

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* By this firft theorem, r being radius, the logarithm of the fine s, is between $r-s$ and $\frac{r-s}{s}r$; and therefore, when s differs but little from r, the logarithm of s will be nearly equal to $\frac{\overline{r+s}\times\overline{r-s}}{2s}$, the arithmetical mean between the limits $r-s$ and $\frac{r-s}{s}r$; but ftill nearer to $\overline{r-s}\sqrt{\frac{r}{s}}$ or $\frac{r-s}{s}\sqrt{rs}$, the geometrical mean between the faid limits.

† By this fecond theorem, the difference between the logarithms of the two fines S and s, lying between the limits $\frac{S-s}{s}r$ and $\frac{S-s}{S}r$, will, when thofe fines differ but little, be nearly equal to $\frac{S^2-s^2}{2Ss}r$ or $\frac{\overline{S+s}\times\overline{S-s}}{2Ss}r$, their arithmetical mean; or nearly $=\frac{S-s}{\sqrt{Ss}}r$, the geometrical mean; or nearly $=\frac{S-s}{S+s}2r$, by fubftituting, in the laft denominator, $\frac{1}{2}\cdot\overline{S+s}$ for $\sqrt{Ss}$, to which it is nearly equal.

the

the laſt logarithm, gives 100.0005000 for the logarithm of the ſaid 2d term, as alſo the common difference of the logarithms of all the proportionals in the 2d table; and therefore by continually adding it, there will be generated the logarithms of all theſe proportionals in the 2d table; the laſt of which is 5000.025, anſwering to 9995001.222927 the laſt term of that table.

Again, by the 2d theorem, the difference between the logarithms of this laſt proportional of the 2d table, and the 2d term in the firſt column of the 3d table, is found to be 1.223,538,7; which being added to the laſt logarithm, gives 5001.248,538,7 for the logarithm of 9,995,00 0, the ſaid 2d term of the 3d table, as alſo the common difference of the logarithms of all the proportionals in the firſt column of that table; and this therefore, being continually added, gives all the logarithms of that firſt column, the laſt of which is 100,024.970,77, the logarithm of 9,900,473.5780, the laſt term of the ſaid column.

Finally, by the 2d theorem again, the difference between the logarithms of this laſt number and 9,900,000, the firſt term in the 2d column, is 478.3502; which being added to the laſt logarithm, gives 100,503.3210 for the logarithm of the ſaid firſt term in the 2d column, as well as the common difference of the logarithms of all the numbers on the ſame line in every line of the table, namely, of all the 1ſt terms, of all the 2d, of all the 3d, of all the 4th, &c. terms in all the columns; and which therefore, being continually added to the logarithms in the firſt column, will give the correſponding logarithms in all the other columns.

And thus is compleated what the author calls the radical table, in which he retains only one decimal place in the logarithms (or *artificials*, as he always calls them in his tract on the conſtruction), and four in the naturals. A ſpecimen of the table is as here follows:

Radical Table.						
Terms	1ſt Column.		2d Column.		69th Column.	
	Naturals.	Artificials.	Naturals.	Artificials.	Naturals.	Artificials.
1	10,000,000.0000	0	9,900,000.0000	100,503.3	5,048,858.8900	6,834,225.8
2	9,995,000.0000	5001.2	9,895,050.0000	105,504.6	5,046,333.4605	6,839,227.1
3	9,990,002.5000	10,002.5	9,890,102.4750	110,505.8	5,043,811.2932	6,844,228.3
4	9,985,007.4987	15,003.7	9,885,157.4237	115,507.1	5,041,289.3879	6,849,229.6
5	9,980,014.9950	20,005.0	9,880,214.8451	120,508.3	5,038,768.7435	6,854,230.8
&c	&c till	&c	&c	&c	&c	&c
21	9,900,473.5780	100,025.0	9,801,468.8423	200,528.2	4,998,609.4034	6,934,250.8

Having thus, in the moſt eaſy manner, compleated the radical table, by little more than mere addition and ſubtraction, both for the natural numbers and logarithms; the logarithmic ſines were eaſily deduced from it by means of the 2d theorem, namely, taking the ſum and difference of each tabular ſine and the neareſt number in the radical table, annex 7 ciphers to the difference, divide the reſult by the ſum, and half the quotient will be the difference between the logarithms of the ſaid numbers, namely, between the tabular ſine and radical number; conſequently, adding or ſubtracting this difference, to or from the given logarithm of the radical number, there will be obtained the logarithmic ſine required. And thus the logarithms of all the ſines from radius to the half of it, or from 90° to 30°, were perfected.

 Next,

Next, for determining the ſines of the remaining 30 degrees, he delivers two methods. In the firſt of theſe he proceeds in this manner: Obſerving that the logarithm of the ratio of 2 to 1, or of half the radius, is 6,931,469.22, of 4 to one is the double of this, of 8 to 1 is triple of it, &c; that of 10 to 1 is 23,025,842.34, of 20 to 1 is the ſum of the logarithms of 2 and 10, and ſo on by compoſition for the logarithms of the ratios between 1 and 40, 80, 100, 200, &c. to 10,000,000; he multiplies any given ſine, for an arc leſs than 30 degrees, by ſome of theſe numbers, till he finds the product nearly equal to one of the tabular numbers; then by means of this and the ſecond theorem, the logarithm of this product is found; to which adding the logarithm that anſwers to the multiple abovementioned, the ſum is the logarithm ſought. But the other method is ſtill much eaſier, and is derived from this property, which he demonſtrates, namely, as half radius is to the ſine of half an arc, ſo is the coſine of the ſaid half arc to the ſine of the whole arc; or as $\frac{1}{2}$ radius : ſine of an arc : : coſine of the arc : ſine of double the arc; hence the logarithmic ſine of an arc is found, by adding together the logarithms of half radius and of the ſine of the double arc, and then ſubtracting the logarithmic coſine from the ſum.

And thus the remainder of the ſines, from 30° down to 0, are eaſily obtained. But in this latter way, the logarithmic ſines for full one half of the quadrant, or from 0 to 45 degrees, he obſerves, may be derived; the other half having already been made by the general method of the radical table, by one eaſy diviſion and addition or ſubtraction for each.

I have dwelt the longer on this work of the inventor of logarithms, becauſe I have not ſeen in any author an account of his method of conſtructing his table, although it is perfectly different from any other method uſed by the later computers, and indeed almoſt peculiar to his ſpecies of logarithms. The whole of this work manifeſts great ingenuity in the deſigner, as well as much accuracy. But notwithſtanding the caution he took to obtain his logarithms true to the neareſt unit in the laſt figure ſet down in the tables, by extending the numbers in the computations to ſeveral decimals, and other means; he had been diſappointed of that end, either by the inaccuracy of his aſſiſtant computers or tranſcribers, or through ſome other cauſe; as the logarithms in the table are commonly very inaccurate. It is remarkable too that in this tract on the conſtruction of the logarithms, Lord Napier never calls them logarithms, bu every where *artificials*, as oppoſed in idea to the natural numbers: and this notion of natural and artificial numbers I take to have been his firſt idea of this matter, and that he altered the word *artificials* to *logarithms* in his firſt book, on the deſcription of them, when he printed it, in the year 1614, and that he would alſo have altered the word every where in this poſthumous work if he had lived to print it: for in the two or three pages of appendix, annexed to the work by his ſon from Napier's papers, he again always calls them logarithms. This appendix relates to the change of the logarithms to that ſcale in which 1 is the logarithm of the ratio of 10 to 1, the logarithm of 1, with or without ciphers, being 0; and it appears to have been written after Briggs had communicated to him his idea of that change.

Napier

Napier here in this appendix alſo briefly deſcribes ſome methods by which this new ſpecies of logarithms may be conſtructed. Having ſuppoſed 0 to be the logarithm of 1, and 1 with any number of ciphers, as 10,000,000,000, the logarithm of 10; he directs to divide this logarithm of 10, and the ſucceſſive quotients, ten times by 5, by which diviſions there will be obtained theſe other ten logarithms, namely 2,000,000,000, 400,000,000, 80,000,000, 16,000,000, 3,200,000, 640,000, 128,000, 25,600, 5,120, 1024: then this laſt logarithm, and its quotients, being divided ten times by 2, will give theſe other ten logarithms 512, 256, 128, 64, 32, 16, 8, 4, 2, 1. And the numbers anſwering to theſe twenty logarithms, we are directed to find in this manner; namely, extract the 5th root of 10 (with ciphers), then the 5th root of that root; and ſo on for ten continual extractions of the 5th root; ſo ſhall theſe ten roots be the natural numbers belonging to the firſt ten logarithms, above found in continually dividing by 5: Next, out of the laſt 5th root we are to extract the ſquare root, then the ſquare root of this laſt root, and ſo on for ten ſucceſſive extractions of the ſquare root; ſo ſhall theſe laſt ten roots be the natural numbers correſponding to the logarithms or quotients ariſing from the laſt ten diviſions by the number 2. And from theſe twenty logarithms, 1, 2, 4, 8, 16, &c. and their natural numbers, the author obſerves that other logarithms and their numbers may be formed, namely, by adding the logarithms and multiplying their correſpondent numbers. It is evident that this proceſs would generate rather an antilogarithmic canon, ſuch as Dodſon's, than the table of Briggs; and that the method would alſo be very laborious, ſince, beſides the very troubleſome original extractions of the 5th roots, all the numbers would be very large, by the multiplication of which the ſucceſſive ſecondary natural numbers are to be found.

Our author next mentions another method of deriving a few of the primitive numbers and their logarithms, namely, by taking continually geometrical means, firſt between 10 and 1, then between 10 and this mean, and again between 10 and the laſt mean, and ſo on; and taking the arithmetical means between their correſponding logarithms. He then lays down various relations between numbers and their logarithms, ſuch as that the products and quotients of numbers anſwer to the ſums and differences of their logarithms; and that the powers and roots of numbers anſwer to the products and quotients of the logarithms by the index of the power or root, &c; as alſo that, of any two numbers, whoſe logarithms are given, if each number be raiſed to the power denoted by the logarithm of the other, the two reſults will be equal. He then delivers another method of making the logarithms to a few of the prime integer numbers, which is well adapted for conſtructing the common table of logarithms. This method eaſily follows from what has been ſaid above, and it depends on this property, "that the logarithm of any number in this ſcale, is 1 leſs than the number of places or figures contained in that power of the given number whoſe exponent is 10,000,000,000, or the logarithm of 10, at leaſt as to integer numbers," for they really differ by a fraction, as is ſhewn by Mr. Briggs in his illuſtrations of theſe properties, printed at the end of this appendix to the con-

 ſtruction

ſtruction of logarithms. I ſhall here ſet down one more of theſe relations, as the manner in which it is expreſſed is exactly ſimilar to that of fluxions and fluents, and it is this: Of any two numbers, as the greater is to the leſs, ſo is the velocity of the increment or decrement of the logarithms at the leſs, to the velocity of the increment or decrement of the logarithms at the greater: that is, in our modern notation, as $X : Y :: \dot{y}$ to $\dot{x}$, where $\dot{x}$ and $\dot{y}$ are the fluxions of the logarithms of X and Y.

Kepler's Conſtruction of Logarithms.

The logarithms of Briggs and Kepler were both printed the ſame year, 1624; but as the latter are of the ſame ſort as Napier's, I ſhall here give the author's conſtruction of them, before we proceed to that of Briggs's.

We have already (p. 34 *&* *ſeq*) deſcribed the nature and form of Kepler's logarithms, ſhewing that they are of the ſame kind as Napier's, but only a little varied in the form of the table. It may alſo be added that, in general, the ideas which theſe two maſters had on this ſubject, were of the ſame nature, only it was more fully and methodically laid down by Kepler, who expanded, and delivered in the form of a regular ſcience, the hints that were given by the illuſtrious inventor. The foundation and nature of their methods of conſtruction are alſo the ſame, but only a little varied in their modes of applying them. Kepler here firſt of any treats of logarithms in the true and genuine way of the meaſures of ratios, or proportions *, as he calls them, and that in a very full and ſcientific manner: and this method of his was afterwards followed and abridged by Mercator, Halley, Cotes, and others, as we ſhall ſee in the proper places. Kepler firſt erects a regular and purely mathematical ſyſtem of proportions, and the meaſures of proportions, treated at conſiderable length in a number of propoſitions, which are fully and chaſtely demonſtrated by genuine mathematical reaſoning, and illuſtrated by numerical examples. This part contains and demonſtrates both the nature and the principles of the ſtructure of logarithms. And in the ſecond part he applies thoſe principles in the actual conſtruction of his table, which contains only 1000 numbers and their logarithms, in the form which we before deſcribed: and in this part he indicates the various contrivances made uſe of in deducing the logarithms of proportions one from another, after a few of the leading ones had been firſt formed by the general and more remote principles. He uſes the name *logarithms*, given them by the inventor, being the moſt proper, as expreſſing the very nature and eſſence of thoſe artificial numbers, and containing, as it were, a definition in the very name of them; but without taking any notice of the inventor, or of the origin of thoſe uſeful numbers.

* Kepler almoſt always uſes the term *proportion* inſtead of *ratio*, which I alſo ſhall do in my account of his work, as well as conform in expreſſions and notations to his other peculiarities. It may alſo be here remarked, that I obſerve the ſame practice in deſcribing the works of other authors, the better to convey the idea of their ſeveral methods and ſtyle. And this may ſerve to account for ſome ſeeming inequalities in the language of this hiſtory.

As

As this tract is very curious and important in itself, and is besides very rare and little known, instead of a particular description only, I shall here give a brief translation of both the parts, omitting only the demonstrations of the propositions, and some rather long illustrations of them.

The book is dedicated to Philip, landgrave of Hesse, but is without either preface or introduction, and commences immediately with the subject of the first part, which is intitled *The Demonstration of the Structure of Logarithms*; and the contents of it are as follows:

Postulate 1. That all proportions equal among themselves, by whatever variety of couplets of terms they may be denoted, are measured or expressed by the same quantity.

Axiom 1. If there be any number of quantities of the same kind, the proportion of the extremes is understood to be composed of all the proportions of every adjacent couplet of terms, from the first to the last.

1 *Proposition*. The mean proportional between two terms divides the proportion of those terms into two equal proportions.

Axiom 2. Of any number of quantities, regularly increasing the means, divide the proportion of the extremes into one proportion more than the number of the means.

Postulate 2. That the proportion between any two terms is divisible into any number of parts, until those parts become less than any proposed quantity.

An example of this section is then inserted in a small table, in dividing the proportion which is between 10 and 7 into 1,073,741,824 equal parts, by as many mean proportionals wanting one, namely, by taking the mean proportional between 10 and 7, then the mean between 10 and this mean, and the mean between 10 and the last, and so on for 30 means, or 30 extractions of the square-root, the last, or 30th, of which roots is 9.999,999,996,678,205,690,0 and the 30th power of 2, which is 1,073,741,824, shews into how many parts the proportion between 10 and 7, or between 1000 &c and 700 &c, is divided by 1,073,741,824 means, each of which parts is equal to the proportion between 10.00 &c, and the 30th mean 9.99 &c, that is, the proportion between 10.00 &c and 9.99 &c, is the 1,073,741,824th part of the proportion between 10 and 7. Then by assuming the small difference 0.000,000,003,321,794,310,0, for the measure of the very small element of the proportion of 10 to 7, or for the measure of the proportion of 10.00 &c to 9.99 &c, or for the logarithm of this last term, and multiplying it by 1,073,741,824 the number of parts, the product will give 3.566,749,481,372,221,440,0 for the logarithm of the less term 7 or 700 &c.

Postulate 3. That the extremely small quantity or element of a proportion, may be measured or denoted by any quantity whatever; as for instance, by the difference of the terms of that element.

2 *Proposition*. Of three continued proportionals, the difference of the two first has to the difference of the two latter, the same proportion which the first term has to the 2d, or the 2d to the 3d.

3 *Prop*. Of any continued proportionals, the greatest terms have the greatest difference, and the least terms the least.

4 *Prop*. In any continued proportionals, if the difference of the greatest terms be made the measure of the proportion between *them*, the difference of any other couplet will be less than the true measure of *their* proportion.

5 *Prop*. In continued proportionals, if the difference of the greatest terms be made the measure of their proportion, then the measure of the proportion of the greatest to any other term will be greater than *their* difference.

6 *Prop*.

6 *Prop.* In continued proportionals, if the difference of the greateſt term and any one of the leſs, taken not immediately next to it, be made the meaſure of their proportion; then the meaſure of the proportion which is between the greateſt and any other term greater than the one before taken, will be leſs than the difference of thoſe terms; but the meaſure of the proportion which is between the greateſt term and any one leſs than that firſt taken, will be greater than their difference.

7 *Prop.* Of any quantities placed according to the order of their magnitudes, if any two ſucceſſive proportions be equal, the three ſucceſſive terms which conſtitute them, will be continued proportionals.

8 *Prop.* Of any quantities placed in the order of their magnitudes, if the intermediates lying between any two terms, be not among the mean proportionals which can be interpoſed between the ſaid two terms, then ſuch intermediates do not divide the proportion of thoſe two terms into commenſurable proportions.

Beſides the demonſtrations, as uſual, ſeveral definitions are here given; as of commenſurable proportions, &c.

9 *Prop.* When two expreſſible lengths are not to one another as two figurate numbers of the ſame ſpecies, ſuch as two ſquares, or two cubes; there cannot fall between them other expreſſible lengths, which ſhall be mean proportionals, and as many in number as that ſpecies requires, namely, one in the ſquares, two in the cubes, three in the biquadrates, &c.

10 *Prop.* Of any expreſſible quantities, following in the order of their magnitudes, if the two extremes be not in the proportion of two ſquare numbers, or two cubes, or two other powers of the ſame kind; none of the intermediates divide the proportion into commenſurables.

11 *Prop.* All the proportions, taken in order, which are between expreſſible terms that are in arithmetical proportion, are incommenſurable to one another. As between 8, 13, 18.

12 *Prop.* Of any quantities placed in the order of magnitude, if the difference of the greateſt terms be made the meaſure of their proportion, then the difference between any two others will be leſs than the meaſure of *their* proportion; and if the difference of the two leaſt terms be made the meaſure of their proportion, then the differences of the reſt will be greater than the meaſure of the proportion between their terms.

Corol. If the meaſure of the proportion between the greateſt exceed their difference, then the proportion of this meaſure to the ſaid difference, will be leſs than that of a following meaſure to the difference of its terms. Becauſe proportionals have the ſame ratio.

13 *Prop.* If three quantities follow one another in the order of magnitude; the proportion of the two laſt will be contained in the proportion of the extremes, a leſs number of times than the difference of the two leaſt is contained in the difference of the extremes: And on the contrary, the proportion of the two greateſt will be contained in the proportion of the extremes, oftener than the difference of the former is contained in that of the latter.

Corol. Hence if the difference of the two greater be equal to the difference of

of the two lefs terms, the proportion between the two greater will be lefs than the proportion between the two lefs.

14 *Prop*. Of three equidifferent quantities taken in order, the proportion between the extremes is more than double the proportion between the two greater terms.

Corol. Hence it follows, that half the proportion of the extremes, is greater than the proportion of the two greateft terms, but lefs than the proportion of the two leaft.

15 *Prop*. If two quantities conftitute a proportion, and each quantity be leffened by half the greater; the remainders will conftitute a proportion greater than double the former.

16 *Prop*. The aliquot parts of incommenfurable proportions, are incommen furable to each other.

17 *Prop*. If one thoufand numbers follow one another in the natural order, beginning at 1000, and differing all by unity, viz. 1000, 999, 998, 997, &c; and the proportion between the two greateft 1000, 999, by continual bifection, be cut into parts that are fmaller than the excefs of the proportion between the next two 999, 998, over the faid proportion between the two greateft 1000, 999; and then for the meafure of that fmall element of the proportion between 1000, 999, there be taken the difference of 1000 and that mean proportional which is the other term of the element. Again, if the proportion between 1000, 998, be likewife cut into double the number of parts which the former proportion between 1000, 999, was cut into; and then for the meafure of the fmall element in this divifion, be taken the difference of its terms, of which the greater is 1000. And, in the fame manner, if the proportion of 1000 to the following numbers, as 997, &c, by continual bifection, be cut into particles of fuch magnitude as may be between $\frac{1}{2}$ and $\frac{3}{4}$ of the element arifing from the fection of the firft proportion between 1000 and 999; the meafure of each element will be given from the difference of its terms. Then, this being done, the meafure of any one of the 1000 proportions, will be compofed of as many meafures of its element, as there are of thofe elements in the faid divided proportion. And all thefe meafures, for all the proportions, will be fufficiently exact for the niceft calculations.

All thefe fections and meafures of proportions are performed in the manner of that defcribed at poftulate 2, and the operation is abundantly explained by numerical calculations.

18 *Prop*. The proportion of any number to the firft term 1000 being known; there will alfo be known the proportion of the reft of the numbers in the fame continued proportion, to the faid firft term.

So from the known proportion between 1000 and 900,
there is alfo known the proportion of 1000 to 810, and to 729;
And from 1000 to 800, alfo 1000 to 640, and to 512;
And from 1000 to 700, alfo 1000 to 490, and to 343;
And from 1000 to 600, alfo 1000 to 360, and to 216;
And from 1000 to 500, alfo 1000 to 250, and to 125.

Corol. Hence arifes the precept for fquaring, cubing, &c; as alfo for extracting the fquare root, cube root, &c, out of the firft figures of numbers.

 For

For it will be, as the greateſt number of the chiliad as a denominator, is to the number propoſed as a numerator, ſo is this to the ſquare of the fraction, and ſo is this to the cube.

19 *Prop.* The proportion of a number to the firſt, or 1000, being known; if there be two other numbers in the ſame proportion to each other, then the proportion of one of theſe to 1000 being known, there will alſo be known the proportion of the other to the ſame 1000.

Corol. 1. Hence from the 15 proportions mentioned in prop. 18, will be known 120 others below 1000, to the ſame 1000.

For ſo many are the proportions, equal to ſome one or other of the ſaid 15, that are among the other integer numbers which are leſs than 1000.

Corol. 2. Hence ariſes the method of treating the Rule-of-Three, when 1000 is one of the given terms.

For this is effected by adding to, or ſubtracting from, each other, the meaſures of the two proportions of 1000 to each of the other two given numbers, according as 1000 is, or is not, the firſt term in the Rule-of-Three.

20 *Prop.* When four numbers are proportional, the firſt to the ſecond as the third to the fourth, and the proportions of 1000 to each of the three former are known, there will alſo be known the proportion of 1000 to the fourth number.

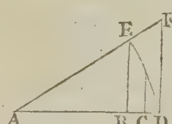
Corol. 1. By this means other chiliads are added to the former.

Corol. 2. Hence ariſes the method of performing the Rule-of-Three, when 1000 is not one of the terms. Namely, from the ſum of the meaſures of the proportions of 1000 to the ſecond and third, take that of 1000 to the firſt, and the remainder is the meaſure of the proportion of 1000 to the fourth term.

Definition. The meaſure of the proportion between 1000 and any leſs number, as before deſcribed, and expreſſed by a number, is ſet oppoſite to that leſs number in the chiliad, and is called its LOGARITHM, that is, the number (αριθμὸς) indicating the proportion (λογον) which 1000 bears to that number, to which the logarithm is annexed.

21 *Prop.* If the firſt or greateſt number be made the radius of a circle, or *ſinus totus*; every leſs number, conſidered as the coſine of ſome arc, has a logarithm greater than the verſed ſine of that arc, but leſs than the difference between the radius and ſecant of the arc; except only in the term next after the radius, or greateſt term, the logarithm of which by the hypotheſis is made equal to the verſed ſine.

That is, if C D be made the logarithm of A C, or the meaſure of the proportion of A C to A D; then the meaſure of the proportion of A B to A D, that is the logarithm of A B, will be greater than B D, but leſs than E F. And this is the ſame as Napier's firſt rule in page 45.



22 *Prop.* The ſame things being ſuppoſed; the ſum of the verſed ſine and exceſs of the ſecant over the radius, is greater than double the logarithm of the coſine of an arc.

Corol.

Corol. The log. coſine is leſs than the arithmetic mean between the verſed ſine and the exceſs of the ſecant.

Precept 1. Any ſine being found in the canon of ſines, and its defect below radius to the exceſs of the ſecant above radius; then ſhall the logarithm of the ſine be leſs than half that ſum, but greater than the ſaid defect or co-verſed ſine.

Let there be the ſine 99,970.1490 of an arc:
Its defect below radius is 29.8510 the co-vers. and leſs than logarithm ſine;
Add the exceſs of the ſecant 29.8599

Sum 59.7109
its half or 29.8555 greater than the logarithm.

Therefore the logarithm is between $\begin{cases} 29.8510 \text{ and} \\ 29.8555. \end{cases}$

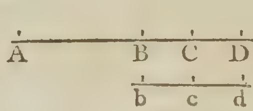
Precept 2. The logarithm of the ſine being found, you will alſo find nearly the logarithm of the round or integer number which is next leſs than your ſine with a fraction, by adding that fractional exceſs to the logarithm of the ſaid ſine.

Thus the logarithm of the ſine 99,970.149 is found to be about 29.854; if now the logarithm of the round number 99,970.000 be required, add 149 the fractional part of the ſine to its logarithm, obſerving the point, thus

$$\left.\begin{array}{r} 29.854 \\ 149 \\ \hline \text{the ſum } 30.003 \end{array}\right\}\text{is the logarithm of the round number 99,970.000 nearly.}$$

23 *Prop.* Of three equidifferent quantities, the meaſure of the proportion between the two greater terms, with the meaſure of the proportion between the two leſſer terms, will conſtitute a proportion, which will be greater than the proportion of the two greater terms, but leſs than the proportion of the two leſſer terms.

Thus if AB, AC, AD be three quantities having the equal differences BC, CD; and if the meaſure of the proportion of AD, AC be cd, and that of AC, AB be bc; then the proportion of cd to cb will be greater than the proportion of AC to AD, but leſs than the proportion of AB to AC.



24 *Prop.* The ſaid proportion between the two meaſures, is leſs than half the proportion between the extreme terms. That is, the proportion between bc, cd, is leſs than half the proportion between AB, AD.

Corol. Since therefore the arithmetical mean divides the proportion into unequal parts, of which the one is greater, and the other leſs, than half the whole; if it be enquired what proportion is between theſe proportions, the anſwer is, that it is a little leſs than the ſaid half.

An example of finding nearly the limits, greater and leſs, to the meaſure of any propoſed proportion.

It being known that the meaſure of the proportion between 1000 and 900 is 10,536.05, required the meaſure of the proportion 900 to 800, where the terms 1000, 900, 800, have equal differences.

 Therefore

Therefore as 9 is to 10 so is 10,536.05 to 11,706.72, which is less than 11,778.30 the measure of the proportion 9 to 8. Again, as the mean proportional between 8 and 10 (which is 8.944,271,9) is to 10 so is 10,536.05 to 11,779.66, which is greater than the measure of the proportion between 9 and 8.

Axiom. Every number denotes an expressible quantity.

25. *Prop.* If the 1000 numbers, differing by 1, follow one another in the natural order; and there be taken any two adjacent numbers, as the terms of some proportion; the measure of this proportion will be to the measure of the proportion between the two greatest terms of the chiliad, in a proportion greater than that which the greatest term 1000 bears to the greater of the two terms first taken, but less than the proportion of 1000 to the less of the said two selected terms.

So of the first 1000 numbers taking any two successive terms, as 501 and 500, the logarithm of the former being 69114.92, and of the latter 69314.72, the difference of which is 199,80. Wherefore by the definition, the measure of the proportion between 501 and 500 is 199.80. In like manner, because the logarithm of the greatest term 1000 is 0, and of the next 999 is 100.05, the difference of these logarithms, and the measure of the proportion between 1000 and 999, is 100.05. Couple now the greatest term 1000 with each of the selected terms 501 and 500; couple also the measure 199.80 with the measure 100.05; so shall the proportion between 199.80 and 100.05 be greater than the proportion between 1000 and 501, but less than the proportion between 1000 and 500.

Corol. 1. Any number below the first 1000 being proposed, as also its logarithm; the differences of any logarithms antecedent to that proposed, towards the beginning of the chiliad, are to the first logarithm (viz. that which is assigned to 999), in a greater proportion than 1000 to the number proposed; but of those which follow towards the last logarithm, they are to the same in a less proportion.

Corol. 2. By this means the places of the chiliad may easily be filled up, which have not yet had logarithms adapted to them by the former propositions.

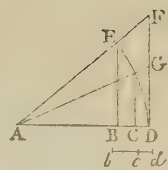
26 *Prop.* The difference of two logarithms, adapted to two adjacent numbers, is to the difference of these numbers, in a proportion greater than 1000 bears to the greater of those numbers, but less than that of 1000 to the less of the two numbers.

This 26 prop. is the same as Napier's second rule at page 47.

27 *Prop.* Having given two adjacent numbers of the 1000 natural numbers, with their logarithmic indices, or the measures of the proportions which those absolute, or round, numbers constitute with 1000 the greatest; the increments or differences of these logarithms will be to the logarithm of the small element of the proportions, as the secants of the arcs whose cosines are the two absolute numbers, are to the greatest number, or the radius of the circle: so that, however, of the said two secants, the less will have to the radius a less proportion, than the proposed difference has to the first of all, but the greater will have a greater proportion, and so also will the mean proportional between the said secants have a greater proportion.

Thus

Thus if BC, CD be equal, alſo *b d* the logarithm of AB, and *c d* the logarithm of AC; then the proportion of *b c* to *c d* will be greater than the proportion of AG to AD, but leſs than that of AF to AD, and alſo leſs than that of the mean proportional between AF and AG to AD.



Corol. 1. The ſame obtains alſo when the two terms differ, not only by the unit of the ſmall element, but by another unit which may be ten-fold, a hundred-fold, or a thouſand-fold of that.

Corol. 2. Hence the differences will be obtained ſufficiently exact, eſpecially when the abſolute numbers are pretty large, by taking the arithmetical mean between two ſmall ſecants, or (if you will be at the labour) by taking the geometrical mean between two larger ſecants, and then by continually adding the differences, the logarithms will be produced.

Corol. 3. *Precept.* Divide the radius by each term of the aſſigned proportion, and the arithmetical mean (or, ſtill nearer, the geometrical mean) between the quotients will be the required increment, which, being added to the logarithm of the greater term, will give the logarithm of the leſs term.

Example.

Let there be given the logarithm of 700, viz. 35,667.4948, to find the logarithm to 699.
Here radius divided by 700 gives 142.8571 &c.
and divided by 699 gives 143.0672 &c.
the arithmetic mean is 142.962
which added to 35,667.4948
gives the logarithm to 699 35,810.4568

Corol. 4. *Precept* for the logarithms of ſines.

The increment between the logarithms of two ſines, is thus found: find the geometrical mean between the coſecants, and divide it by the difference of the ſines; the quotient will be the difference of the logarithms.

Example.

0° 1′	ſine 2909	coſec.	34,377,468.2	
0 2′	ſine 5818	coſec.	17,188,731.9	
	dif. 2909	geom. mean	24,280,000.0	nearly.

The quotient 8000.0 exceeds the required increment of the logarithms, becauſe the ſecants are here ſo large.

Appendix. Nearly in the ſame manner it may be ſhewn, that the ſecond differences are in the duplicate proportion of the firſt, and the third in the duplicate of the ſecond. Thus, for inſtance, in the beginning of the logarithms the firſt difference is 100.00000, viz. equal to the difference of the numbers 100,000.00000 and 99,900.00000; the ſecond, or difference of the differences, 10000; the third 20. Again, after arriving at the number 5000.00000, the logarithms have for a difference 200.00000, which is to the firſt difference, as the number 100,000.00000 to 50,000.00000; but the ſecond difference is 40,000, in which 10,000 is contained four times; and the third 328, in which 20 is contained ſixteen times. But ſince in treating of new matters we labour under the

 want

want of proper words, therefore, lest we should become too obscure, the demonstration is omitted untried.

28 *Prop.* No number expresses exactly the measure of the proportion, between two of the 1000 numbers, constituted by the foregoing method.

29 *Prop.* If the measures of all proportions be expressed by numbers or logarithms; all proportions will not have assigned to them their due portion of measure, to the utmost accuracy.

30 *Prop.* If to the number 1000, the greatest of the chiliad, be referred others that are greater than it, and the logarithm of 1000 be made 0, the logarithms belonging to those greater numbers will be negative.

This concludes the first, or scientific, part of the work, the principles of which Kepler applies, in the second part, to the actual construction of the first 1000 logarithms, which is pretty minutely described. This part is intitled *A very compendious method of constructing the Chiliad of Logarithms*; and it is not improperly so called, the method being very concise and easy. The fundamental principles are briefly these: That at the beginning of the logarithms, their increments, or differences, are equal to those of the natural numbers: that the natural numbers may be considered as the decreasing cosines of increasing arcs: and that the secants of those arcs at the beginning have the same differences as the cosines, and therefore the same differences as the logarithms. Then, since the secants are the reciprocals of the cosines, by these principles and the third corol. to the twenty-seventh proposition, he establishes the following method of constituting the 100 first or smallest logarithms to the 100 largest numbers, 1000, 999, 998, 997, &c. to 900. viz. Divide the radius 1000, increased with seven ciphers, by each of these numbers separately, disposing the quotients in a table, and they will be the secants of those arcs which have the divisors for their cosines; continuing the division to the 8th figure, as it is in that place only that the arithmetical and geometrical means differ. Then by adding successively the arithmetical means between every two successive secants, the sums will be the series of logarithms. Or by adding continually every two secants, the successive sums will be the series of the double logarithms.

Besides these 100 logarithms, thus constructed, he constitutes two others by continual bisection, or extractions of the square root, after the manner described in the second postulate. And first he finds the logarithm which measures the proportion between 100,000.00 and 97,656.25, which latter term is the third proportional to 1024 and 1000, each with two ciphers; and this is effected by means of twenty-four continual extractions of the square root, determining the greatest term of each of twenty-four classes of mean proportionals; then the difference between the greatest of these means and the first or whole number 1000, with ciphers, being as often doubled, there arises 2371.6526 for the logarithm sought, which made negative is the logarithm of 1024. Secondly, the like process is repeated for the proportion between the numbers 1000 and 500, from which arises 69,314.7193 for the logarithm of 500; which he also calls the logarithm of duplication, being the measure of the proportion of 2 to 1.

Then from the foregoing he derives all the other logarithms in the chiliad, beginning with those of the prime numbers 1, 2, 3, 5, 7, &c, in the first 100.

6 And

And firſt, ſince 1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1, are all in the continued proportion of 1000 to 500, therefore the proportion of 1024 to 1 is decuple of the proportion of 1000 to 500, and conſequently the logarithm of 1 would be decuple of the logarithm of 500, if 0 were taken as the logarithm of 1024; but ſince the logarithm of 1024 is applied negatively, the logarithm of 1 muſt be diminiſhed by as much: diminiſhing therefore 10 times the logarithm of 500, which is 693,147.1928, by 2371.6526, the remainder 690,775.5422 is the logarithm of 1, or of 100.00 what is ſet down in the table.

And becauſe 1, 10, 100, 1000, are continued proportionals, therefore the proportion of 1000 to 1 is triple of the proportion of 1000 to 100, and conſequently $\frac{1}{3}$ of the logarithm of 1 is to be put for the logarithm of 100, viz. 23,025.85141, and this is alſo the logarithm of decuplication, or of the proportion of 10 to 1, And hence multiplying this logarithm of 100 ſucceſſively by 2, 3, 4, 5, 6, and 7, there ariſe the logarithms to the numbers in the decuple proportion, as in the little table here annexed.

Numbers.	Logarithms.
100	230,258.5141
10	460,517.0282
1	690,775.5422
.1	921,034.0563
.01	1,151,292.5703
.001	1,381,551.0844
.0001	1,611,809.5985

Alſo if the logarithm of duplication, or of the proportion of 2 to 1, be taken from the logarithm of 1, there will remain the logarithm of 2; and from the logarithm of 2 taking the logarithm of 10, there remains the logarithm of the proportion of 5 to 1; which taken from the logarithm of 1, there remains the logarithm of 5. See the little table here annexed.

Log. of 1	690,775.5422
of 2 to 1	69,314.7193
log. of 2	621,460.8229
log. of 10	460,517.0281
of 5 to 1	160,943.7948
log. of 5	529,831.7474

For the logarithms of other prime numbers he has recourſe to thoſe of ſome of the firſt or greateſt century of numbers, before found, viz. of 999, 998, 997, &c. And firſt, taking 960, whoſe logarithm is 4082.2001; then by adding to this logarithm the logarithm of duplication, there will ariſe the ſeveral logarithms of all theſe numbers, which are in duplicate proportion continued from 960, namely 480, 240, 120, 60, 30, 15. Hence the logarithm of 30 taken from the logarithm of 10, leaves the logarithm of the proportion of 3 to 1; which taken from the logarithm of 1, leaves the logarithm of 3, viz. 580,914.3106. And the double of this diminiſhed by the logarithm of 1, gives 471,053.0790 for the logarithm of 9.

Next, from the logarithm of 990, or 9 × 10 × 11, which is 1005.0331, he finds the logarithm of 11, namely, ſubtract the ſum of the logarithms of 9 and 10 from the ſum of the logarithm of 990 and double the logarithm of 1, there remains 450,986.0106 the logarithm of 11.

Again, from the logarithm of 980, or 2 × 10 × 7 × 7, which is 2020.2711, he finds 496,184.5228 for the logarithm of 7.

And from 5129.3303 the logarithm of 950 or 5 × 10 × 19, he finds 396,331.6392 for the logarithm of 19.

In like manner the logarithm

to 998 or 4 × 13 × 19, gives the logarithm of 13;
to 969 or 3 × 17 × 19, gives the logarithm of 17;
to 986 or 2 × 17 × 29, gives the logarithm of 29;
to 966 or 6 × 7 × 23, gives the logarithm of 23;
to 930 or 3 × 10 × 31, gives the logarithm of 31.

And

And ſo on for all the primes below 100, and for many of the primes in the other centuries up to 900. After which he directs to find the logarithms of all numbers compoſed of theſe, by the proper addition and ſubtraction of their logarithms, namely, in finding the logarithm of the product of two numbers, from the ſum of the logarithms of the two factors take the logarithm of 1, the remainder is the logarithm of the product. In this way he ſhews that the logarithms of all numbers under 500 may be derived, except thoſe of the following 36 numbers, namely, 127, 149, 167, 173, 179, 211, 223, 251, 257, 263, 269, 271, 277, 281, 283, 293, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449. Alſo, beſides the compoſite numbers between 500 and 900, made up of the products of ſome numbers whoſe logarithms have been before determined, there will be 59 primes not compoſed of them; which with the 36 above mentioned, make 95 numbers in all not compoſed of the products of any before them, and the logarithms of which he directs to be derived in this manner; namely, by conſidering the differences of the logarithms of the numbers interſperſed among them; then by that method by which were conſtituted the differences of the logarithms of the ſmalleſt 100 numbers in a continued ſeries, we are to proceed here in the diſcontinued ſeries, that is, by prop. 27, corol. 3, and eſpecially by the appendix to it, if it be rightly uſed, from whence thoſe differences will be very eaſily ſupplied.

This cloſes the ſecond part, or the actual conſtruction of the logarithms; after which follows the table itſelf, which has been before deſcribed, p. 35. Before I diſmiſs Kepler's work, however, it may not be improper in this place to take notice of an erroneous property laid down by him in the appendix to the 27th prop. juſt now referred to; both becauſe it is an error in principle, tending to vitiate the practice, and becauſe it ſerves to ſhew that Kepler was unacquainted with the true nature of the orders of differences of the logarithms, notwithſtanding what he ſays above with reſpect to the conſtruction of them by means of their ſeveral orders of differences, and that conſequently he has no legal claim to any ſhare in the diſcovery of the differential method, known at that time to Briggs, and (it would ſeem) to him alone, it being publiſhed in his logarithms in the ſame year 1624, as Kepler's book, together with the true nature of the logarithmic orders of differences, as we ſhall preſently ſee in the following account of his works. Now this error of Kepler's, here alluded to, is in that expreſſion where he ſays the third differences are in the *duplicate* ratio of the ſecond differences, like as the ſecond differences are in the duplicate ratio of the firſt; or, in other words, that the third differences are as the *ſquares* of the ſecond differences, as well as the ſecond differences as the ſquares of the firſt; or that the third differences are as the *fourth powers* of the firſt differences. Whereas in truth the third differences are only as the *cubes* of the firſt differences. Kepler ſeems to have been led into this error by a miſtake in his numbers, viz. when he ſays in that appendix, that *the third difference is* 328, *in which* 20 *is contained* 16 *times*; for when the numbers are accurately computed, the third difference comes out only 161, in which therefore 20 is contained only 8 times, which is the cube of 2, the number of times the one firſt difference contains the other. It would hence ſeem that Kepler had haſtily drawn the above erroneous principle from this

this one numerical example, or little more, falſe as it is: for had he made the trial in many inſtances, although erroneouſly computed, they could not eaſily have been ſo uniformly ſo, as to afford the ſame falſe concluſion. And therefore from hence, and what he ſays at the concluſion of that appendix, it may be inferred that he either never attempted the demonſtration of the property in queſtion, or elſe that he found himſelf embarraſſed with it, and unable to accompliſh it, and therefore diſpatched it in the ambiguous manner in which it appears.

But it may eaſily be ſhewn, not only that the third differences of the logarithms at different places, are as the cubes of the firſt differences; but, in general, that the numbers in any one and the ſame order of differences, at different places, are as that power of the numbers in the firſt differences, whoſe index is the ſame as that of the order; or that the ſecond, third, fourth, &c, differences, will be as the ſecond, third, fourth, &c. powers of the firſt differences. For the ſeveral orders of differences, when the abſolute numbers differ by indefinitely ſmall parts, are as the ſeveral orders of fluxions of the logarithms; but if x be any number, then $\frac{m\dot{x}}{x}$ is the fluxion of the logarithm of x, to the modulus m, and the ſecond fluxion, or the fluxion of this fluxion, is $-\frac{m\dot{x}^2}{x^2}$ ſince $\dot{x}$ is conſtant; and the third, fourth, &c, fluxions, are $\frac{2m\dot{x}^3}{x^3}$, $-\frac{2.3m\dot{x}^4}{x^4}$, &c; that is, the firſt, ſecond, third, fourth, fifth, ſixth, &c, orders of fluxions, are equal to the modulus m multiplied into each of theſe terms, $\frac{\dot{x}}{x}$, $-\frac{1\dot{x}^2}{x^2}$, $\frac{1.2\dot{x}^3}{x^3}$, $-\frac{1.2.3\dot{x}^4}{x^4}$, $\frac{1.2.3.4\dot{x}^5}{x^5}$, $-\frac{1.2.3.4.5\dot{x}^6}{x^6}$ &c, where it is evident that the fluxion of any order, is as that power of the firſt fluxion, whoſe index is the ſame as the number of the order. And theſe quantities would actually be the ſeveral terms of the differences themſelves, if the differences of the numbers were indefinitely ſmall. But they vary the more from them, as the differences of the abſolute numbers differ from $\dot{x}$, or as the ſaid conſtant numerical difference 1, approaches towards the value of x the number itſelf. However, upon the whole, the ſeveral orders vary proportionably, ſo as ſtill ſenſibly to preſerve the ſame analogy, namely, that two nth differences are in proportion as the nth powers of their reſpective firſt differences.

Of Briggs's Conſtruction of his Logarithms.

Nearly according to the methods deſcribed in page 47, Mr. Briggs conſtructed the logarithms of the prime numbers, as appears from his relation of this buſineſs in the *Arithmetica Logarithmica*, printed in 1624, where he details, in an ample manner, the whole conſtruction and uſe of his logarithms. The work is divided into thirty-two chapters or ſections. In the firſt of theſe, logarithms in a general ſenſe are defined, and ſome properties of them illuſtrated. In the ſecond chapter he remarks, that it is moſt convenient to make o the logarithm of 1; and on that ſuppoſition he exemplifies theſe following properties, namely, that

that the logarithms of all numbers are either the indices of powers, or proportional to them; that the ſum of the logarithms of two or more factors, is the logarithm of their product; and that the difference of the logarithm of two numbers, is the logarithm of their quotient. In the third ſection he ſtates the other aſſumption which is neceſſary to limit his ſyſtem of logarithms, namely, making 1 the logarithm of 10, as that which produces the moſt convenient form of logarithms; he hence alſo takes occaſion to ſhew that the powers of 10, namely 100, 1000, &c, are the only numbers which can have rational logarithms. The fourth ſection treats of the characteriſtic; by which name he diſtinguiſhes the integral, or firſt part of a logarithm towards the left-hand, which expreſſes one leſs than the number of integer places or figures in the number belonging to that logarithm, or how far the firſt figure of this number is removed from the place of units; namely, that 0 is the characteriſtic of the logarithms of all numbers from 1 to 10; and 1 the characteriſtic of all thoſe from 10 to 100; and 2 that of thoſe from 100 to 1000; and ſo on.

He begins the fifth chapter with remarking, that his logarithms may chiefly be conſtructed by the two methods which were mentioned by Napier, as above related, and for the ſake of which he here premiſes ſeveral *lemmata*, concerning the powers of numbers and their indices, and how many places or figures are in the products of numbers, obſerving that the product of two numbers will conſiſt of as many figures as there are in both factors, unleſs perhaps the product of the firſt figures in each factor be expreſſed by one figure only, which often happens, and then commonly there will be one figure in the product leſs than in the two factors; as alſo that, of any two of the terms, in a ſeries of geometricals, the reſults will be equal by raiſing each term to the power denoted by the index of the other; or any number raiſed to the power denoted by the logarithm of the other, will be equal to this latter number raiſed to the power denoted by the logarithm of the former; and conſequently if the one number be 10, whoſe logarithm is 1 with any number of ciphers, then any number raiſed to the power whoſe index is 1000 &c, or the logarithm of 10, will be equal to 10 raiſed to the power whoſe index is the logarithm of that number; that is, the logarithm of any number in this ſcale, where 1 is the logarithm of 10, is the index of that power of 10 which is equal to the given number. But the index of any integral power of 10, is one leſs than the number of places in that power, conſequently the logarithm of any other number, which is no integral power of 10, is not quite one leſs than the number of places in that power of the given number whoſe index is 1000, &c, or the logarithm of 10.

Find therefore the 10th, or 100th, or 1000th, &c, power of any number, as ſuppoſe 2, with the number of figures in ſuch power; then ſhall that number of figures always exceed the logarithm of 2, although the exceſs will be conſtantly leſs than 1.

An

An example of this process is here given in the margin; where the first column contains several of the powers of 2, the 2d their corresponding indices, and the 3d contains the number of places in the powers in the first column; and of these numbers in the third column, such as are on the lines of those indices that consist of 1 with ciphers, are continual approximations to the logarithm of 2, being always too great by less than 1 in the last figure, that logarithm being. 301,029,995,663,98 &c.

Powers of 2	Indices.	No. of places, or logs.
2	1	1
4	2	1
16	4	2
256	8	3
1024	10	4 log. of 2
10486	20	7 log. of 4
10995	40	13 log of 16
12089	80	25 log. of 256
12676	100	31 log. of 2
16069	200	61 log. of 4
25823	400	121 log. 16
66680	800	241 log. 256
10715	1000	302 log. 2
11481	2000	603 log. 4
13182	4000	1205 log 16
17377	8000	2409 log. 256
19950	10,000	3011 log. 2
39803	20,000	6021 log. 4
15843	40,000	12042 log. 16
25099	80,000	24083 l. 256
99900	100,000	30103 log. 2
99801	200,000	60206 l. 4
99601	400,000	120412 &c.
99204	800,000	240824
99006	1,000,000	301030
98023	2,000,000	602060
96085	4,000,000	1204120
92323	8,000,000	2408240
90498	10,000,000	3010300
81899	20,000,000	6020600
67075	40,000,000	12041200
44990	80,000,000	24082400
36846	100,000,000	30103000
13577	200,000,000	60206000
18433	400,000,000	120411999
33977	800,000,000	240823997
46129	1000,000,000	301029996

And here, since the exact powers of 2 are not required, but only the number of figures they consist of, as shewn by the third column, only a few of the first figures of the powers in the first column are retained, those being sufficient to determine the number of places in them; and the multiplications in raising these powers are performed in a contracted way, so as to have the fifth, or last, figure in them true to the nearest unit. Indeed these multiplications might be performed in the same manner, retaining only the first three figures, and those to the nearest unit in the third place; which would make this a very easy way indeed of finding the logarithms of a few prime numbers.

It may also be remarked, that those several powers, whose indices are 1 with ciphers, are raised by thrice squaring from the former powers, and multiplying the first by the third of these squares: making also the corresponding doublings and additions of their indices: thus, the square of 2 is 4, the square of 4 is 16, the square of 16 is 256, and 256 multiplied by 4 is 1024; in like manner, the double of 1 is 2, the double of 2 is 4, the double of 4 is 8, and 8 added to 2 makes 10. And the same for all the following powers and indices. The numbers in the third column, which shew how many places are in the corresponding powers in the first column, are produced in the very same way as those in the second column, namely, by three duplications and one addition; only observing to subtract 1

when the product of the firſt figures is expreſſed by one figure, or when the firſt figures exceed thoſe of the number or power next above them. It may farther be obſerved that, like as the firſt number in each quaternion, or ſpace of four lines or numbers, in the third column, approximates to the logarithm of 2, the firſt number in the firſt quaternion of the firſt column; ſo the ſecond, third, and fourth terms of each quaternion in the third column, approximate to the logarithm of 4, 16, and 256, the ſecond, third, and fourth numbers in the firſt quaternion of the firſt column. And moreover, by cutting off one, two, three, &c, figures, as the index or integral part, from the ſaid logarithms of 2, 4, 16, and 256, the firſt, ſecond, third and fourth numbers in the firſt quaternion of the firſt column, the remaining figures will be the decimal part of the logarithms of the correſponding firſt, ſecond, third, and fourth numbers in the following ſecond, third, fourth, &c, quaternions: the reaſon of which is, that any number of any quaternion in the firſt column, is the tenth power of the correſponding term in the next preceding quaternion. So that the third column contains the logarithms of all the numbers in the firſt column: a property which, if Dr. Newton had been aware of, he could not well have committed ſuch groſs miſtakes as are found in a table of his ſimilar to that above given, in which moſt of the numbers in the latter quaternions are totally erroneous; and his confuſed and imperfect account of this method, would induce one to believe that he did not well underſtand it.

In the ſixth chapter our illuſtrious author begins to treat of the other general method of finding the logarithms of prime numbers, which he thinks is an eaſier way than the former, at leaſt when the logarithm is required to a great many places of figures. This method conſiſts in taking a great number of continued geometrical means between 1 and the given number whoſe logarithm is required; that is, firſt extracting the ſquare root of the given number, then the root of the firſt root, the root of the ſecond root, the root of the third root, and ſo on till the laſt root ſhall exceed 1 by a very ſmall decimal, greater or leſs according to the intended number of places to be in the logarithm ſought: then finding the logarithm of this ſmall number, by methods deſcribed below, he doubles it as often as he made extractions of the ſquare root, or, which is the ſame thing, he multiplies it by ſuch power of 2 as is denoted by the ſaid number of extractions, and the reſult is the required logarithm of the given number; as is evident from the nature of logarithms. The rule to know how far to continue this extraction of roots is, that the number of decimal places in the laſt root be double the number of true places required to be found in the logarithm, and that the firſt half of them be ciphers; the integer being 1: The reaſon of which is, that then the ſignificant figures in the decimal, after the ciphers, are directly proportional to thoſe in the correſponding logarithms; ſuch figures in the natural number being the half of thoſe in the next preceeding number, like as the logarithm of the laſt number is the half of the preceeding logarithm. Therefore, any one ſuch ſmall number, with its logarithm, being once found, by the continual extractions of ſquare roots out of a given number, as 10, and correſponding biſections of its given logarithm 1; the logarithm for any other ſuch ſmall number, derived by like continual extractions from another given number, whoſe logarithm

logarithm is ſought, will be found by one ſingle proportion : which logarithm is then to be doubled according to the number of extractions, or multiplied at once by the like power of 2, for the logarithm of the number propoſed. To find the firſt ſmall number and its logarithm, our author begins with the number 10, and its logarithm 1, and extracts continually the root of the laſt number, and biſects its logarithm, as here regiſtered in the annexed table, but to far more places of figures, till he arrives at the 53d and 54th roots, with their annexed logarithms, as here below :

	10, given no.	1, its log.
1	3·162,277 &c.	0·5
2	1·778,279	0·25
3	1·333,521	0·125
4	1·154,781	0·062,5
5	1·074,607	0·031,25
	&c.	&c.

	Numbers.	Logarithms.
53	1·000,000,000,000,000,255,638,298,640,064,70	0·000,000,000,000,000,111,022,302,462,515,654,04
54	1·000,000,000,000,000,127,819,149,320,032,35	0·000,000,000,000,000,055,511,151,231,257,827,02

where the decimals in the natural numbers are to each other in the ratio of the logarithms, namely, in the ratio of 2 to 1 : and therefore any other ſuch ſmall number being found, by continual extraction or otherwiſe, it will then be as 0.000,000,000,000,000,127,819,149,320,032,35 &c, is to 0.000,000,000,000,000,055,511,151,231,257,827,02 &c, ſo is that other ſmall decimal, to the correſponding ſignificant figures of its logarithm. But as every repetition of this proportion requires both a very long multiplication and a very long diviſion, he reduces this conſtant ratio to another equivalent ratio whoſe antecedent is 1, by which all the diviſions are ſaved : thus,

as 0.000,000,000,000,000,127,819,149,320,032, 35 &c, is to 0.000,000,000,000,000,055,511,151,231,257,827,0 , &c, ſo is 1.000,000,000,000,000,000, to 0.434,294,481,903,251,804, &c, that is, the logarithm of 1·000,000,000,000,000,1, is 0·000,000,000,000,000,043,429,448,190,325,180,4 ; and therefore this laſt number being multiplied by any ſuch ſmall decimal, found as above by continual extraction, the product will be the correſponding logarithm of ſuch laſt root.

But as the extraction of ſo many roots is a very troubleſome operation, our author deviſes ſome ingenious contrivances to abridge that labour. And firſt, in the 7th chapter, by the following device, to have fewer and eaſier extractions to perform : namely, raiſing the powers from any given prime number, whoſe logarithm is ſought, till a power of it be found ſuch that its firſt figure on the left hand is 1, and the next to it either one or more ciphers ; then, having divided this power by 1 with as many ciphers as it has figures after the firſt, or ſuppoſing all after the firſt to be decimals, the continual roots from this power are extracted till the decimal become ſufficiently ſmall, as when the firſt fifteen places are ciphers ; and then by multiplying the decimal by 43429 &c, we have the logarithm of this laſt root : which logarithm multiplied by the like power of the number 2, gives the logarithm of the firſt number from which the extraction was begun : to this logarithm prefixing a 1, or 2, or 3, &c, according as this number was found by dividing the power of the given prime number by 10, or 100, or 1000, &c ; and laſtly, dividing the reſult by the index of that power, the quotient will be the required logarithm of the given prime number. Thus, to

 find

find the logarithm of 2: it is firſt raiſed to the 10th power, as in the margin, before the firſt figures come to be 10; then, dividing by 1000, or cutting off for decimals, all the figures after the firſt or 1, the root is continually extracted from the quotient 1,024, till the 47th extraction, which gives 1 000,000,000,000,000,168,516,057, 053,949,77; the decimal part of which multiplied by 43429, &c, gives 0.000,000,000,000,000,073,185,593,690,623,936,8 for its logarithm; and this being continually doubled for 47 times, will give the logarithms of all the roots up to the firſt number: or being at once multiplied by the 47th power of 2, viz. 140,737,488, 355,328, which is raiſed as in the annexed table, it gives 0.010,299,956,639,811,952,652,774,4 for the logarithm of the number 1,024, true to 17 or 18 decimals: to this prefix 3, ſo ſhall 3.010,2 &c, be the logarithm of 1024: and laſtly, becauſe 2 is the tenth root of 1024, divide by 10, ſo ſhall 0.301,029,995,663,981,195,2 be the logarithm required to the given number 2.

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
1,048,576	20
1,073,741,824	30
1,099,511,627,776	40
140,737,488,355,328	47

The logarithms of 1, 2, and 10 being now known; it is remarked that the logarithm of 5 becomes known; for ſince 10 ÷ 2 is = 5, therefore log. 10 — log. 2 = log. 5, which is 0.698,970,004,336,018,805,8; and that from the multiplications and diviſions of theſe three 2, 5, 10, with the correſponding additions and ſubtractions of their logarithms, a multitude of other numbers and their logarithms are produced; ſo from the powers of 2 are obtained 4, 8, 16, 32, 64, &c; from the powers of 5 theſe 25, 125, 625, 3125, &c; alſo the powers of 5 by thoſe of 10 give 250, 1250, 6250, &c; and the powers of 2 by thoſe of 10 give 20, 200, 2000, &c; 40, 400, 80, 800, &c; likewiſe by diviſion are obtained $2\frac{1}{2}$, $1\frac{1}{4}$, $12\frac{1}{2}$, $6\frac{1}{4}$, $1\frac{3}{5}$, $3\frac{1}{5}$, $6\frac{2}{5}$, &c.

He then obſerves that the logarithm of 3, the next prime number, will be beſt derived from that of 6, in this manner: 6 raiſed to the 9th power becomes 10,077,696, which divided by 10,000,000, gives 1.007,769,6, and the root from this continually extracted till the 46th, is 1.000,000,000,000,000,109,985, 934,588,155,718,66; the decimal part of which multiplied by 43429 &c, gives 0.000,000,000,000,000,047,766,284,478,608,030,4 for its logarithm; and this 46 times doubled, or multiplied by the 46th power of 2, gives 0.003,361,253, 452,792,69 for the logarithm of 1.007,769,6; to which adding 7, the logarithm of the diviſor 10,000,000, and dividing by 9, the index of the power of 6, there reſults 0.778,151,250,383,643,63 for the logarithm of 6; from which ſubtracting the logarithm of 2, there remains 0.477,121,254,719,662,44 for the logarithm of 3.

In the eighth chapter our ingenious author deſcribes an original and eaſy method of conſtructing, by means of differences, the continual mean proportionals which were before found by the extraction of roots. And this, with the other methods of generating logarithms by differences, in this book as well as in our author's *Trigonometria Britannica*, are, I believe, the firſt inſtances that are to

be

be found of making ſuch uſe of differences, and ſhew him to have been the inventor of what may be called the *Differential Method.* He ſeems to have diſcovered this method in the following manner: Having obſerved that theſe continual means between 1 and any number propoſed, found by the continual extraction of the ſquare roots, approach always nearer and nearer to the halves of each preceeding root, as is viſible when they are placed together under each other; and indeed it is found that as many of the ſignificant figures of each decimal part, as there are ciphers between them and the integer 1, agree with the half of thoſe above them; I ſay, having obſerved this evident approximation, he ſubtracted each of theſe decimal parts, which he called A or the firſt differences, from half the next preceeding one, and by comparing together the remainders or ſecond differences, called B, he found that the ſucceeding were always nearly equal to $\frac{1}{4}$ of the next preceeding ones; then taking the difference between each ſecond difference and $\frac{1}{4}$ of the preceeding one, he found that theſe third differences, called C, were nearly in the continual ratio of 8 to 1; again taking the difference between each C and $\frac{1}{8}$ of the next preceeding, he found that theſe fourth differences, called D, were nearly in the continual ratio of 16 to 1; and ſo on, the 5th (E), 6th (F), &c, differences, being nearly in the continual ratio of 32 to 1, of 64 to 1, &c: theſe plain obſervations being made, they

		1,00776,96	
1		1,00387,72833,36962,45663,84655,1	
2		1,00193,67661,36946,61675,87022,9	
3		1,00096,79146,39099,01728,89072,0	
4		1,00048,38402,68846,62985,49253,5	A
5		1,00024,18908,78824,68563,80872,7	A
		24,19201,34423,31492,74626,7	$\frac{1}{2}$A
		292,55598,62928,93754,0	B
6		1,00012,09381,26397,13459,43919,4	A
		12,09454,39412,34281,90436,3	$\frac{1}{2}$A
		73,13015,20822,26516,9	B
		73,13899,65732,23438,5	$\frac{1}{4}$B
		884,44909,76921,5	C
7		1,00006,04672,35055,30968,01600,5	A
		6,04690,63198,56729,71959,7	$\frac{1}{2}$A
		18,28143,25761,70359,2	B
		18,28253,80205,61629,2	$\frac{1}{4}$B
		110,54443.91270,0	C
		110,55613,72115,2	$\frac{1}{8}$C
		1169,80843,2	D
8		1,00003,02331,60505,65775,90479,4	A
		3,02336,17527,65484,00800,2	$\frac{1}{4}$A
		4,57021,99708,04320,8	B
		4,57035,81440,42589,8	$\frac{1}{4}$B
		13,81732,38269,0	C
		13,81805,48908,7	$\frac{1}{8}$C
		73,10639,7	D
		73,11302,8	$\frac{1}{16}$D
		663,1	E
9		1,00001,51164,65999,05672,95048,8	A
		1,51165,80252,82887,98239,7	$\frac{1}{2}$A
		1,14253,77215,03190,9	B
	Hitherto the	1,14255,49927,01080,2	$\frac{1}{4}$B
	ſmaller differences	1,72711,97889,3	C
	are found by ſub-	1,72716,54783,6	$\frac{1}{8}$C
	tracting the larger from	4,56894,3	D
	the parts of the like pre-	4,56915,0	$\frac{1}{16}$D
	ceding ones.	20,7	E
		20,7	$\frac{1}{32}$E
	Here the greater differences	65	$\frac{1}{32}$E
	remain after ſubtracting	28555,89	$\frac{1}{16}$D
	the ſmaller from the parts	28555,24	D
	of the difference of	21588,99736,16	$\frac{1}{8}$C
	the next preceding	21588,71180,92	C
	number.	28563,34303,75797,72	$\frac{1}{4}$B
		28563,22715,04616,80	B
		75582,32999,52836,47523,40	$\frac{1}{2}$A
10		1.0000,75582,03436,30121,42907,60	A
		2	$\frac{1}{32}$E
		1784,70	$\frac{1}{16}$D
		1784,68	D
		2698,58897,62	$\frac{1}{8}$C
		2698,57112,94	C
		7140,80678,76154,20	$\frac{1}{4}$B
		7140,77980,19041,26	B
		37791,02218,15060,71453,80	$\frac{1}{2}$A
11		1,00000,37790,95077,37080,52412,54	A

very

very naturally and clearly suggested to him the notion and method of constructing all the remaining numbers from the differences of a few of the first, found by extracting the roots in the usual way. This will evidently appear from the annexed specimen of a few of the first numbers in the last example for finding the logarithm of 6; where after the 9th number the rest are supposed to be constructed from the preceeding differences of each, as here shewn in the 10th and 11th. And it is evident that, in proceeding, the trouble will become always less and less, the differences gradually vanishing, till at last only the first differences remain. And that generally each less difference is shorter than the next greater, by as many places as there are ciphers at the beginning of the decimal in the number to be generated from the differences.

He then concludes this chapter with an ingenious, but not obvious, method of finding the differences B, C, D, E, &c. belonging to any number, as suppose the 9th from that number itself, independent of any of the preceeding 8th, 7th, 6th, 5th, &c; and it is this: Raise the decimal A to the 2d, 3d, 4th, 5th, &c powers; then will the 2d (B), 3d (C), 4th (D), &c differences, be as here below, viz.

$B = \frac{1}{2}A^2,$

$C = \frac{1}{2}A^3 + \frac{1}{8}A^4,$

$D = \frac{7}{8}A^4 + \frac{7}{8}A^5 + \frac{7}{16}A^6 + \frac{1}{8}A^7 + \frac{1}{64}A^8$

$E = 2\frac{5}{8}A^5 + 7A^6 + 10\frac{15}{16}A^7 + 12\frac{69}{128}A^8 + 11\frac{11}{64}A^9 + 7\frac{105}{128}A^{10},$

$F = 13\frac{9}{16}A^6 + 81\frac{3}{8}A^7 + 296\frac{87}{256}A^8 + 834\frac{43}{128}A^9 + 1953\frac{285}{512}A^{10}$ &c.

$G = 122\frac{1}{16}A^7 + 1510\frac{67}{128}A^8 + 11475\frac{73}{128}A^9 + 68372\frac{79}{2048}A^{10}$ &c.

$H = 1937\frac{95}{128}A^8 + 47151\frac{93}{128}A^9 + 706845\frac{1493}{8192}A^{10}$ &c.

$I = 54902\frac{89}{128}A^9 + 2558465\frac{23587}{32768}A^{10}$ &c.

$K = 2805527A^{10}$ &c.

&c,

Thus in the 9th number of the foregoing example, omitting the ciphers at the beginning of the decimals, we have

$A = 1,51164,65999,05672,95048,8$

$A^2 = 2,28507,54430,06381,6726$

$A^3 = 3,45422,65239,48546,2$

$A^4 = 5,22156,97802,288$

$A^5 = 7,89316,8205$

$A^6 = 11,93168,1$

&c.

Consequently

$\frac{1}{2}A^2 = 1,14253,77215,03190,8363 = B$

$\frac{1}{2}A^3$ - 1,72711,32619,74273

$\frac{1}{8}A^4$ - - - 65269,62225

$\frac{1}{2}A^3 + \frac{1}{8}A^4$ 1,72711,97889,36498 $= C$

$\frac{7}{8}A^4$ 4,56887,35577

$\frac{7}{8}A^5$ - - 6,90652

$\frac{7}{16}A^6$ - - - - 5

$\frac{7}{8}A^4 + \frac{7}{8}A^5 + \frac{7}{16}A^6$ 4,56894,26234 $= D$

$2\frac{5}{8}A^5$ - - 20,71957

$7A^6$ - - - 83

$2\frac{5}{8}A^5 + 7A^6$ - - - 20,72040 $= E$

which agree with the like differences in the foregoing specimen.

In the ninth chapter, after obferving that from the logarithms of 1, 2, 3, 5, and 10, before found, are to be determined, by addition and fubtraction, the logarithms of all other numbers which can be produced from thefe by multiplication and divifion; for finding the logarithms of other prime numbers, inftead of that in the feventh chapter, our author then fhews another ingenious method of obtaining numbers beginning with 1 and ciphers, and fuch as to bear a certain relation to fome prime number by means of which its logarithm may be found. The method is this: Find three products having the common difference 1, and fuch that two of them are produced from factors having given logarithms, and the third produced from the prime number, whofe logarithm is required, either multiplied by itfelf, or by fome other number whofe logarithm is given; then the greateft and leaft of thefe three products being multiplied together, and the mean by itfelf, there arife two other products alfo differing by 1, of which the greater divided by the lefs, gives for a quotient 1 with a fmall decimal, having feveral ciphers at the beginning. Then the logarithm of this quotient being found as before, from thence will be deduced the required logarithm of the given prime number. Thus, if it be propofed to find the logarithm of the prime number 7; here $6 \times 8 = 48$, $7 \times 7 = 49$, and $5 \times 10 = 50$ will be the three products, of which the logarithms of 48 and 50, the 1ft and 3d, will be given from thofe of their factors 6, 8, 5, 10; alfo $48 \times 50 = 2400$, and $49 \times 49 = 2401$ are the two new products, and $2401 \div 2400 = 1{,}00041\frac{2}{3}$ their quotient: then the leaft of 44 means between 1 and this quotient is 1,00000,00000,00000, 02367,98249,04333,6405, which multiplied by 43429 &c, produces 0.0000, 00000,0000,01028,40172,88387,29715 for its logarithm; which being 44 times doubled, or multiplied by 17592186044416, produces 0,00018,09183, 45421,30 for the logarithm of the quotient $1{,}00041\frac{2}{3}$; which being added to the logarithm of the divifor 2400, gives the logarithm of the dividend 2401; then the half of this logarithm is the logarithm of 49 the root of 2401, and the half of this again gives 0,84509,80400,14256,82 for the logarithm of 7 which is the root of 49.——The author adds another example to illuftrate this method; and then fets down the requifite factors, products, and quotients for finding the logarithms of all other prime numbers up to 100.

The 10th chapter is employed in teaching how to find the logarithms of fractions, namely by fubtracting the logarithm of the denominator from that of the numerator, then the logarithm of the fraction is the remainder; which therefore is either abundant or defective, that is pofitive or negative, as the fraction is greater or lefs than 1.

In the 11th chapter we are fhewn an ingenious contrivance for very accurately finding intermediate numbers to given logarithms, by the proportional parts. On this occafion it is remarked, that while the abfolute numbers increafe uniformly, the logarithms increafe unequally, with a decreafing increment; for which reafon it happens, that either logarithms or numbers corrected by means of the proportional parts, will not be quite accurate, the logarithms fo found being always too fmall, and the abfolute numbers fo found too great; but yet fo however as that they approach much nearer to accuracy towards the end of the table, where the increments or differences become much nearer to equality, than in the former parts

parts of the table. And from this property our author, ever fruitful in happy expedients to obviate natural difficulties, contrives a device to throw the proportional part, to be found from the numbers and logarithms, always near the end of the table, in whatever part they may happen naturally to fall. And it is this: Rejecting the characteriſtic of any given logarithm, whoſe number is propoſed to be found, take the arithmetical complement of the decimal part, by ſubtracting it from 1,000 &c, the logarithm of 10; then find in the table the logarithm next leſs than this arithmetical complement, together with its abſolute number; to this tabular logarithm add the logarithm that was given, and the ſum will be a logarithm neceſſarily falling among thoſe near the end of the table; find then its abſolute number, corrected by means of the proportional part, which will not be very inaccurate, as falling near the end of the table; this being divided by the abſolute number, before found for the logarithm next leſs than the arithmetical complement, the quotient will be the required number anſwering to the given logarithm; which will be ~~much more~~ correct than if it had been found from the proportional part of the difference where it naturally happened to fall: and the reaſon of this operation is evident from the nature of logarithms. But as this diviſor, when taken as the number anſwering to the logarithm next leſs than the arithmetical complement, may happen to be a large prime number; it is farther remarked, that inſtead of this number and its logarithm, we may uſe the next leſs compoſite number which has ſmall factors, and *its* logarithm; becauſe the diviſion by thoſe ſmall factors, inſtead of by the number itſelf, will be performed by the ſhort and eaſy way of diviſion in one line. And for the more eaſy finding proper compoſite numbers and their factors, our author here ſubjoins an abacus or liſt of all ſuch numbers, with their logarithms and component factors, from 1000 to 10000; from which the proper logarithms and factors are immediately obtained by inſpection. Thus, for example, to find the root of 10800, or the mean proportional between 1 and 10800: The logarithm of 10800 is 4.033,423,755,486,95, the half of which is 2.016,711,877,743,47, the logarithm of the number ſought. The arithmetical complement of the decimal part of this logarithm, to wit, the decimal fraction 0.016,711,877,743, 47, to 1, (the logarithm of 10,) is 0.983,288,122,256,53. Now the neareſt logarithm to this in the *abacus* is 0.982,271,233,039,57, and its annexed number is 9600, the factors of which are 2, 6, 8, beſides the ciphers. To this laſt logarithm, 0.982,271,233,039,57, if we add the decimal part of the logarithm ſought, to wit, 0.016,711,877,743,47, the ſum will be 0.998,983,110,783,04, whoſe abſolute number, corrected by the proportional part, is 9.976,612,651,652,1; which being divided continually by 2, 6, 8, the factors of 96, the laſt quotient is 103.923,048,454,71; which is pretty correct, the true number being 103. 923,048,454,133 = $\sqrt{}$ 10800.

We now arrive at the 12th and 13th chapters, in which our ingenious author firſt of all teaches the rules of the *Differential Method*, in conſtructing logarithms by interpolation from differences. This is the ſame method which has ſince been more largely treated of by later authors, and particularly by the learned Mr. Cotes, in his *Canonotechnia*. How Mr. Briggs came by it, does not well appear, as he only delivers the rules, without laying down the principles or inveſtigation of

of them. He divides the method into two cafes, namely, when the fecond differences are equal, or nearly equal, and when the differences run out to any length whatever. The former of thefe is treated in the 12th chapter; and he particularly adapts it to the interpolating 9 equidiftant means between two given terms, evidently for this reafon, that then the powers of 10 become the principal multipliers or divifors, and fo the operations may be performed mentally. The fubftance of his procefs is this: Having given two abfolute numbers with their logarithms, to find the logarithms of 9 arithmetical means between the given numbers: Between the given logarithms take the 1ft difference, as well as between each of them and their next or equidiftant greater and lefs logarithms; and likewife the 2d differences, or the two differences of thefe three 1ft differences; then if thefe 2d differences be equal, multiply one of them feverally by the numbers 45, 35, &c, in the annexed tablet, dividing each product by 1000, that is, cutting off three figures from each; laftly, to $\frac{1}{10}$ of the 1ft difference of the given logarithms add feverally the firft five quotients, and fubtract the other five: fo fhall the ten refults be the refpective 1ft differences to be continually added, to compofe the required feries of logarithms. Now this amounts to the fame thing as what is at this day taught in the like cafe: we know that if A be any term of an equidiftant feries of terms, and a, b, c, &c, the firft of the 1ft, 2d, 3d, &c, order of differences; then the term z, whofe diftance from A is expreffed by x, will be thus, $z = A + x\,a + x.\frac{x-1}{2}b + x.\frac{x-1}{2}.\frac{x-2}{3}c +$ &c. And if now, with our author, we make the 2d differences equal, then c, d, e, &c, will all vanifh, or be equal to 0, and z will become barely $= A + x\,a + x.\frac{x-1}{2}b$. Therefore if we take x fucceffively equal to $\frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$, &c, we fhall have the annexed feries of terms with their differences. Where it is to be obferved, that our author had reduced the differences from the 1ft to the 2d form, as he thought it eafier to multiply by 5 than to divide by 2. Alfo all the laft terms $(x.\frac{x-1}{2}b)$ are fet down pofitive, becaufe in the logarithms b is negative. If the two 2d differences be only nearly equal, take an arithmetical mean between them, and proceed with it the fame as above with one of the equal 2d differences. He alfo fhews how to find any one fingle term, independent of the reft; and concludes the chapter with pointing out a method of finding the proportional part more accurately than before.

1	45	Additive products.
2	35	
3	25	
4	15	
5	5	
6	5	Subductive products.
7	15	
8	25	
9	35	
10	45	

Series of terms.	The Differences.
A	
$A+\frac{1}{10}a+\frac{9}{200}b$	$\frac{1}{10}a+\frac{9}{200}b=\frac{1}{10}a+\frac{45}{1000}b$
$A+\frac{2}{10}a+\frac{16}{200}b$	$\frac{1}{10}a+\frac{7}{200}b=\frac{1}{10}a+\frac{35}{1000}b$
$A+\frac{3}{10}a+\frac{21}{200}b$	$\frac{1}{10}a+\frac{5}{200}b=\frac{1}{10}a+\frac{25}{1000}b$
$A+\frac{4}{10}a+\frac{24}{200}b$	$\frac{1}{10}a+\frac{3}{200}b=\frac{1}{10}a+\frac{15}{1000}b$
$A+\frac{5}{10}a+\frac{25}{200}b$	$\frac{1}{10}a+\frac{1}{200}b=\frac{1}{10}a+\frac{5}{1000}b$
$A+\frac{6}{10}a+\frac{24}{200}b$	$\frac{1}{10}a-\frac{1}{200}b=\frac{1}{10}a-\frac{5}{1000}b$
$A+\frac{7}{10}a+\frac{21}{200}b$	$\frac{1}{10}a-\frac{3}{200}b=\frac{1}{10}a-\frac{15}{1000}b$
$A+\frac{8}{10}a+\frac{16}{200}b$	$\frac{1}{10}a-\frac{5}{200}b=\frac{1}{10}a-\frac{25}{1000}b$
$A+\frac{9}{10}a+\frac{9}{200}b$	$\frac{1}{10}a-\frac{7}{200}b=\frac{1}{10}a-\frac{35}{1000}b$
$A+\quad a$	$\frac{1}{10}a-\frac{9}{200}b=\frac{1}{10}a-\frac{45}{1000}b$

In the 13th chapter our author remarks, that the beft way of filling up the intermediate

termediate chiliads of his table, namely, from 20,000 to 90,000, is by quinquiſection, or interpoſing four equidiſtant means between two given terms; the method of performing which he thus particularly deſcribes. Of the given terms, or logarithms, and two or three others on each ſide of them, take the 1ſt, 2d, 3d, &c, differences, till the laſt differences come out equal, which ſuppoſe to be the 5th differences: divide the 1ſt differences by 5, the 2d by 25, the 3d by 125, the 4th by 625, and the 5th by 3125, and call the reſpective quotients the 1ſt, 2d, 3d, 4th, 5th *mean* differences; or, inſtead of dividing by theſe powers of 5, multiply by their reciprocals $\frac{2}{10}$, $\frac{4}{100}$, $\frac{8}{1000}$, $\frac{16}{10000}$, $\frac{32}{100000}$, that is multiplied by 2, 4, 8, 16, 32, cutting off reſpectively one, two, three, four, five figures from the end of the products, for the ſeveral mean differences: then the 4th and 5th of theſe mean differences are ſufficiently accurate, but the 1ſt, 2d, and 3d, are to be corrected in this manner: From the mean third differences ſubtract three times the 5th difference, and the remainders are the *correct* 3d differences; from the mean 2d differences ſubtract double the 4th differences, and the remainders are the correct 2d differences; laſtly, from the mean 1ſt differences take the correct 3d differences, and $\frac{1}{5}$ of the 5th difference, and the remainders will be the correct firſt differences. Such are the corrections when the differences extend as far as the 5th. However, in compleating thoſe chiliads in this way, there will be only 3 orders of differences, as neither the 4th nor 5th will enter the calculation, but will vaniſh through their ſmallneſs: therefore the mean 2d and 3d difference will need no correction, and the mean 1ſt differences will be corrected by barely ſubtracting the 3d from them. Theſe preparatory numbers being thus found, all the 2d differences of the logarithms required, will be generated by adding continually, from the leſs to the greater, the conſtant 3d difference; and the ſeries of 1ſt differences will be found by adding the ſeveral 2d differences; and, laſtly, by adding continually theſe 1ſt differences to the 1ſt given logarithm, &c, the required logarithmic terms will be generated.

Theſe eaſy rules being laid down, Mr. Briggs next teaches how by them the remaining chiliads may beſt be compleated: namely, having here the logarithms for all numbers up to 20,000, find the logarithms to every 5 beyond this, or of 20,005, 20,010, 20,015, &c, in this manner; to the logarithms of the 5th part of each of thoſe, namely 4001, 4002, 4003, &c. add the conſtant logarithm of 5, and the ſums will be the logarithms of all the terms of the ſeries 20,005, 20,010, 20,015, &c: and theſe logarithms will have the very ſame differences as thoſe of the ſeries 4001, 4002, 4003, &c; by means of which therefore interpoſe 4 equidiſtant terms by the rules above; and thus the whole canon will be eaſily compleated.

He here alſo extends the rules for correcting the mean differences in quinquiſection, as far as the 20th difference; he alſo lays down ſimilar rules for triſection, and ſpeaks of general rules for any other ſection, but which are omitted as being leſs eaſy. So that he appears to have been poſſeſſed of all that Cotes afterwards delivered in his *Canonotechnia, ſive Conſtructio Tabularum per Differentias*, drawn from the *Differential Method*, as their general rules exactly agree; Briggs's mean and correct differences being by Cotes called round and quadrat differences, becauſe he expreſſes them by the numbers 1, 2, 3, &c. written reſpectively in a ſmall circle and ſquare.

Mr.

Mr. Briggs alſo obſerves that the ſame rules equally apply to the conſtruction of equidiſtant terms of any other kind, ſuch as ſines, tangents, ſecants, the powers of numbers, &c: and farther remarks, that of the ſines of three equidifferent arcs, all the remote differences may be found by the rule of proportion, becauſe the ſines and their 2d, 4th, 6th, 8th, &c, differences are continued proportionals, as are alſo the 1ſt, 3d, 5th, 7th, &c, differences among themſelves; and like as the 2d, 4th, 6th, &c, differences are proportional to the ſines of the mean arcs, ſo alſo are the 1ſt, 3d, 5th, &c, differences proportional to the coſines of the ſame arcs. Moreover, with regard to the powers of numbers, he remarks the following curious properties: 1ſt, that they will each have as many orders of differences as are denoted by the index of the power, the ſquares having two orders of differences, the cubes three, the 4th powers four, &c: ſecond, that the laſt differences will be all equal, and each equal to the common difference of the ſides or roots raiſed to the given power, and multiplied by $1 \times 2 \times 3 \times 4$ &c, continued to as many terms as there are units in the index; ſo if the roots differ by 1, the 2d difference of the ſquares will be each 1×2 or 2, the third differences of the cubes each $1 \times 2 \times 3$ or 6, the 4th differences of the 4th powers each $1 \times 2 \times 3 \times 4$ or 24, and ſo on; and if the common difference of the roots be any other number n, then the laſt differences of the ſquares, cubes, 4th powers, 5th powers, &c, will be reſpectively $2n^2$, $6n^3$, $24n^4$, $120n^5$, &c.

Beſides what was ſhewn in the eleventh chapter concerning the taking out the logarithms of large numbers by means of proportional parts, he employs the next or 14th chapter in teaching how, from the firſt ten chiliads only, and a ſmall table of one page here given, to find the number anſwering to any logarithm, and the logarithm to any number conſiſting of fourteen places of figures *.

Having thus fully ſhewn the conſtruction and chief properties of his logarithms, our ingenious author, in the remaining eighteen chapters, exemplifies their uſes in various curious and important ſubjects: ſuch as the Rule-of-three, or rule of proportion; finding the roots of given numbers; finding any number of mean proportionals between two given terms; with other arithmetical rules: Alſo various geometrical ſubjects; as, 1ſt, Having given the ſides of any plane triangle, to find the area, perpendicular, angles, and diameters of the inſcribed and circumſcribed circles; 2d, In a right-angled triangle, having given any two of theſe, to find the reſt, viz. one leg and the hypotenuſe, one leg and the ſum or

* It is no more than a large exemplification of this method of Briggs's that has been printed ſo late as 1771, in a 4to. tract by Mr. Rob. Flower, under the title of *The Radix, a New Way of making Logarithms*; although Briggs's work might not be known to this writer.—Since this was written I have been favoured with the following anecdote concerning Mr. Flower and his work, by the Rev. Dr. Horſley, the learned editor of the works of Sir I. Newton. "This Robert Flower was a very obſcure, and probably an illiterate man. He was maſter of a writing ſchool in the town of Biſhop Stortford in Hertfordſhire. He communicated his Radix, before he publiſhed it, to my late learned friend Math. Raper, Eſq. of Thorley Hall. I was at Thorley at the time, upon a viſit to my father, who was rector of the pariſh; and I well remember that Mr. Raper told me, with great ſurprize, that Flower (who was known to us both by name as the writing-maſter of the neighbouring market town) had fallen upon Briggs's way of finding all logarithms from the firſt ten chiliads. And he was ſo well perſuaded that Flower had made the diſcovery for himſelf, without any light from Briggs, that with his accuſtomed munificence he rewarded the man's ingenuity with a preſent of ten guineas; informing him, I believe, that his work had been done before, and diſſuading the publication."

difference of the hypotenufe and the other leg, the two legs, one leg and the area, the area and the fum or difference of the legs, the hypotenufe and fum or difference of the legs, the hypotenufe and area, and the perimeter and area; 3d, Upon a given bafe to defcribe a triangle equal and ifoperimetrical to another triangle given; 4th, To defcribe the circumference of a circle fo, that the three diftances from any point in it to the three angles of a given plane triangle, fhall be to one another in a given ratio; 5th, Having given the bafe, the area, and the ratio of the two fides of a plane triangle, to find the fides; 6th, Given the bafe, difference of the fides, and area of a triangle, to find the fides; 7th, To find a triangle whofe area and perimeter fhall be expreffed by the fame number; 8th, Of four given lines, of which the fum of any three is greater than the fourth, to form a quadrilateral figure about which a circle may be defcribed; 9th, Of the diameter, circumference, and area of a circle, and the furface and folidity of the fphere generated by it, having any one given, to find any of the reft; 10th, Concerning the ellipfe, fpheroid, and gauging; 11th, To cut a line or a number in extreme and mean ratio; 12th, Given the diameter of a circle, to find the fides and areas of the infcribed and circumfcribed regular figures of 3, 4, 5, 6, 8, 10, 12, and fixteen fides; 13th, Concerning the regular figures of 7, 9, 15, 24, and 30 fides; 14th, Of ifoperimetrical regular figures; 15th, Of equal regular figures; and 16th, Of the fphere and the 5 regular bodies; which clofes this introduction. Such of thefe problems as can admit of it, are determined by elegant geometrical conftructions, and they are all illuftrated by accurate arithmetical calculations performed by logarithms; for the exemplification of which they are purpofely given.

At the end he remarks, that the chief and moft neceffary ufe of logarithms, is in the doctrine of fpherical trigonometry, which he here promifes to give in a future work, and which was accomplifhed in his *Trigonometria Britannica*, to the defcription of which we now proceed.

Of BRIGGS's *Trigonometria Britannica.*

At the clofe of the account of writings on the natural fines, tangents, and fecants, I omitted the defcription of this work of our learned author, although it is perhaps the greateft of this kind, all things confidered, that ever was executed by one perfon; purpofely referving my account of it to this place, not only as it is connected with the invention and conftruction of logarithms, but thinking it deferved more peculiar and diftinguifhed notice, on account of the importance and originality of its contents. The divifion of the quadrant, and the mode of conftruction, are both new; and the numbers are far more accurate, and are extended to more places, than they had ever been before. The circular arcs had always been divided in a fexagefimal proportion; but here the quadrant is divided into degrees and decimals, as this is a much eafier mode of computation than by 60ths; the divifion being completed only to 100ths of degrees, though his defign was to have extended it to 1000ths of degrees. And, befides his own private opinion, he was induced to adopt this method of decimal divifions, partly at

at the requeſt of other perſons, and partly perhaps from the authority of Vieta, p. 29 *Calendarii Gregoriani*. And it is probable that computations by this decimal diviſion would have come into general uſe, had it not been for the publication of Vlacq's tables, which were extended to every 10 ſeconds, or 6th parts of minutes. But beſides this method by a decimal diviſion of the degrees, of which the whole circle contains 360, or the quadrant 90, in the 14th chapter he remarks that ſome other perſons were inclined rather to adopt a complete decimal diviſion of the whole circle, firſt into 100 parts, and each of theſe into 1000 parts; and for *their* ſakes he ſubjoins a ſmall table of the ſines of every 40th part of the quadrant, and remarks that from theſe few the whole may be made out by continual quinquiſections; namely, 5 times theſe 40 make 200; then 5 times theſe give 1000; thirdly, 5 times theſe give 5000; and, laſtly, 5 times theſe give 25,000 for the whole quadrant, or 100,000 for the whole circumference.

But to return. Our author's large table conſiſts of natural ſines to 15 places, natural tangents and ſecants each to 10 places, logarithmic ſines to 14 places, and logarithmic tangents to 10 places, each beſides the characteriſtic: a moſt ſtupendous performance! The table is preceeded by an introduction, divided into two books, the one containing an account of the truly ingenious conſtruction of the table, by the author himſelf; and the other its uſes in trigonometry, &c, by Henry Gellibrand, profeſſor of aſtronomy in Greſham College, who remarks in the preface that the work was compoſed by the author about the year 1600; though it was only publiſhed by the direction of Gellibrand in 1633, it having been printed at Gouda under the care of Vlacq, and by the printer of his *Trigonometria Artificialis*, which came out the ſame year.

After briefly mentioning the common methods of dividing the quadrant, and conſtructing the tables of ſines, &c, from the ancients down to his own time, he haſtens to the deſcription of his own peculiar and truly ingenious method, which is briefly this: Having firſt divided the quadrant into a ſmall number of parts, as 72, he finds the ſine of one of thoſe parts; then from it the ſines of the double, triple, quadruple, &c, up to the quadrant, or 72 parts. He next quinquiſects each of theſe parts, by interpoſing four equidiſtant means, by differences; he then quinquiſects each of theſe; and finally each of theſe again; which completes the diviſion as far as degrees and centeſms. The rules for performing all theſe things, he inveſtigates and illuſtrates in a very ample manner. In treating of multiple and ſubmultiple arcs, he gives general algebraical expreſſions, for the ſine or chord of any multiple whatever of a given arc, which he deduced from a geometrical figure, by finding the law for the ſeries of ſucceſſive multiple chords or ſines, after the manner of Vieta, who was the firſt perſon that I know of, who laid down general rules for the chords of multiples and ſubmultiples of arcs or angles; and the ſame was afterwards improved by Sir I. Newton, to ſuch form, that radius, and double the coſine of the firſt given angle, are the firſt and ſecond terms of all the proportions for finding the ſines and coſines of the multiple angles. For aſſigning the coefficients of the terms in the multiple expreſſions, our author here delivers the conſtruction of figurate or polygonal numbers, inſerts a large table of them, and teaches their ſeveral uſes; one of which is, that every other number taken in the diagonal lines, furniſhes the coefficients of the terms of

the general equation by which the ſines and chords of multiple arcs are expreſſed, which he amply illuſtrates; and another, that the ſame diagonal numbers conſtitute the coefficients of the terms of any power of a binomial; which property was alſo mentioned by Vieta in his *Angulares Sectiones*, *theor.* 6, 7: and this is the firſt mention I have ſeen made of this law of the coefficients of the powers of a binomial, commonly called Sir I. Newton's binomial theorem, although it is very evident that Sir Iſaac was not the firſt inventor of it; the part of it properly belonging to him ſeems to be only the extending of it to fractional indexes, which was indeed an immediate effect of the general method of denoting all roots like powers with fractional exponents, the theorem being not at all altered.

Of the binomial theorem.

However it appears that our author Briggs was the firſt who taught the rule for generating the coefficients of the terms, ſucceſſively one from another, of any power of a binomial, independent of thoſe of any other power. For having ſhewn, in his *Abacus* Παγχρηστος (which he ſo calls on account of its frequent and excellent uſe, and of which a ſmall ſpecimen is here annexed), that the numbers

ABACUS ΠΑΓΧΡΗΣΤΟΣ.

H	G	F	E	D	C	B	A
− 8	− 7	+ 6	+ 5	− 4	− 3	+ 2	1
1	1	1	1	1	1	1	1
9	8	7	6	5	4	3	2
	36	28	21	15	10	6	3
		84	56	35	20	10	4
			126	70	35	15	5
				126	56	21	6
					84	28	7
						36	8
							9

in the diagonal directions, aſcending from right to left, are the coefficients of the powers of a binomial, the indexes being the figures in the firſt perpendicular column A, which are alſo the coefficients of the 2d terms of each power (thoſe of the firſt terms, being 1, are here omitted); and that any one of theſe diagonal numbers is in proportion to the next higher in the diagonal, as the vertical of the former is to the marginal of the latter, that is, as the uppermoſt number in the column of the former is to the firſt or right-hand number in the line of the latter; having ſhewn theſe things, I ſay, he thereby teaches the generation of the coefficients of any power, independently of all other powers, by the very ſame law or rule which we now uſe in the binomial theorem. Thus, for the 9th power; 9 being the coefficient of the 2d term, and 1 always that of the firſt, to find the 3d coefficient we have 2 : 8 : : 9 : 36; for the 4th term, 3 : 7 : : 36 : 84; for the 5th term, 4 : 6 : : 84 : 126; and ſo on for the reſt. That is to ſay, the coefficients of the terms in any power m, are inverſely as the vertical numbers or firſt line 1, 2, 3, 4, ... m, and directly as the aſcending numbers m, $m - 1$, $m - 2$, $m - 3$, ... 1, in the firſt column A; and that conſequently thoſe coefficients

cients are found by the continual multiplication of these fractions $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, ... $\frac{1}{m}$, which is the very theorem as it stands at this day, and as applied by Newton to roots or fractional exponents, as it had before been used for integral powers. This theorem then being thus plainly taught by Briggs about the year 1600, I am surprised how a man of such general reading as Dr. Wallis was, could possibly be ignorant of it, as he plainly appears to be by the 85th chapter of his Algebra, where he fully ascribes the invention to Newton, and adds that he himself had formerly sought after such a rule, but without success: or how Mr. John Bernoulli, not half a century since, could himself first dispute the invention of this theorem with Newton, and then give the discovery of it to M. Pascal, who was not born till long after it had been taught by Briggs. *See* Bernoulli's *Works, vol.* 4. *p.* 173. But I do not wonder that Briggs's remark was unknown to Newton, who owed almost every thing to genius, and very little to reading: and I have no doubt that he made the discovery himself, without any light from Briggs; and that he thought it was new for all powers in general, as it was indeed for roots and quantities with fractional and irrational exponents.

When the above table of the sums of figurate numbers is used by our author in determining the coefficients of the terms of the equation, whose root is the chord of any submultiple of an arc, as when the section is expressed by any uneven number, he remarks that the powers of that chord or root will be the 1st, 3d, 5th, 7th, &c, in the alternate uneven columns, A, C, E, G, &c, with their signs + or −, as marked to the powers, continued till the highest power be equal to the index of the section; and that the coefficients of those powers are the sums of two continuous numbers in the same column with the powers, beginning with 1 at the highest power, and gradually descending one line obliquely to the right at each lower power: so for a trisection, the numbers are 1 in C, and 1 + 2 = 3 in A; and therefore the terms are − 1 ③ + 3 ①: for a quinquisection, the numbers are 1 in E, 1 + 4 = 5 in C, 2 + 3 = 5 in A; so that the terms are 1 ⑤ − 5 ③ + 5 ①: for a septisection, the numbers are 1 in G, 1 + 6 = 7 in E, 4 + 10 = 14 in C, and 3 + 4 = 7 in A; and so the terms are − 1 ⑦ + 7 ⑤ − 14 ③ + 7 ①: and so on; the sum of all these terms being always equal to the chord of the whole or multiple arc. But when the section is denominated by an even number, the squares of the chords enter the equation instead of the first powers as before, and the dimensions of all the powers are doubled, the coefficients being found as before, and therefore the powers and numbers will be those in the 2d, 4th, 6th, &c, columns: and the uneven sections may also be expressed the same way: hence for a bisection the terms will be − 1 ④ + 4 ②; for a trisection 1 ⑥ — 6 ④ + 9 ②; for the quadrisection − 1 ⑧ + 8 ⑥ − 20 ④ + 16 ②; for the quinquisection 1 ⑩ − 10 ⑧ + 35 ⑥ − 50 ④ + 25 ②; and so on.

Our

Our author alſo ſubjoins another table, a ſmall ſpecimen of which is here annexed, in which the firſt column conſiſts of the uneven numbers, 1, 3, 5, &c; the reſt being found by addition as before, and the alternate diagonal numbers themſelves are the coefficients.

F	E	D	C	B	A
+⑥	+⑤	−④	−③	+②	①
1	1	1	1	1	1
	7	6	5	4	3
		20	14	9	5
			30	16	7
				25	9
					11

The method is quite different from that of Vieta, who gives another table for the like purpoſe, a ſmall part of which is here annexed, which is formed by adding from the number 2 downwards obliquely towards the right; and the coefficients of the terms ſtand upon the horizontal line.

VIETA's *Table.*

1ſt					
2					
3	2d				
4	2				
5	5	3d			
6	9	2			
7	14	7	4th		
8	20	16	2		
9	27	30	9	5th	
10	35	50	25	2	6th

Theſe angular ſections were afterwards further diſcuſſed by Oughtred and Wallis. And the ſame theorems of Vieta and Briggs have been ſince given in a different form, by Meſſrs. Herman, and the Bernoullis in the *Leipſic Acts*, and the *Memoirs of the Royal Academy of Sciences*. Theſe theorems they expreſſed by the alternate terms of the power of a binomial, whoſe exponent is that of the multiple angle or ſection. And Mr. De Lagny, in the ſame Memoirs, firſt ſhewed that the tangents and ſecants of multiple angles are alſo expreſſed by the terms of a binomial, in the form of a fraction, of which ſome of thoſe terms form the numerator, and others the denominator. Thus, if r expreſs the radius, s the ſine, c the coſine, t the tangent, and ſ the ſecant of the angle A; then the ſine, coſine, tangent, and ſecant of n times the angle are expreſſed thus, *viz.*

$$\text{Sin.}\,nA = \frac{1}{r^{n-1}} \times : \frac{n}{1} c^{n-1} s - \frac{n.n-1.n-2}{1\,.\,2\,.\,3} c^{n-3} s^3 + \frac{n.n-1.n-2.n-3.n-4}{1\,.\,2\,.\,3\,.\,4\,.\,5} c^{n-5} s^5 \ \&c.$$

$$\text{Coſine}\ nA = \frac{1}{r^{n-1}} \times : c^n - \frac{n.n-1}{1\,.\,2} c^{n-2} s^2 + \frac{n.n-1.n-2.n-3}{1\,.\,2\,.\,3\,.\,4} c^{n-4} s^4 \ \&c.$$

$$\text{Tang.}\,nA = r \times \frac{\frac{n}{s} r^{n-t} t - \frac{n.n-1.n-2}{1\,.\,2\,.\,3} r^{n-3} t^3 + \frac{n.n-1.n-2.n-3.n-4}{1\,.\,2\,.\,3\,.\,4\,.\,5} r^{n-5} t^5 \ \&c.}{r^n - \frac{n.n-1}{1\,.\,2} r^{n-2} t^2 + \frac{n.n-1.n-2.n-3}{1\,.\,2\,.\,3\,.\,4} r^{n-4} t^4 \ \&c.}$$

$$\text{Sec.}\,nA = r \times \frac{\text{ſ}^2 \text{ or } r^2 + t^2}{r^n - \frac{n.n-1}{1\,.\,2} r^{n-2} t^2 + \frac{n.n-1.n-2.n-3}{1\,.\,2\,.\,3\,.\,4} r^{n-4} t^4 \ \&c.}$$

where it is evident that the ſeries in the value of the ſine of the multiple angle $n\,A$ conſiſts of the even terms of the ſeries that is equal to $\overline{c+s}|^n$, or the *nth* power of the binomial $c + s$, and the ſeries in the value of the coſine of the ſame multiple angle conſiſts of the uneven terms of the ſame power; alſo the ſeries in the numerator of the tangent conſiſts of the even terms of the power $\overline{r+t}|^n$, and the

the denominator, both of the tangent and ſecant, conſiſts of the uneven terms of the ſame power $\overline{r+t}|^n$. And, if the diameter, chord, and chord of the ſupplement, be ſubſtituted for the radius, ſine, and coſine, in the expreſſions for the ſine and coſine of the multiple arc, the reſult will give the chord and chord of the ſupplement of n times the arc, or angle, A. Theſe and various other expreſſions for multiple and ſubmultiple arcs, with other improvements in trigonometry, have alſo been given by Euler and other eminent writers on the ſubject.

The before-mentioned M. De Lagny offered a project for ſubſtituting, inſtead of the common logarithms, a binary arithmetic, which he called the *natural logarithms*, and which he and M. Leibnitz ſeem to have both invented about the ſame time, independently of each other: but the project came to nothing. Mr. De Lagny alſo publiſhed, in ſeveral Memoirs of the Royal Academy, a new method of determining the angles of figures, which he called *Goniometry*. It conſiſts in meaſuring with a pair of compaſſes the arc which ſubtends the angle in queſtion; however this arc is not meaſured by applying its extent to any preconſtructed ſcale, but by examining what part it is of half the circumference of the ſame circle, in this manner: from the propoſed angular point as a center, with a ſufficiently large radius, a ſemicircle being deſcribed, a part of which is the arc intercepted by the ſides of the propoſed angle, the extent of this arc is taken with a fine pair of compaſſes, and applied continually upon the arc of the ſemicircle, by which he finds how often it is contained in the ſemicircle, with uſually a ſmall arc remaining; in the ſame manner he meaſures how often this remaining arc is contained in the firſt arc, and what remains again is applied continually to the firſt remainder, and ſo the 3d remainder to the 2d, the 4th to the 3d, and ſo on till there be no remainder, or elſe till it become inſenſibly ſmall. By this proceſs he obtains a ſeries of quotients, or fractional parts, one of another, which being properly reduced into one, give the ratio of the firſt arc to the ſemicircumference, or the ratio of the propoſed angle to two right angles or 180 degrees, and conſequently the magnitude of that angle in degrees, minutes, &c, if required, and that commonly to a degree of accuracy far exceeding the calculation of the ſame by means of any tables of ſines, tangents, or ſecants, notwithſtanding the apparent paradox in this expreſſion at firſt ſight. Thus, if the 1ſt arc be 4 times contained in the ſemicircle, the remainder once contained in the firſt arc, the next five times in the ſecond, and finally the fourth two times in the third: here the quotients are 4, 1, 5, 2; conſequently the fourth or laſt arc was $\frac{1}{2}$ the 3d, therefore the 3d was $\frac{1}{5\frac{1}{2}}$ or $\frac{2}{11}$ of the 2d, and the 2d was $\frac{1}{1\frac{2}{11}}$ or $\frac{11}{13}$ of the 1ſt, and the firſt or arc ſought, was $\frac{1}{4\frac{11}{13}}$ or $\frac{13}{63}$ of the ſemicircle; and conſequently it contains $37\frac{1}{7}$ degrees, or $37^\circ\ 8'\ 34''\ \frac{2}{7}$.

But to return from this long digreſſion, Mr. Briggs next treats of interpolation by differences, and chiefly of quinquiſection, after the manner uſed in the 13th chapter of his conſtruction of logarithms before deſcribed. He here proves that curious property of the ſines and their ſeveral orders of differences, before mentioned, namely, that, of equidifferent arcs, the ſines with the 2d, 4th, 6th, &c, differences, are continued proportionals; as alſo the coſines of the means be-

tween thofe arcs, and the 1ft, 3d, 5th, &c, differences And to this treatife on interpolation by differences, he adds a marginal note, complaining that this 13th chapter of his *Arithmetica Logarithmica* had been omitted by Vlacq in his edition of it; as if he were afraid of an intention to deprive him of the honour of the invention of interpolation by fucceffive differences. The note is this: *Modus correctionis à me traditus eft in Arithmeticæ Logarithmicæ capite* 13, *in editione Londinenfi: Iftud autem caput, unà cum fequenti in editione Batavâ, me inconfulto et infcio omiffum fuit: nec in omnibus editionis illius author (vir alioqui induftrius et non indoctus) meam mentem videtur affequutus. Ideóque, ne quicquam defit cuiquam qui integrum canonem conficere cupiat, quædam maximè neceffaria illinc huc transferenda cenfui.*

A large fpecimen of quinquifection by differences is then given, and he fhews how it is to be applied to the conftruction of the whole canon of fines, both for 100th and 1000th parts of degrees; namely, for centefms, divide the quadrant firft into 72 equal parts, and find their fines by the primary methods; then thefe quinquifected give 360 parts, a fecond quinquifection gives 1800 parts, and a third gives 9000 parts, or centefms of degrees: but for millefms, divide the quadrant into 144 equal parts; then one quinquifection gives 720, a fecond gives 3600, a third 18,000, and a fourth gives 90,000 parts or millefms.

He next proceeds to the natural tangents and fecants, which are directed to be raifed in the fame manner, by interpolations from a few primary ones, conftructed from the known proportions between fines, tangents, and fecants; excepting that half the tangents and fecants are to be formed by addition and fubtraction only, by means of fome fuch theorems as thefe, namely, 1ft, the fecant of an arc is equal to the fum of the tangent of the fame arc, and the tangent of half its complement, which will find every other fecant; 2d, Double the tangent of an arc, added to the tangent of half its complement, is equal to the tangent of the fum of that arc and the faid half complement, by which rule half the tangents will be found; &c.

In the two remaining chapters of this book are treated the conftruction of the logarithmic fines, tangents, and fecants. This is preceeded by fome remarks on the origin and invention of them. Our author here obferves that logarithms may be of various kinds: that others had followed the plan of Baron Napier, the firft inventor; among whom Benjamin Urfinus is efpecially commended, who applied Napier's logarithms to every ten feconds of the quadrant; but that he himfelf, encouraged by the noble inventor, devifed other logarithms that were much eafier and more excellent *. He fays he put 10, with cyphers, for the logarithm of radius; 9 for the logarithm fine of 5° 44′, whofe natural fine is one 10th of the radius; 8 for that of 34′, whofe natural fine is one 100th of the radius, &c: thereby making 1 the logarithm of the ratio of 10 to 1, which is the characteriftic of his fpecies of logarithms.

To conftruct the logarithmic fines, he directs firft to divide the quadrant into 72 equal parts as before, and to find the logarithms of their natural fines, as in

* His words are "Ego verò, ipfius inventoris primi cohortatione adjutus, alios logarithmos applicandos cenfui, qui multo faciliorem ufum habent, præftantiorémque. Logarithmus radii circularis vel finus totius, a me ponitur 10 &c."

the

the 14th chapter of his *Arithmetica Logarithmica*; after which this number will be increased by quinquisection, first to 360, then to 1800, and lastly to 9000, or centesms of degrees. But if millesms of degrees be required, divide the quadrant first into 144 equal parts, and then by four quinquisections these will be extended to the following parts, 720, 3600, 18,000, and 90,000, or millesms of degrees. He remarks however that the logarithmic sines of only half the quadrant need be found in this manner; as the other half may be found by mere addition, or subtraction, by means of this theorem, as the sine of half an arc is to half the radius, so is the sine of the whole arc to the cosine of the said half arc. This theorem he illustrates with examples; and then adds a table of the logarithmic sines of the primary 72 parts of the quadrant, from which the rest are to be made out by quinquisection.

In the next chapter our author shews the construction of the natural tangents and secants more fully than he had done before, demonstrating and illustrating several curious theorems for the easy finding of them He then concludes this chapter, and the book, with pointing out the very easy construction of the logarithmic tangents and secants by means of these three theorems.

1st, As cosine : sine : : radius : tangent,
2d, As tangent : radius : : radius : cotangent,
3d, As cosine : radius : : radius : secant.

So that in logarithms, the tangents are found by subtracting the cosines from the sines, adding always 10 or the radius; the cotangents are found by subtracting always the tangents from 20 or double the radius; and the secants are found by subtracting the cosines from 20 the double radius.

The 2d book, by Gellibrand, contains the use of the canon in plane and spherical trigonometry.

Besides Briggs's methods of constructing logarithms, above described, no others were given about that time. For as to the calculations made by Vlacq, his numbers being carried to comparatively but few places of figures, they were performed by the easiest of Briggs's methods, and in the manner which this ingenious man had pointed out in his two volumes. Thus, the 70 chiliads of logarithms, from 20,000 to 90,000, computed by Vlacq, and published in 1628, being extended only to 10 places, yield no more than two orders of mean differences, which are also the correct differences, in quinquisection, and therefore will be made out thus; namely, one-fifth of them by the mere addition of the constant logarithm of 5; and the other four-fifths of them by two easy additions of very small numbers, namely, of the 1st and 2d differences, according to the directions given in Briggs's *Arith. Log.* c. 13. p. 31. And as to Vlacq's logarithmic sines and tangents to every 10 seconds, they were easily computed thus: the sines for half the quadrant were found by taking the logarithms to the natural sines in Rheticus's canon; and then from these the logarithmic sines to the other half quadrant were found by mere addition and subtraction; and from these all the tangents by one single subtraction. So that all these operations might easily be performed by one person, as quickly as a printer could set up the types; and thus the computation and printing might both be carried on together. And hence it appears that there is no reason for admiration at the expedition with which these tables were said to have been brought out.

 Of

Of certain Curves related to Logarithms.

About this time the mathematicians of Europe began to confider fome curves which have properties analogous to logarithms. Edmund Gunter, it has been faid, firft gave the idea of a curve, whofe abfciffes are in arithmetical progreffion, while the correfponding ordinates are in geometrical progreffion, or whofe abfciffes are the logarithms of their ordinates; but I cannot find it noticed in any part of his writings. The fame curve was afterwards confidered by others, and named the *Logarithmic* or *Logiftic* curve by Huygens in his *Differtatio de Caufâ Gravitatis*, where he enumerates all the principal properties of this curve, fhewing its analogy to logarithms *. Many other learned men have alfo treated of its properties; particularly Le Seur and Jacquier in their comment on Newton's Principia; Dr. John Keill in the elegant little tract on logarithms fubjoined to his edition of Euclid's Elements; and Francis Maferes, Efq. Curfitor Baron of the Exchequer, in his ingenious treatife on Trigonometry; in which books the doctrine of logarithms is copioufly and learnedly treated, and their analogy to the logarithmic curve, &c, fully difplayed.—It is indeed rather extraordinary that this curve was not fooner announced to the public; fince it refults immediately from baron Napier's manner of conceiving the generation of logarithms, by only fuppofing the lines which reprefent the natural numbers to be placed at right angles to that upon which the logarithms are taken. This curve greatly facilitates the conception of logarithms to the imagination, and affords an almoft intuitive proof of the very important property of their fluxions, or very fmall increments, to wit, that the fluxion of the number is to the fluxion of the logarithm, as the number is to the fubtangent; as alfo of this property, that, if three numbers be taken very nearly equal, fo that their ratios to each other may differ but a little from a ratio of equality, as for example, the three numbers 10,000,000, 10,000,001, 10,000,002, their differences will be very nearly proportional to the logarithms of the ratios of thofe numbers to each other: all which follows from the logarithmic arcs being very little different from their chords, when they are taken very fmall. And the conftant fubtangent of this curve is what was afterwards by Cotes called the *Modulus* of the fyftem of logarithms: and fince, by the former of the two properties above mentioned, this fubtangent is a 4th proportional to the fluxion of the number, the fluxion of the logarithm, and the number; this property afforded occafion to Mr. Baron Maferes to give the following definition of the modulus, which is the fame in effect as Cotes's, but more clearly expreffed, namely, that it is the limit of the magnitude of a 4th proportional to thefe three quantities, to wit, the difference of any two natural numbers that are very nearly equal to each other, either of the faid numbers, and the logarithm or meafure of the ratio they have to each other. Or we may define the modulus to be the natural number at that part of the fyftem of logarithms, where the fluxion of the number is equal to the fluxion of the logarithm, or where the numbers and logarithms have equal differences. And

* Mr. James Gregory alfo fpeaks of this curve in the higheft terms of approbation, on account of its utility in illuftrating the nature of ratios and logarithms. See his Preface, or Proœmium, to his *Geometriæ Pars Univerfalis*, publifhed at *Padua* in Italy in the year 1668.

hence

hence it follows, that the logarithms of equal numbers or of equal ratios, in different ſyſtems, are to one another as the *moduli* of thoſe ſyſtems. Moreover, the ratio whoſe meaſure, or logarithm, is equal to the modulus, and which is therefore by Cotes called the *ratio modularis*, is by calculation found to be the ratio of 2·718,281,828, 459, &c, to 1, or of 1 to ·367,879,441,171, &c: the calculation of which number may be ſeen at full length in Mr. Baron Maſeres's treatiſe on the Principles of Life-annuities, p. 274, 275 *.

The hyperbolic curve alſo afforded another ſource for developing and illuſtrating the properties and conſtruction of logarithms. For the hyperbolic areas lying between the curve and one aſymptote, when they are bounded by ordinates parallel to the other aſymptote, are analogous to the logarithms of their abſciſſes or parts of the aſymptote. And ſo alſo are the hyperbolic ſectors; any ſector bounded by an arc of the hyperbola and two radii, being equal to the quadrilateral ſpace bounded by the ſame arc, the two ordinates to either aſymptote from the extremities of the arc, and the part of the aſymptote intercepted between them. And although Napier's logarithms are commonly ſaid to be the ſame as hyberbolic logarithms, it is not to be underſtood that hyperbolas exhibit Napier's logarithms only, but indeed all other poſſible ſyſtems of logarithms whatever. For, like as the right-angled hyperbola, the ſide of whoſe ſquare inſcribed at the vertex is 1, gives us Napier's logarithms; ſo any other ſyſtem of logarithms is expreſſed by the hyperbola whoſe aſymptotes form a certain oblique angle, the ſide of the rhombus inſcribed at the vertex of the hyperbola in this caſe alſo being ſtill 1, the ſame as the ſide of the ſquare in the right-angled hyperbola. But the areas of the ſquare and rhombus, and conſequently the logarithms of any one and the ſame number or ratio, will differ according to the ſine of the angle of the aſymptotes. And the area of the ſquare or rhombus, or any inſcribed parallelogram, is alſo the ſame thing as what was by Cotes called the modulus of the ſyſtem of logarithms; which modulus will therefore be expreſſed by the numerical meaſure of the ſine of the angle formed by the aſymptotes, to the radius 1; as that is the ſame with the number expreſſing the area of the ſaid ſquare or rhombus, the ſide being 1: which is another definition of the modulus to be added to thoſe we before remarked above, in treating of the logarithmic curve. And the evident reaſon of this is, that in the beginning of the generation of theſe areas from the vertex of the hyperbola, the naſcent increment of the abſciſſe drawn into the altitude 1, is to the increment of the area, as radius is to the ſine of the angle of the ordinate and abſciſſe, or of the aſymptotes; and at the beginning of the logarithms, the naſcent increment of the natural numbers is to the increment of the logarithms, as 1 is to the modulus of the ſyſtem. Hence we eaſily diſcover that the angle formed by the aſymptotes of the hyperbola exhibiting Briggs's ſyſtem of logarithms, will be 25 deg. 44 min. $25\frac{1}{2}$ ſec. this being the angle whoſe ſine is 0·434,294,481,9, &c, the modulus of this ſyſtem.

Or indeed any one hyperbola, as has been remarked by Mr. Baron Maſeres, will expreſs all poſſible ſyſtems of logarithms whatever, namely, if the ſquare or rhom-

* This calculation is alſo ſet forth in one of the tracts contained in this preſent volume, pages 358, 359, 360, and 371, 372, 373, 374, 375.

bus

bus inſcribed at the vertex, or, which is the ſame thing, any parallelogram inſcribed between the aſymptotes and the curve at any other point, be expounded by the modulus of the ſyſtem ; or, which is the ſame, by expounding the area, intercepted between two ordinates which are to each other in the ratio of 10 to 1, by the logarithm of that ratio in the propoſed ſyſtem.

As to the firſt remarks on the analogy between logarithms and the hyperbolic ſpaces, it having been ſhewn by Gregory St. Vincent, in his *Quadratura Circuli & Sectionum Coni*, publiſhed at Antwerp in 1647, that if one aſymptote be divided into parts in geometrical progreſſion, and from the points of diviſion ordinates be drawn parallel to the other aſymptote, they will divide the ſpace between the aſymptote and curve into equal portions ; from hence it was ſhewn by Merſennus, that by taking the continual ſums of thoſe parts, there would be obtained areas in arithmetical progreſſion, adapted to abſciſſes in geometrical progreſſion, and which therefore were analogous to a ſyſtem of logarithms. And the ſame analogy was remarked and illuſtrated ſoon after by Huygens, and many others, who ſhew how to ſquare the hyperbolic ſpaces by means of the logarithms.

*Of *Gregory's Computation of Logarithms.*

On the other hand, Mr. James Gregory, in his *Vera Circuli et Hyperbolæ Quadratura*, firſt printed at Patavium, or Padua, in the north of Italy, in the year 1667, having approximated to the aſymptotic ſpaces of an hyperbola by means of a ſeries of inſcribed and circumſcribed polygons, from thence ſhews how to compute the logarithms, which are analogous to thoſe areas : and thus the quadrature of the hyperbolic ſpaces became the ſame thing as the computation of the logarithms. He here alſo lays down various methods to abridge the computation, with the aſſiſtance of ſome properties of numbers themſelves, by which we are enabled to compoſe the logarithms of all prime numbers under 1000, each by one multiplication, two diviſions, and the extraction of the ſquare root. And the ſame ſubject is farther purſued in his *Exercitationes Geometricæ*, to be deſcribed hereafter.

There are are alſo innumerable other geometrical figures having properties analogous to logarithms ; ſuch as the equi-angular ſpiral (which is alſo called the *Logarithmic Spiral*), the figure of tangents, and the figure of ſecants, &c ; which it is not to our purpoſe to diſtinguiſh more particularly.

Of † Mercator's Logarithmotechnia.

In 1668, Mr. Nicholas Mercator publiſhed his *Logarithmotechnia, ſive Methodus*

* James Gregory was born at Aberdeen in Scotland in 1639, where he was educated. He was profeſſor of Mathematics in the college of St. Andrew ; and died of a fever in December 1675, being only 36 years of age.

† Nicholas Mercator, a learned mathematician, and an ingenious member of the Royal Society, was a native of Holſtein in Germany, but ſpent moſt of his time in England, where he died in the year 1690, at about 50 years of age. He was the author of many other works in Geometry, Geography, Aſtronomy, Aſtrology, &c.

conſtruendi Logarithmos nova, accurata, & facilis; in which he delivers a new and ingenious method for computing the logarithms upon principles purely arithmetical; which being curious, and very accurately performed, I ſhall here give a rather full and particular account of that little tract, as well as of the ſmall ſpecimen of the quadrature of curves by infinite ſeries which is ſubjoined to it; and the rather, as this work gave occaſion to the public communication of ſome of Sir Iſaac Newton's earlieſt pieces, to evince that he had not borrowed them from this publication. So it appears that theſe two ingenious men had, independent of each other, in ſome inſtances fallen upon the ſame diſcoveries.

Our author begins this work with remarking that the word *Logarithm* is compoſed of the two Greek words λογὸς and ἀριθμὸς, which anſwer to the words *ratio* and *number*, being as much as to ſay, the *number of ratios*; which he obſerves is quite agreeable to the nature of them, for that a logarithm is nothing elſe but the number of *ratiunculæ*, or very ſmall ratios, contained in the ratio which any number bears to unity. He then makes a very learned and critical diſſertation on the nature of ratios, their magnitude and meaſure, conveying a clearer idea of the nature of logarithms than had been given by either Napier or Briggs, or any other writer except the famous Kepler, in his work before deſcribed; although thoſe other writers ſeem indeed to have had in their own minds the ſame ideas on the ſubject as Kepler and Mercator, but without having expreſſed them ſo clearly. Our author indeed pretty cloſely follows Kepler in his modes of thinking and expreſſion, and after him in plain and expreſs terms calls logarithms the meaſures of ratios; and, in order to the right underſtanding that definition of them, he explains what he means by the magnitude of a ratio. This he does pretty fully, but not too fully, conſidering the nicety and ſubtlety of the ſubject of ratios, and their magnitude, with their addition to, and ſubtraction from, each other, which are points that have often been miſconceived by very learned mathematicians, who have thence been led into conſiderable miſtakes. Witneſs the overſight of Gregory St. Vincent, which Huygens animadverted upon in the Ἐξέτασις *Cyclometriæ Gregorii, à Sancto Vincentio*, and which aroſe from not underſtanding, or not adverting to, the nature of ratios, and their proportions one to another. And many other ſimilar miſtakes might here be adduced of other eminent writers. From all which we muſt commend the propriety of our author's attention, in ſo judiciouſly diſcriminating between the magnitude of a ratio, as of a to b, and the fraction $\frac{a}{b}$, or quotient ariſing from the diviſion of one term of the ratio by the other; which latter method of conſidering ratios is always attended with danger of errors and confuſion on the ſubject: though in the 5th definition of the 6th book of Euclid this quotient is accounted the quantity of the ratio; but this definition is probably not genuine, and therefore very properly omitted by profeſſor Simſon in his edition of the Elements. And in theſe ideas on the ſubject of logarithms, Kepler and Mercator have been followed by Dr. Halley, and Mr. Cotes, and moſt other eminent writers ſince that time.

Purely from the above idea of logarithms, namely, as being the meaſures of ratios, and as expreſſing the number of *ratiunculæ* contained in any ratio, or into which it may be divided, the number of the like equal *ratiunculæ* contained in ſome

ſome one ratio, as of 10 to 1, being ſuppoſed given, our author ſhews how the logarithm, or meaſure, of any other ratio may be found. But this, however, he ſhews only by-the-bye, as not being the principal method he intends to teach, as his laſt and beſt, and which we arrive not at till near the end of the book, as we ſhall ſee below. Having ſhewn, then, that theſe logarithms, or numbers of ſmall ratios, or meaſures of ratios, may be all properly repreſented by numbers, and that (the logarithm, or meaſure, of the ratio of 1 to 1, or of the ratio of equality, or, as it is often called, the logarithm of 1, being always 0), the logarithm of 10, or the meaſure of the ratio of 10 to 1, is moſt conveniently repreſented by 1, with any number of cyphers: he then proceeds to ſhew how the meaſures of all other ratios may be found from this laſt ſuppoſition. And he explains the principles by the two following examples.

Firſt, to find the logarithm of 100·5*, or to find how many *ratiunculæ* are contained in the ratio of 100·5 to 1, the number of *ratiunculæ* in the decuple ratio, or ratio of 10 to 1, being 10,000,000.

The given ratio 100·5 to 1, he firſt divides into its parts, namely, the ratios of 100.5 to 100, of 100 to 10, and of 10 to 1; the laſt two of which being decuples, it follows that the characteriſtic of the logarithm of the whole ratio of 100·5 to 1 will be 2, and it only remains to find how many parts of the next decuple belong to the firſt of theſe three ratios, to wit, the ratio of 100·5 to 100. Now if each term of this ratio be multiplied by itſelf, the products will be in the duplicate ratio of the firſt terms, or this laſt ratio will contain a double number of parts; and if theſe be multiplied by the firſt terms again, the ratio of the laſt products will contain three times the number of parts; and ſo on, the number of times of the firſt parts contained in the ratio of any like powers of the firſt terms, being always denoted by the exponent of the power. If therefore the firſt terms, 100·5 and 100, be continually multiplied till the ſame powers of them have to each other a ratio whoſe meaſure is known, as ſuppoſe the decuple ratio 10 to 1, whoſe meaſure is 10,000,000; then the exponent of that power ſhews what multiple this meaſure 10,000,000, of the decuple ratio, is of the required meaſure of the firſt ratio 100·5 to 100; and conſequently, dividing 10,000,000, by that exponent, the quotient is the meaſure of the ratio 100·5 to 100 ſought. The operation for finding this he ſets down as here follows; where the ſeveral multiplications are all performed in the contracted way by inverting the figures of the multiplier, and retaining only the firſt number of decimals in each product.

* Mercator diſtinguiſhes his decimals from integers thus 100[5, or thus 100|5.

	power
100·5000 - -	1
5001 - -	1
1005000	
5025	
1010025 - -	2
5200101 - -	2
1010025	
10100	
20	
5	
1020150 - -	4
0510201 - -	4
1020150	
20403	
102	
51	
1040706 - -	8
6070401 - -	8
1083068 - -	16
8603801 - -	16
1173035 - -	32
5303711 - -	32
1376011 - -	64
1106731 - -	64
1893406 -	128
6043981 -	128
3584985 -	256
5894853 -	256
12852116 -	512

This power being greater than the decuple of the like power of 100, which muſt always be 1 with ciphers, reſume therefore the 256th power, and multiply it, not by itſelf, but by the next before it, viz. by the 128th, thus :

3584985 - - -	256
6043981 - - -	128
6787831 - - -	384
1106731 - - -	64
9340130 - - -	448
5303711 - - -	32
10956299 - - -	480

This power again exceeding the ſame power of 100 more than 10 times, I therefore draw the ſame 448th, not into the 32d, but the next preceding, thus :

9340130 - - -	448
8603801 - - -	16
10115994 - - -	464

This being again too much, inſtead of the 16th, draw it into the 8th or next preceeding, thus :

9340130 - - -	448
6070401 - - - -	8
9720329 - - -	456
0510201 - - - -	4
9916193 - - -	460
5200101 - - - -	2
10015603 - - -	462

Which power again exceeds the limit; therefore draw the 460th into the 1ſt, thus :

9916193 - - -	460
5001 - - -	1
9965774 - - -	461

Since therefore the 462d power of 100·5 is greater, and the 461ſt power is leſs, than the decuple of the ſame power of 100; I find that the ratio of 100·5 to 100 is contained in the decuple more than 461 times, but leſs than 462 times. Again,

Since the		power is		and the differences	
	460		9916193		
	461		9965774	49581	nearly
	462		10015603	49829	equal;

therefore the proportional part which the exact power, or 10000000, exceeds the next leſs 9965774, will be eaſily and accurately found by the Golden Rule, thus :

The juſt power - - -	10000000
and the next leſs - -	9965774
the difference - -	34226 ; then

As 49829 the dif. between the next leſs and greater,
: To 34226 the dif. between the next leſs and juſt,
: : So is 10000 : to 6868, the decimal parts ; and therefore the ratio of 100·5 to 100, is 461·6868 times contained in the decuple or ratio of 10 to 1. Dividing

now 1,0000000, the meaſure of the decuple ratio, by 461·6868, the quotient ·002,165,97 is the meaſure of the ratio of 100·5 to 100; which being added to 2, the meaſure of 100 to 1, the ſum 2·002,165,97 is the meaſure of the ratio of 100·5 to 1, that is, the log. of 100·5 is 2·002,165,97.

In the ſame manner he next inveſtigates the log. of 99·5, and finds it to be 1·997,823,07.

A few obſervations are then added, calculated to generalize the conſideration of ratios, their magnitude and affections. It is here remarked that he conſiders the magnitude of the ratio between two quantities as the ſame, whether the antecedent be the greater or the leſs of the two terms; ſo the magnitude of the ratio of 8 to 5, is the ſame as of 5 to 8; that is, by the magnitude of the ratio of either to the other, is meant the number of *ratiunculæ* between them, which will evidently be the ſame whether the greater or leſs term be the antecedent. And he farther remarks that of different ratios, when we divide the greater term of each ratio by the leſs, that ratio is of the greater maſs, or magnitude, which produces the greater quotient, *et vice verſâ*; although thoſe quotients are not proportional to the maſſes, or magnitudes, of the ratios. But when he conſiders the ratio of a greater term to a leſs, or of a leſs to a greater, that is to ſay, the ratio of greater or leſs inequality, as abſtracted from the magnitude of the ratio, he diſtinguiſhes it by the word *affection*, as much as to ſay, greater or leſs affection, ſomething in the manner of poſitive and negative quantities, or ſuch as are affected with the ſigns + and − . . . . . The remainder of this work he delivers in ſeveral propoſitions, as follows.

Prop. 1. In ſubtracting from each other two quantities of the ſame affection, to wit, both poſitive or both negative; if the remainder be of the ſame affection with the two given, then is the quantity ſubtracted the leſs of the two, or expreſſed by the leſs number; but if the contrary, it is the greater.

Prop. 2. In any continued ratios, as $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, &c, (by which is meant the ratios of a to $a+b$, of $a+b$ to $a+2b$, of $a+2b$ to $a+3b$, &c) of equidifferent terms, the antecedent of each ratio being equal to the conſequent of the next preceeding one, and proceeding from leſs terms to greater, the meaſure of each ratio will be expreſſed by a greater quantity than that of the next following; and the ſame through all their orders of differences, namely, the 1ſt, 2d, 3d, &c, differences; but the contrary when the terms of the ratios decreaſe from greater to leſs.

Prop. 3. In any continued ratios of equidifferent terms, if the 1ſt or leaſt be a, the difference between the 1ſt and 2d b, and c, d, e, &c, the reſpective firſt terms of their 2d, 3d, 4th, &c, differences; then ſhall the ſeveral quantities themſelves be as in the annexed ſcheme; where each term is compoſed of the firſt term, together with as many of the differences as it is diſtant from the firſt term, and to thoſe differences joining, for coefficients, the numbers in the ſloping or oblique lines contained in the annexed table of figurate numbers, in the ſame manner, he obſerves, as the ſame figurate numbers compleat the powers raiſed from a binomial root, as had long before been taught by others. He alſo remarks that this rule not only gives any one term, but alſo the ſum of any number of ſucceſſive terms from the beginning, making the 2d coefficient the firſt, the 3d the 2d, and ſo on; thus, the ſum of the firſt 5 terms is $5a + 10b + 10c + 5d + e$.

1ſt term	a
2d - -	$a + b$
3d - -	$a + 2b + c$
4th - -	$a + 3b + 3c + d$
5th - -	$a + 4b + 6c + 4d + e$
&c.	&c.

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
1	3	6	10	15	21	28	36	
1	4	10	20	35	56	84		
1	5	15	35	70	126			
1	6	21	56	126				
1	7	28	84					
1	8	36						
1	9							

In the 4th *prop.* it is ſhewn that if the terms decreaſe, proceeding from the greater to the leſs, the ſame theorems hold good, by only changing the ſign of every other term, as in the margin.

1ſt term	a
2d - -	$a - b$
3d - -	$a - 2b + c$
4th - -	$a - 3b + 3c - d$
5th - -	$a - 4b + 6c - 4d + e$
&c.	&c.

Prop. 6 and 7 treat of the approximate multiplication and diviſion of ratios, or, which is the ſame thing, the finding nearly any powers or any roots of a given fraction, in an eaſy manner. The theorem for raiſing any power, when reduced to a ſimpler form, is this: the m power of $\frac{a}{b}$, or $\left.\frac{a}{b}\right|^{m}$, is $= \frac{s \mp md}{s \pm md}$ nearly, where s is $= a + b$, and $d = a \backsim b$, the ſum and difference of the two numbers, and the upper or under ſigns take place according as $\frac{a}{b}$ is a proper or an improper fraction, that is, according as a is leſs or greater than b. And the theorem for extracting the mth root of $\frac{a}{b}$ is $\sqrt[m]{\frac{a}{b}}$ or $\left.\frac{a}{b}\right|^{\frac{1}{m}} = \frac{ms \mp d}{ms \pm d}$ nearly; which latter rule is alſo the ſame as the former, as will be evident by ſubſtituting $\frac{1}{m}$ inſtead of m in the firſt theorem. So that univerſally $\left.\frac{a}{b}\right|^{\frac{m}{n}}$ is $= \frac{ns \mp md}{ns \pm md}$ nearly. Theſe theorems however are nearly true only in ſome certain caſes, namely, when $\frac{a}{b}$ and $\frac{m}{n}$ do not differ greatly from unity. And in the 7th *prop.* the author ſhews how to find nearly the error of the theorems.

In

In the 8th *prop.* it is ſhewn that the meaſures of ratios of equidifferent terms, are nearly reciprocally as the arithmetical means between the terms of each ratio. So of the ratios $\frac{16}{18}$, $\frac{33}{35}$, $\frac{50}{52}$, the mean between the terms of the firſt ratio is 17, of the 2d 34, of the 3d 51, and the meaſures of the ratios are nearly as $\frac{1}{17}$, $\frac{1}{34}$, $\frac{1}{51}$.

From this property he proceeds, in the 9th *prop.* to find the meaſure of any ratio leſs than $\frac{99\cdot5}{100\cdot5}$, which has an equal difference (1) of terms. In the two examples, mentioned near the beginning, our author found the logarithm or meaſure of the ratio, of $\frac{99\cdot5}{100}$, to be 21769, $\frac{3}{10}$, and that of $\frac{100}{100\cdot5}$ to be $21659\frac{7}{10}$; therefore the ſum 43429 is the logarithm of $\frac{99\cdot5}{100\cdot5}$, or $\frac{99\cdot5}{100} \times \frac{100}{100\cdot5}$; or the logarithm of $\frac{99\cdot5}{100\cdot5}$ is nearer 43430, as found by other more accurate computations.— Now to find the logarithm of $\frac{100}{101}$, having the ſame difference of terms (1) with the former; it will be, by prop. 8, as 100.5 (the mean between 101 and 100) : 100 (the mean between 99·5 and 100·5) : : 43430 : 43213 the logarithm of $\frac{100}{101}$, or the difference between the logarithms of 100 and 101. But the log. of 100 is 2; therefore the logarithm of 101 is 2·004,321,3.——Again, to find the logarithm of 102, we muſt firſt find the logarithm of $\frac{101}{102}$; the mean between its terms being 101·5, therefore as 101·5 : 100 : : 43430 : 42788 the logarithm of $\frac{101}{102}$, or the difference of the logarithms of 101 and 102. But the logarithm of 101 was found above to be 2·004,321,3; therefore the logarithm of 102 is 2·008,600,1.—So that dividing continually 868596 (the double of 434298 the logarithm of $\frac{99\cdot5}{100\cdot5}$ or $\frac{199}{201}$) by each number of the ſeries 201, 203, 205, 207, &c. then add 2 to the 1ſt quotient, to the ſum add the 2d quotient, and ſo on, adding always the next quotient to the laſt ſum, the ſeveral ſums will be the reſpective logarithms of the numbers in this ſeries 101, 102, 103, 104, &c.

The next, or *prop.* 10, ſhews that, of two pair of continued ratios whoſe terms have equal differences, the difference of the meaſures of the firſt two ratios, is to the difference of the meaſures of the other two, as the ſquare of the common term in the two latter, is to the ſquare of the common term in the two former, nearly. Thus in the four ratios $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+3b}{a+4b}$, $\frac{a+4b}{a+5b}$, as the meaſure of $\frac{aa+2ab}{\overline{a+b}|^2}$ (the difference of the firſt two or the quotient of the two fractions) : the meaſure of $\frac{aa+8ab+15bb}{\overline{a+4b}|^2}$: : $\overline{a+4b}|^2$: $\overline{a+b}|^2$, nearly.

In *prop.* 11, the author ſhews that ſimilar properties take place among two ſets of ratios conſiſting each of 3 or 4 &c continued numbers.

Prop. 12 ſhews that, of the powers of numbers in arithmetical progreſſion, the orders of differences which become equal, are the 2d differences in the ſquares, the 3d differences in the cubes, the 4th differences in the 4th powers, &c. And from hence it is ſhewn how to conſtruct all thoſe powers by the continual addition of their differences. As had been long before more fully explained by Briggs.

In

In the next or 13th *prop.* our author explains his compendious method of raiſing the tables of logarithms, ſhewing how to conſtruct the logarithms by addition only, from the properties contained in the 8th, 9th and 12th propoſitions. For this purpoſe he makes uſe of the quantity $\frac{a}{b-c}$, which by diviſion he reſolves into this infinite ſeries $\frac{a}{b} + \frac{ac}{bb} + \frac{ac^2}{b^3} + \frac{ac^3}{b^4}$ &c (*in infin.*). Putting then $a = 100$ the arithmetical mean between the terms of the ratio $\frac{99\cdot5}{100\cdot5}$, $b = 100000$, and c ſucceſſively equal to 0·5, 1·5, 2·5, &c. that ſo $b - c$ may be reſpectively equal to 99999·5, 99998·5, 99997·5, &c, (the correſponding means between the terms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c), it is evident that $\frac{a}{b-c}$ will be the quotient of the 2d term divided by the 1ſt in the proportions mentioned in the 8th and 9th propoſitions; and when each of theſe quotients are found, it remains then only to multiply them by the conſtant 3d term 43429, or rather 43429·8, of the proportion, to produce the logarithms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c, till $\frac{10000}{10001}$; then adding theſe continually to 4 (the logarithm of 10000 the leaſt number), or ſubtracting them from 5 (the logarithm of the higheſt term 100000), there will reſult the logarithms of all the abſolute numbers from 10000 to 100000. Now when c is $= 0.5$, then $\frac{a}{b} = \cdot001$, $\frac{ac}{bb} = \cdot000,000,005$, $\frac{ac^2}{b^3} = \cdot000,000,000,000,025$, $\frac{ac^3}{b^4} = \cdot000,000,000,000,000,000,125$, &c; therefore $\frac{a}{b-c} = \frac{a}{b} + \frac{ac}{bb} + \frac{ac^2}{b^3}$ &c is $= \cdot001,000,005,000,025,000,125$. In like manner, if $c = 1.5$, then $\frac{a}{b+c}$ will be $= \cdot001,000,015,000,225,003,375$: and if $c = 2\cdot5$, then $\frac{a}{b-c}$ will be $= \cdot001,000,025,000,625,015,625$; &c. But inſtead of conſtructing all the values of $\frac{a}{b-c}$ in the uſual way of raiſing the powers, he directs them to be found by addition only, as in the laſt propoſition. Having thus found all the values of $\frac{a}{b-c}$, the author then ſhews that they may be drawn into the conſtant logarithm 43429 by addition only, by the help of the annexed table of the firſt nine products of it.

1	43429
2	86858
3	130287
4	173716
5	217145
6	260574
7	304003
8	347432
9	390861

The author then diſtinguiſhes which of the logarithms it may be proper to find in this way, and which from their component parts. Of theſe the logarithms of all even numbers need not be thus computed, being compoſed from the number 2; which cuts off one half of the numbers: neither are thoſe numbers to be computed which end in 5, becauſe 5 is one of their factors: theſe laſt are $\frac{1}{10}$ of the numbers; and the two together $\frac{1}{2} + \frac{1}{10}$ make $\frac{3}{5}$ of the whole: and of the other $\frac{2}{5}$, the $\frac{1}{3}$ of them, or $\frac{2}{15}$ of the whole, are compoſed of 3; and hence $\frac{3}{5} + \frac{2}{15}$, or $\frac{11}{15}$ of the numbers, are made up of ſuch as are compoſed of 2, 3, and 5. As to the

the other numbers which may be composed of 7, of 11, &c; he recommends to find *their* logarithms in the general way, the same as if they were incomposites, as it is not worth while to separate them in so easy a mode of calculation. So that of the 90 chiliads of numbers from 10000 to 100000, only 24 chiliads are to be computed. Neither indeed are all of these to be calculated from the foregoing series for $\frac{a}{b-c}$, but only a few of them in that way, and the rest by the proportion in the 8th proposition. Thus having computed the logarithms of 10003 and 10013, omitting 10023 as being divisible by 3, estimate the logarithms of 10033 and 10043, which are the 30th numbers from 10003 and 10013; and again omitting 10053, a multiple of 3, find the logarithms of 10063 and 10073. Then by prop. 8,

As 10048, the arithmetical mean between 10033 and 10063,
to 10018, the arithmetical mean between 10003 and 10033,
so 13006, the difference between the logarithms of 10003 and 10033,
to 12967, the difference between the logarithms of 10033 and 10063;

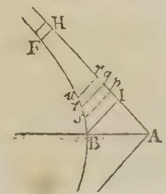
$$\text{That is, 1st - - As} \left\{ \begin{array}{c} 10048 \\ 10078 \\ 10108 \end{array} \right\} : 10018 :: 13006 : \left\{ \begin{array}{l} 12967 \\ \text{\&c.} \\ \end{array} \right.$$

$$\text{Again, As} \left\{ \begin{array}{c} 10058 \\ 10088 \\ 10118 \end{array} \right\} : 10028 :: 12992 : \left\{ \begin{array}{l} 12953 \\ \text{\&c.} \\ \end{array} \right.$$

$$\text{And 3dly, As} \left\{ \begin{array}{c} 10068 \\ 10098 \\ \text{\&c.} \end{array} \right\} : 10038 :: 12979 : \left\{ \begin{array}{l} 12940 \\ \text{\&c.} \\ \end{array} \right.$$

And with this our author concludes his compendium for constructing the tables of logarithms.

He afterwards shews some applications and relations of the doctrine of logarithms to geometrical figures: in order to which, in *prop.* 14, he proves algebraically that, in the right-angled hyperbola ABFH, if from the vertex B and from any other point F there be drawn the right lines BI, FH, perpendicular to the asymptote AH, or parallel to the other asymptote; then will AH : AI : : BI : FH. And,



In *prop.* 15, if AI = BI = 1, and HI = a; then will FH = $\frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - a^5$ &c, *in infinitum*, by a continual algebraic division, the process of which he describes step by step, as a thing that was new or uncommon. But that method of division had been taught before by Dr. Wallis, in his *Opus Arithmeticum*.

Prop. 16 is this: Any given number being supposed to be divided into innumerable small equal parts, it is required to assign the sum of any powers of the continual sums of those innumerable parts. For which purpose he lays down this rule: If the next higher power of the given number above that power whose sum is sought, be divided by its exponent, the quotient will be the sum of the powers sought. That is, if N be the given number, and a one of its innumerable

ble equal parts, then will $a^n + \overline{2a}|^n + \overline{3a}|^n + \overline{4a}|^n$ &c .... N^n be $= \frac{N^{n+1}}{n+1}$: which theorem he demonſtrates by a method of induction. And this, it is evident, is the finding the ſum of any powers of an infinite number of arithmeticals, of which the greateſt term is a given quantity, and the leaſt indefinitely ſmall. It is alſo remarkable that the above expreſſion is ſimilar to the rule for finding the fluent to the given fluxion of a power, as afterwards taught by Sir I. Newton.

Our author then applies this rule in *prop.* 17, to the quadrature of the hyperbola. Thus, putting AI = 1, conceive the aſymptote to be divided from I into innumerable equal parts, namely Ip = pq = qr = a; then by the 14th and 15th,

$$\left.\begin{array}{l} ps = 1 - a + a^2 - a^3 \text{ \&c} \\ qt = 1 - 2a + 4a^2 - 8a^3 \text{ \&c} \\ ru = 1 - 3a + 9a^2 - 27a^3 \text{ \&c} \end{array}\right\}$$ But the area BIru is = the ſum ps + qt + ru, which is =

$3 - 6a + 14a^2 - 36a^3$ &c, that is, equal to the number of terms contained in the line Ir, minus the ſum of thoſe terms, plus the ſum of the ſquares of the ſame, minus the ſum of their cubes, plus the ſum of the 4th powers, &c. Putting now IA = 1, as before, and Ip = 0·1 the number of terms, to find the area BIps; by prop. 16 the ſum of the terms will be $\frac{0 \cdot 1^2}{2} = \cdot 005$, the ſum of their ſquares = ·000,333,333, the ſum of their cubes ·000,025, the ſum of the 4th powers = ·000,002, the ſum of the 5th powers = ·000,000,166, the ſum of the 6th powers = ·000,000,014, &c. Therefore the area BIps is = 1 — ·005 + ·000,333,333 — ·000,025 + ·000,002 — ·000,000,166 + ·000,000,014 &c = ·100,335,347 — ·005,025,166 = ·095,310,181 &c.

Again, putting Iq = ·21 the number of terms, he finds in like manner the area BIqt = ·21 — ·022,05 + ·003,087 — ·000,486,202 + ·000,081,682 — ·000,014,294 + ·000,002,572 — ·000,000,472 + ·000,000,088 &c = ·213, 171,345, — ·022,550,984 = ·190,620,361, &c.

He then adds, hence it appears that as the ratio of AI to Ap, or 1 to 1·1, is half, or ſubduplicate, of the ratio of AI to Aq, or 1 to 1·21; ſo the area BIps is here found to be half of the area BIqt. Theſe areas he computes to 44 places of figures, and finds them ſtill in the ratio of 2 to 1.

The foregoing doctrine amounts to this, that if the rectangle BI × Ir, which in this caſe is expreſſed by Ir only, be put = A, AI being = 1 as before; then the area BIru, or the hyperbolic logarithm of $1 + A$, or of the ratio of 1 to $1 + A$ will be equal to the infinite ſeries $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \frac{1}{5}A^5$ &c; and which therefore may be conſidered as Mercator's quadrature of the hyperbola, or his general expreſſion of an hyperbolic logarithm in an infinite ſeries. And this method was farther improved by Dr. Wallis in the Philoſ. Tranſ. for the year 1668.

In *prop.* 18, our author compares the hyperbolic *areolæ* with the *ratiunculæ* of equidifferent numbers, and obſerves that

the areola BIps is the meaſure of the ratiuncula of AI to Ap,
the areola ſpqt is the meaſure of the ratiuncula of Ap to Aq,
the areola tqru is the meaſure of the ratiuncula of Aq to Ar, &c.

Finally, in the 19th *prop.* he ſhews how the ſums of logarithms may be taken

after the manner of the ſums of the *areolæ*. And from hence infers as a corollary, how the continual product of any given numbers in arithmetical progreſſion ma be obtained: for the ſum of the logarithms is the logarithm of the continual product. He then remarks that from the premiſes it appears in what manner Merſennus's problem may be reſolved, if not geometrically, at leaſt in figures to any number of places. And thus cloſes this ingenious tract.

In the Philoſ. Tranſ. for 1668 are alſo given ſome farther illuſtrations of this work by the author himſelf. And in various places alſo in a ſimilar manner are logarithms and hyperbolic areas treated of by Lord Brouncker, Dr. Wallis, Sir I. Newton, and many other learned perſons.

Of Gregory's Exercitationes Geometricæ.

In the ſame year 1668 came out Mr. James Gregory's *Exercitationes Geometricæ*, in which are contained,

1, Appendicula ad veram circuli et hyperbolæ quadraturam:

2, Nicolai Mercatoris quadratura hyperbolæ geometricè demonſtrata:

3, Analogia inter lineam meridianam planiſphærii nautici et tangentes artificiales, geometricè demonſtrata; ſeu quòd ſecantium naturalium additio efficiat tangentes artificiales:

4, Item quòd tangentium naturalium additio efficiat ſecantes artificiales:

5, Quadratura conchoidis:

6, Quadratura ciſſoidis: &

7, Methodus facilis et accurata componendi ſecantes et tangentes artificiales.

The firſt of theſe pieces, or the *Appendicula*, contains ſome farther extenſion and illuſtration of his *Vera circuli et hyperbolæ quadratura*, occaſioned by the animadverſions made on that work by the famous mathematician and philoſopher Huygens.

In the 2d is demonſtrated geometrically the quadrature of the hyperbola, by which he finds a ſeries ſimilar to Mercator's for the logarithm, or the hyperbolic ſpace beyond the firſt ordinate (BI, *fig. pa.* 94). In like manner he finds another ſeries for the ſpace at an equal diſtance within that ordinate. Theſe two ſeries having all their terms alike, but all the ſigns of the one being plus, and thoſe of the other being alternately plus and minus, it is evident that by adding the two together, every other term will be cancelled, and the double of the reſt will denote the ſum of both ſpaces. He then applies theſe properties to the logarithms; the concluſion from all which may be thus briefly expreſſed:

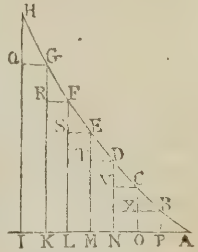
$$\text{ſince } A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 - \tfrac{1}{4}A^4 \text{ \&c.} = \text{the log. of } \frac{1+A}{1},$$

$$\text{and } A + \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 + \tfrac{1}{4}A^4 \text{ \&c.} = \text{the log. of } \frac{1}{1-A},$$

therefore $2A + \frac{2}{3}A^3 + \frac{2}{5}A^5 + \frac{2}{7}A^7$ &c. $=$ the log. of $\frac{1+A}{1-A}$, or of the ratio of $1 - A$ to $1 + A$. Which may be accounted Mr. James Gregory's method of making logarithms.

The remainder of this little volume is chiefly employed about the nautical meridian,

meridian, and the logarithmic tangents and ſecants. It does not appear by whom, nor by what accident, was diſcovered the analogy between a ſcale of logarithmic tangents and Wright's protraction of the nautical meridian line, which conſiſted of the ſums of the ſecants. It appears, however, to have been firſt publiſhed, and introduced into the practice of navigation, by Mr. Henry Bond, who mentions this property in an edition of Norwood's Epitome of Navigation, printed about 1645; and he again treats of it more fully in an edition of Gunter's works printed in 1653, where he teaches, from this property, to reſolve all the caſes of Mercator's ſailing by the logarithmic tangents, independent of the table of meridional parts. This analogy had only been found to be nearly true by trials, but not demonſtrated to be a mathematical property. Such demonſtration ſeems to have been firſt diſcovered by Mr. Nicholas Mercator, who, deſirous of making the moſt advantage of this and another concealed invention of his in navigation, invited the publick, by a paper in the Philoſophical Tranſactions for June 4, 1666, to enter into a wager with him on his ability to prove the truth or falſehood of the ſuppoſed analogy. This mercenary propoſal, however, ſeems not to have been taken up by any one, and Mercator reſerved his demonſtration. The propoſal, however, excited the attention of mathematicians to the ſubject itſelf, and a demonſtration was not long wanting. The firſt was publiſhed about two years after by Gregory in the tract now under conſideration, and from thence and other ſimilar properties here demonſtrated, he ſhews in the laſt article how the tables of logarithmic tangents and ſecants may eaſily be computed from the natural tangents and ſecants. The ſubſtance of which is as follows:



Let AI be the arc of a quadrant extended in a right line, and let the figure AHI be compoſed of the natural tangents of every arc from the point A erected perpendicular to AI at their reſpective points: let AP, PO, ON, NM, &c, be the very ſmall equal parts into which the quadrant is divided, namely, each $\frac{1}{60}$, or $\frac{1}{100}$, of a degree; and let PB, OC, ND, ME, &c be drawn perpendicular to AI, and equal to the tangents of the ſeveral arcs AP, AO, AN, AM, &c, reſpectively. Then it is manifeſt from what had been demonſtrated, that the figures ABP, ACO, &c, are the artificial ſecants of the arcs AP, AO, &c, putting o for the artificial radius. It is alſo manifeſt that the rectangles BO, CN, DM, &c, will be found from the multiplication of the ſmall part AP of the quadrant by each natural tangent. But, he proceeds, there is a little more difficulty in meaſuring the figures ABP, BCX, CDV, &c; for if the firſt differences of the tangents be equal, AB, BC, CD, &c, will not differ from right lines, and then the figures ABP, BCX, CDV, &c, will be right-angled triangles, and therefore any one, as HQG, will be $= \frac{1}{2}QH \times QG$: but if the ſecond differences be equal, the ſaid figures will be portions of trilineal quadratrices; for example HQG will be a portion of a trilineal quadratrix, whoſe axis is parallel to QH; and each of the laſt differences being Z, it will be $QHG = \frac{1}{2}QH \times QG - \frac{1}{12}Z \times QG$: and if the 3d differences be equal, the ſaid figures will be portions of trilineal cubices, and then ſhall QHG be equal $\frac{1}{2}QH \times QG - \sqrt{\frac{1}{72}QH \times Z \times QG^2 - \frac{1}{1728}Z^2 \times QG^2}$: when the 4th differences are equal, the ſaid figures are portions of trilineal quadrato-quadratrices, and the 4th differences are equal to 24 times the 4th power of QG

 divided

divided by the cube of the latus rectum; alſo when the 5th differences are equal, the ſaid figures are portions of trilineal ſurſolids, and the 5th differences are equal to 120 times the ſurſolid of QG divided by the 4th power of the latus rectum: and ſo on *in infinitum*. What has been here ſaid of the compoſition of artificial ſecants from the natural tangents, it is remarked, may in like manner be underſtood of the compoſition of artificial tangents from the natural ſecants, according to what was before demonſtrated. It is alſo obſerved that the artificial tangents and ſecants are computed, as above, on the ſuppoſition that 0 is the logarithm of 1, and 1,000,000,000,000,000 the radius, and 2,302,585,092,994,045,624,017,870 the logarithm of 10; but that they may be more eaſily computed, namely, by addition only, by putting $\frac{1}{60}$ of a degree $= QG = AP = 1$, and the logarithm of $10 = 7,915,704,467,897,819$; for by this means $\frac{1}{2}QH \times QG$ is $= \frac{1}{2}QH = QHG$, and $\frac{1}{2}QH \times QG - \frac{1}{12}Z \times QG = \frac{1}{2}QH - \frac{1}{12}Z = QHG$, alſo $\frac{1}{2}QH \times QG - \sqrt{\frac{1}{72}QH \times Z \times QG^2 - \frac{1}{1728}Z^2 \times QG^2} = \frac{1}{2}QH - \sqrt{\frac{1}{72}QH \times Z - \frac{1}{1728}Z^2} = QHG$: And finally by one diviſion only are found the artificial tangents and ſecants to 1,000,000,000,000,000 the logarithm of 10, putting ſtill 1 for radius, which are the differences of the artificial tangents and ſecants in the table from that artificial radius; and to make the operations eaſier in multiplying by the number 7,915,704,467,897,819, or logarithm of 10, a table is ſet down of its products by the firſt 9 figures. But if AP or QG be $= \frac{1}{100}$ of a degree, the artificial tangents and ſecants will anſwer to 13,192,840,779,829,703 as the logarithm of 10, whoſe firſt 9 multiples are alſo placed in the table. But to repreſent the numbers by the artificial radius rather than by the logarithm of 10, the author directs to add cyphers, &c.—And ſo much for Gregory's *Exercitationes Geometricæ*.

The ſame analogy between the logarithmic tangents and the meridian line, as alſo other ſimilar properties, were afterwards more elegantly demonſtrated by Dr. Halley in the Philoſophical Tranſactions for February, 1696, and various methods given for computing the ſame, by examining the nature of the ſpirals into which the rhumbs are transformed in the ſtereographical projection of the ſphere on the plane of the equator: the doctrine of which was rendered ſtill more eaſy and elegant by the ingenious Mr. Cotes in his *Logometria*, firſt printed in the Philoſophical Tranſactions for 1714, and afterwards in the collection of his works publiſhed in 1732, by his couſin Dr. Robert Smith, who ſucceeded him in the Plumian profeſſorſhip of philoſophy in the Univerſity of Cambridge.

The learned Dr. Iſaac Barrow alſo, in his *Lectiones Geometricæ*, *Lect.* XI. *Append.* firſt publiſhed in 1672, delivers a ſimilar property, namely, that the ſum of all the ſecants of any arc, is analogous to the logarithm of the ratio of $r + s$ to $r - s$, or radius plus ſine to radius minus ſine; or (which is the ſame thing) that the meridional parts anſwering to any degree of latitude, are as the logarithms of the ratios of the verſed ſines of the diſtances from the two poles.

Mr. Gregory's method for making logarithms was farther exemplified in numbers, in a ſmall tract on this ſubject, printed in 1688, by one Euclid Speidell, a ſimple and illiterate perſon, and ſon of John Speidell before mentioned among the firſt writers on logarithms.

Mr. Gregory alſo invented many other infinite ſeries, and among them theſe here following, viz. a being an arc, t its tangent, and s the ſecant, to the radius r; then is

$$a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} \text{ \&c.}$$
$$t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} \text{ \&c.}$$
$$s = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} \text{ \&c.}$$

And if τ and σ be the artificial, or logarithmic, tangent and ſecant of the ſame arc a, the whole quadrant being q, and $e = 2a - q$; then

$$e = \tau - \frac{\tau^3}{6r^2} + \frac{\tau^5}{24r^4} - \frac{61\tau^7}{5040r^6} + \frac{277\tau^9}{72576r^8} \text{ \&c.}$$
$$\tau = e + \frac{e^3}{6r^2} + \frac{e^5}{24r^4} + \frac{61e^7}{5040r^6} + \frac{277e^9}{72576r^8} \text{ \&c.}$$
$$\sigma = \frac{a^2}{2r} + \frac{a^4}{12r^3} + \frac{a^6}{45r^5} + \frac{17a^8}{2520r^7} + \frac{62a^{10}}{28350r^9} \text{ \&c.}$$

Alſo if f be the artificial ſecant of $45°$, and $f + l$ the artificial ſecant of any arc a, he artificial radius being 0; then is

$$a = \tfrac{1}{2}q + l - \frac{l^2}{r} + \frac{4l^3}{3r^2} - \frac{7l^4}{3r^3} + \frac{14l^5}{3r^4} - \frac{452l^6}{45r^5} \text{ \&c.}$$

The inveſtigation of all which ſeries may be ſeen at pages 298 *et ſeq.* vol. 1. of Dr. Horſley's learned and elegant commentary on Sir Iſaac Newton's works, they having been given without demonſtration, in the *Commercium Epiſtolicum*, No. xx. where the number 2 is alſo wanting in the denominator of the firſt term of the ſeries expreſſing the value of σ.

Such then were the ways in which Mercator and Gregory applied theſe their very ſimple ſeries $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4$ &c, and $A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4$ &c, for the purpoſe of computing logarithms. But they might, as I apprehend, have applied them to this purpoſe in a ſhorter and more direct manner, by computing, by their means, only a few logarithms of ſmall ratios, in which the terms of the ſeries would have decreaſed by the powers of 10 or ſome greater number, the numerators of all the terms being unity, and their denominators the powers of 10 or ſome greater number, and then employing theſe few logarithms, ſo computed, to the finding of the logarithms of other and greater ratios by the eaſy operations of mere addition and ſubtraction. This might have been done for the logarithms of the ratios of the firſt ten numbers, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, to 1, in the following manner, communicated by Mr. Baron Maſeres.— In the firſt place the logarithm of the ratio of 10 to 9, or of 1 to $\frac{9}{10}$, or of 1 to $1 - \frac{1}{10}$, is equal to the ſeries $\frac{1}{10} + \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} + \frac{1}{4 \times 10000} + \frac{1}{5 \times 100000}$ &c. In like manner are eaſily found the logarithms of the ratios of 11 to 10; and then, by the ſame ſeries, thoſe of 121 to 120, and of 81 to 80, and of 2401 to 2400; in all which caſes the ſeries would converge ſtill faſter than in the two firſt caſes. We may then proceed by mere addition and ſubtraction of logarithms, as follows.

Log. $\frac{11}{9} =$ L. $\frac{11}{10}$ + L. $\frac{10}{9}$,	L. $\frac{120}{80} =$ L. $\frac{3}{2}$,	L. $\frac{80}{16} =$ L. $\frac{81}{16}$ − L. $\frac{81}{80}$
L. $\frac{121}{81} =$ 2L. $\frac{11}{9}$	L. $\frac{9}{4} =$ 2L. $\frac{3}{2}$,	L. $\frac{5}{1} =$ L. $\frac{80}{16}$,
L. $\frac{121}{80} =$ L. $\frac{121}{81}$, + L. $\frac{81}{80}$,	L. $\frac{10}{4} =$ L. $\frac{10}{9}$ + L. $\frac{9}{4}$,	L. $\frac{5}{2} =$ L. $\frac{10}{4}$,
L. $\frac{120}{80} =$ L. $\frac{121}{80}$ − L. $\frac{121}{120}$,	L. $\frac{81}{16} =$ 2L. $\frac{9}{4}$	L. $\frac{2}{1} =$ L. $\frac{5}{1}$ − L $\frac{5}{2}$

Having

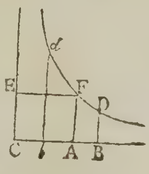
Having thus got the logarithm of the ratio of 2 to 1, or, in common language, the logarithm of 2, the logarithms of all ſorts of even numbers may be derived from thoſe of the odd numbers which are their coefficients with 2 or its powers. I would then proceed as follows.

L. $4 = 2$L. 2,	L. $100 = 2$L. 10,	L. $2401 =$ L. $\frac{2401}{2400} +$ L. 2400,
L. $10 =$ L. $\frac{10}{4} +$ L. 4,	L. $8 = 3$L. 2,	L. $7 = \frac{1}{4}$L. 2401,
L. $9 =$ L. $\frac{9}{4} +$ L. 4.	L. $24 =$ L. 8, $+$ L. 3,	L. $11 =$ L. $\frac{11}{9} +$ L. 9,
L. $3 = \frac{1}{2}$L. 9	L. $2400 =$ L. $100 +$ L. 24,	L. $6 =$ L. $2 +$ L. 3.

Thus we have got the logarithms of 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. And this is, upon the whole, perhaps the beſt method of computing logarithms that can be taken*. There have been indeed ſome methods diſcovered by Dr. Halley, and other mathematicians, for computing the logarithms of the ratios of prime numbers to the next adjacent even numbers, that are ſtill ſhorter than the application of the foregoing ſeries. But thoſe methods are leſs ſimple and eaſy to underſtand and apply than theſe ſeries; and the computation of logarithms by theſe ſeries, when the terms of them decreaſe by the powers of 10, or of ſome greater number, is ſo very ſhort and eaſy (as we have ſeen in the foregoing computations of the logarithms of the ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, &c,) that it is not worth while to ſeek for any ſhorter methods of computing them. And this method of computing logarithms is very nearly the ſame with that of Sir Iſaac Newton in his ſecond letter to Mr. Oldenburg, dated October 1676, as will be ſeen in the following article.

Of Sir Iſaac Newton's Methods.

The excellent Sir I. Newton greatly improved the quadrature of the hyperbolical aſymptotic ſpaces by infinite ſeries derived from the general quadrature of curves by his method of fluxions; or rather indeed he invented that method himſelf, and the conſtruction of logarithms derived from it, in the year 1665 or 1666, before the publication of either Mercator's or Gregory's books, as appears by his letter to Mr. Oldenburg, dated Oct. 24, 1676, printed in p. 634 *et ſeq.* vol. 3. of Wallis's works, and elſewhere. The quadrature of the hyperbola, thence tranſlated, is to this effect. Let dFD be an equilateral hyperbola, whoſe center is C, vertex F, and interpoſed ſquare CAFE $= 1$. In CA take AB and Ab, on each ſide of the point A, $= \frac{1}{10}$ or 0·1 : And, erecting the perpendiculars BD, bd ; half the ſum of the ſpaces AD and Ad will be $= 0.1 + \frac{0.001}{3} + \frac{0.000,01}{5} + \frac{0.000,000,1}{7}$ &c. and the



* This method of making logarithms is very copiouſly explained, and illuſtrated by examples, in one of the tracts contained in this volume, intitled, " *Remarks on the two infinite Serieſes* $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ *&c, and* $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ *&c, which were found by Mr. Nicholas Mercator and Dr. John Wallis, &c,*" in pages 235, 236, &c—344.

half

half-difference will be $= \frac{0.01}{2} + \frac{0.0001}{4} + \frac{0.000,001}{6} + \frac{0.000,000,01}{8}$ &c. Which reduced will ſtand thus,

1.000,000,000,000,0	0·005,000,000,000,0	The ſum of theſe, to wit, 0.105,360,515,657,7, is Ad ;
333,333,333,3	25,000,000,0	and the difference, to wit, 0.095,310,179,804,3, is AD.
2,000,000,0	166,666,6	In like manner putting AB and Ab
14,285,7	1,250,0	each = 0.2, there is obtained
111,1	10,0	Ad = 0.223,143,551,314,2, and
9	1	AD = 0.182,321,556,793, 9.
0·100,335,347,731,0	0·005,025,167,926,7	

Having thus the hyperbolic logarithms of the four decimal numbers 0.8, 0.9, 1.1, and 1.2 ; and ſince $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = 2$, and 0.8 and 0.9 are leſs than unity ; add their logarithms to double the logarithm of 1.2, and you will have 0.693,147, 180,559,7 for the hyperbolic logarithm of 2. To the triple of this add the logarithm of 0.8, becauſe $\frac{2 \times 2 \times 2}{0.8} = 10$, and you will have 2.302,585,092,993,3 the logarithm of 10. Hence by one addition are found the logarithms of 9 and 11 : And thus the logarithms of all the prime numbers 2, 3, 5, 11 are prepared. Moreover, by only depreſſing the numbers, above computed, lower in the decimal places, and adding, are obtained the logarithms of the decimals 0.98, 0.99, 1.01, 1.02 ; as alſo of theſe, 0.998, 0.999, 1.001, 1.002. And hence by addition and ſubtraction will ariſe the logarithms of the primes 7, 13, 17, 37, &c. All which logarithms being divided by the above logarithm of 10, give the common logarithms to be inſerted in the table.

And again a few pages farther on in the ſame letter he reſumes the conſtruction of the logarithms, thus : Having found, as above, the hyperbolic logarithms of 10, 0.98, 0.99, 1.01, 1.02, which may be effected in an hour or two, divide the laſt four logarithms by the logarithm of 10, and adding the index 2, you will have the tabular logarithms of 98, 99, 100, 101, 102. Then by interpolating nine means between each of theſe, will be obtained the logarithms of all numbers between 980 and 1020 ; and, again interpolating 9 means between every two numbers from 980 to 1000, the table will be ſo far conſtructed. Then from theſe will be collected the logarithms of all the primes under 100, together with thoſe of their multiples : all which will require only addition and ſubtraction ; for

$$\sqrt{\frac{9984 \times 1020}{9945}} = 2,\ \frac{10}{2} = 5,\ \sqrt{\frac{98}{2}} = 7,\ \frac{99}{9} = 11,\ \frac{1001}{7 \times 11} = 13,\ \frac{102}{6} = 17,\ \frac{988}{4 \times 13} = 19,\ \frac{9936}{16 \times 27} = 23,\ \frac{986}{2 \times 17} = 29,\ \frac{992}{32} = 31,\ \frac{999}{27} = 37,\ \frac{984}{24} = 41,\ \frac{989}{23} = 43,\ \frac{987}{27} = 47,\ \frac{9911}{11 \times 17} = 53,\ \frac{9971}{13 \times 13} = 59,\ \frac{9882}{2 \times 81} = 61,\ \frac{9849}{3 \times 49} = 67,\ \frac{994}{14} = 71,\ \frac{9928}{8 \times 17} = 73,\ \frac{9954}{7 \times 18} = 79,\ \frac{996}{12} = 83,\ \frac{9968}{7 \times 16} = 89,\ \frac{9894}{6 \times 17} = 97.$$

This quadrature of the hyperbola, and its application to the conſtruction of logarithms, are ſtill farther explained by our celebrated author in his treatiſe on Fluxions, publiſhed by Mr. Colſon, in 1736, where he gives all the three ſeries for

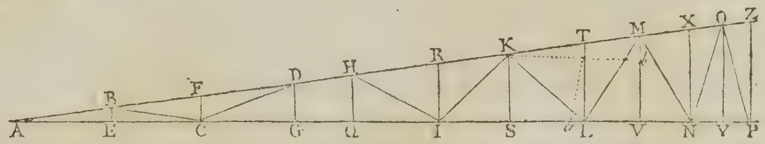
for the areas AD, Ad, Bd, in general terms, the two former of which are the ſame as thoſe publiſhed by Mercator and Wallis, and the laſt is the ſame as that publiſhed by Gregory; and he explains the manner of deriving the laſt ſeries from the two former, namely, by uniting together the two ſeries for the ſpaces on each ſide of an ordinate, bounded by other ordinates at equal diſtances, by which means every 2d term of each ſeries is cancelled, and the reſult is a ſeries converging much quicker than either of the former. And, in this treatiſe on fluxions, as well as in the letter before quoted, he recommends this as the moſt commodious method of conſtructing a canon of logarithms, computing by the ſeries the hyperbolic ſpaces anſwering to the prime numbers 2, 3, 5, 7, 11, &c, and dividing them by 2.302,585,092,994,045,7, which is the area correſponding to the number 10, or elſe multiplying them by its reciprocal 0.434,294, 481,903, 251,8, for the common, or Briggs's, logarithms. "Then the logarithms of all the numbers in the canon which are made by the multiplication of theſe, are to be found by the addition of their logarithms, as is uſual. And the void places are to be interpolated afterwards by the help of this theorem: Let n be a number to which a logarithm is to be adapted, x the difference between that and the two neareſt numbers equally diſtant on each ſide, whoſe logarithms are already found, and let d be half the difference of the logarithms; then the required logarithm of the number n will be obtained by adding $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ &c, to the logarithm of the leſs number." This theorem he demonſtrates by the hyperbolic areas, and then proceeds thus; "The two firſt terms $d + \frac{dx}{2n}$ of this ſeries I think to be accurate enough for the conſtruction of a canon of logarithms, even though they were to be produced to 14 or 15 figures; provided the number whoſe logarithm is to be found be not leſs than 1000. And this can give little trouble in the calculation, becauſe x is generally an unit, or the number 2. Yet it is not neceſſary to interpolate all the places by the help of this rule. For the logarithms of numbers which are produced by the multiplication or diviſion of the number laſt found, may be obtained by the numbers whoſe logarithms were had before, by the addition or ſubtraction of their logarithms. Moreover by the differences of the logarithms, and by their 2d and 3d differences, if there be occaſion, the void places may be more expeditiouſly ſupplied; the foregoing rule being to be applied only when the continuation of ſome full places is wanted, in order to obtain thoſe differences, &c." So that Sir Iſaac Newton of himſelf diſcovered all the ſeries for the above quadrature which were found out, and afterwards publiſhed, partly by Mercator and Dr. Wallis, and partly by Gregory; and theſe we may here exhibit in one view all together, and that in a general manner for any hyperbola; namely, putting CA $= a$, AF $= b$, and AB $=$ Ab $= x$; then will BD $= \frac{ab}{a+x}$, and bd $= \frac{ab}{a-x}$; whence the area

$$\text{AD will be} = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \text{ \&c.}$$

$$\text{and the area Ad will be} = bx + \frac{bx^2}{2a} + \frac{bx^3}{3a^2} + \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \text{ \&c.}$$

$$\text{and the area Bd will be} = 2bx + \frac{2bx^3}{3a^2} + \frac{2bx5}{5^4} + \frac{2bx^7}{7a^6} + \frac{2bx^9}{9a^8} \text{ \&c.}$$

In the ſame letter alſo, above quoted, to Mr. Oldenburg, our illuſtrious author teaches a method of conſtructing the trigonometrical canon of ſines by an eaſier method of multiple angles than that before delivered by Briggs for the ſame purpoſe, becauſe that in Sir Iſaac's way the radius, or 1, is the firſt term, and double the ſine or coſine of the firſt given angle is the ſecond term of all the proportions by which the ſeveral ſucceſſive multiple ſines or coſines are found. The ſubſtance of this method is thus. The beſt foundation for the conſtruction of the table of ſines, is the continual addition of a given angle to itſelf or to another given angle. As if the angle A be to



be added; inſcribe HI, IK, KL, LM, MN, NO, OP, &c, each equal to the radius AB; and to the oppoſite ſides draw the perpendiculars BE, HQ, IR, KS, LT, MV, NX, OY, &c; ſo ſhall the angle A be the common difference of the angles HIQ, IKH, KLI, LMK, &c; their ſines HQ, IR, KS, &c; and their coſines IQ, KR, LS, &c. Now let any one of them LMK, be given, and the reſt will be thus found: Draw Ta and Kb perpendicular to SV and MV; then becauſe of the equiangular triangles ABE, TLa, KMb, ALT, AMV, &c, it wil be AB : AE : : KT : Sa ($= \frac{1}{2}$LV + $\frac{1}{2}$LS) : : LT : Ta ($= \frac{1}{2}$MV + $\frac{1}{2}$KS), and AB : BE : : LT : La ($= \frac{1}{2}$LS $-$ $\frac{1}{2}$LV) : : KT ($= \frac{1}{2}$KM) : $\frac{1}{2}$Mb ($= \frac{1}{2}$MV $-$ $\frac{1}{2}$KS). Hence are given the ſines and coſines KS, MV, LS, LV. And the method of continuing the progreſſions is evident. Namely,

$$\text{AB} : 2\,\text{AE} : : \left\{ \begin{array}{l} \text{LV} : \text{MT} + \text{MX} : : \text{MX} : \text{NV} + \text{NY}, \&\text{c}. \\ \text{MV} : \text{NX} + \text{LT} : : \text{NX} : \text{OY} + \text{MV}, \&\text{c}. \end{array} \right.$$

$$\text{or AB} : 2\,\text{BE} : : \left\{ \begin{array}{l} \text{LV} : \text{NX} - \text{LT} : : \text{MX} : \text{OY} - \text{MV}, \&\text{c}. \\ \text{MV} : \text{MT} - \text{MX} : : \text{NX} : \text{NV} - \text{NY}, \&\text{c}. \end{array} \right.$$

And on the other hand, AB : : 2 AE : : LS : KT + KR &c. Therefore put AB = 1, and make BE × LT = La, AE × KT = Sa, Sa − La = LV, 2AE × LV − TM = MX, &c.

The ſenſe of theſe general theorems is this, that if P be any one among a ſeries of angles in arithmetical progreſſion, the angle d being their common difference, then as radius or

$$\left. \begin{array}{l} 1 : 2\,\text{coſ.}\,d : : \left\{ \begin{array}{l} \text{coſ.}\,P : \text{coſ.}\,P+d + \text{coſ.}\,P-d \\ \text{ſin.}\,P : \text{ſin.}\,P+d + \text{ſin.}\,P-d \end{array} \right. \\ 1 : 2\,\text{ſin.}\,d : : \left\{ \begin{array}{l} \text{coſ.}\,P : \text{ſin.}\,P+d - \text{ſin.}\,P-d \\ \text{ſin.}\,P : \text{coſ.}\,P+d - \text{coſ.}\,P-d \end{array} \right. \end{array} \right\}$$

where the fourth terms of theſe proportions are the ſums or differences of the ſines or coſines of the two angles next leſs and greater than any angle P in the ſeries; and therefore, ſubtracting the leſs extreme from the ſum, or adding it to the difference, the reſult will be the greater extreme, or next ſine or coſine beyond that of

of the term *P*. And in the ſame manner are all the reſt to be found. This method it is evident, is equally applicable whether the common difference d, or angle A, be equal to one term of the ſeries or not; when it is one of the terms, then the whole ſeries of ſines and coſines becomes thus; As 1 : 2 coſ. d : :

fin.d : ſin. $2d$: : ſin. $2d$: ſin. d + ſin. $3d$: : ſin. $3d$: ſin. $2d$ + ſin. $4d$: : ſin. $4d$: ſin. 3d + ſin. $5d$ &c,
coſ.d : 1 + coſ. $2d$: : coſ. $2d$: coſ. d + coſ. $3d$: : coſ. $3d$: coſ. $2d$ + coſ. $4d$: : coſ. $4d$: coſ. 3d + coſ. $5d$ &c,

which is the very method contained in the directions given by Mr. Abraham Sharp for conſtructing the canon of ſines.

Sir Iſaac Newton remarks that it only remains to find the ſine and coſine of a firſt angle A by ſome other method, and for this purpoſe he directs us to make uſe of ſome of his own infinite ſeries: thus by them will be found 1.57079, &c, for the quadrantal arc, the ſquare of which is 2.4694 &c; divide this ſquare by the ſquare of the number expreſſing the ratio of 90 degrees to the angle A calling the quotient z; then three or four terms of this ſeries $1 - \frac{z}{2} + \frac{z^2}{24} - \frac{z^3}{720} + \frac{z^4}{40320}$ &c, will give the coſine of that angle A. Thus we may firſt find an angle of 5 degrees, and thence the table computed to the ſeries of every 5 degrees; then theſe interpolated to degrees or half degrees by the ſame method; and theſe interpolated again; and ſo on as far as neceſſary. But two-thirds of the table being computed in this manner, the remaining third will be found by addition or ſubſtraction only, as is well known.

Various other improvements in logarithms and trigonometry are owing to the ſame excellent perſonage; ſuch as the ſeries for expreſſing the relation between circular arcs and their ſines, coſines, verſed ſines, tangents, &c; namely, the arc being a, the ſine s, the verſed-ſine v, coſine c, tangent t, radius 1, then is

$$a = s + \tfrac{1}{6}s^3 + \tfrac{3}{40}s^5 + \tfrac{5}{112}s^7 + \tfrac{35}{1152}s^9 + \tfrac{63}{2816}s^{11} \text{ \&c.}$$
$$a = v^{\frac{1}{2}} + \tfrac{1}{6}v^{\frac{3}{2}} + \tfrac{3}{40}v^{\frac{5}{2}} + \tfrac{5}{112}v^{\frac{7}{2}} + \tfrac{35}{1152}v^{\frac{9}{2}} + \tfrac{63}{2816}v^{\frac{11}{2}} \text{ \&c.}$$
$$a = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \tfrac{1}{9}t^9 - \tfrac{1}{11}t^{11} \text{ \&c.}$$
$$s = a - \tfrac{1}{6}a^3 + \tfrac{1}{120}a^5 - \tfrac{1}{5040}a^7 + \tfrac{1}{362,880}a^9 - \tfrac{1}{39,916,800}a^{11} \text{ \&c.}$$
$$c = 1 - \tfrac{1}{2}a^2 + \tfrac{1}{24}a^4 - \tfrac{1}{720}a^6 + \tfrac{1}{40,320}a^8 - \tfrac{1}{3,628,800}a^{10} \text{ \&c.}$$
$$v = \tfrac{1}{2}a^2 - \tfrac{1}{24}a^4 + \tfrac{1}{720}a^6 - \tfrac{1}{40,320}a^8 + \tfrac{1}{3,628,800}a^{10} - \tfrac{1}{479,001,600}a^{12} \text{ \&c.}$$
$$t = a + \tfrac{1}{3}a^3 + \tfrac{2}{15}a^5 + \tfrac{17}{315}a^7 + \tfrac{62}{2835}a^9 + \tfrac{1302}{155,925}a^{11} \text{ \&c.}$$

Of Dr. Halley's Method.

Many other improvements in the conſtruction of logarithms are alſo derived from the ſame doctrine of fluxions, as we ſhall ſhew hereafter. In the mean time proceed we to the ingenious method of the learned Dr. Edmund Halley, Secretary to the Royal Society, and the ſecond Aſtronomer Royal, having ſucceeded Mr. Flamſteed in that honourable office in the year 1719 at the Royal Obſervatory at Greenwich, where he died the 14th of January 1742, in the 86th year of his age. His method was firſt printed in the Philoſophical Tranſactions, for the year 1695, and it is entitled, "A moſt compendious and facile method for con-

ſtructing

ſtructing the logarithms, exemplified and demonſtrated from the nature of numbers, without any regard to the hyperbola, with a ſpeedy method for finding the number from the given logarithm."

Inſtead of the more ordinary definition of logarithms, *numerorum proportionalium æquidifferentes comites*, in this tract our learned author adopts this other, *numeri rationum exponentes*, as being better adapted to the principle on which logarithms are here conſtructed, where thoſe quantities are not conſidered as the logarithms of the numbers, for example, of 2, or of 3, or of 10, but as the logarithms of the ratios of 1 to 2, or of 1 to 3, or of 1 to 10. In this conſideration he firſt purſues the idea of Kepler and Mercator, remarking that any ſuch ratio is proportional to, and is meaſured by, the number of equal *ratiunculæ* contained in each; which *ratiunculæ* are to be underſtood as in a continued ſcale of proportionals, infinite in number, between the two terms of the ratio; which infinite number of mean proportionals is to that infinite number of the like and equal *ratiunculæ* between any other two terms, as the logarithm of the one ratio is to the logarithm of the other: thus, if there be ſuppoſed between 1 and 10 an infinite ſcale of mean proportionals, whoſe number is 100000 &c *in infinitum*; then between 1 and 2 there will be 30102 &c of ſuch proportionals; and between 1 and 3 there will be 47712 &c of them; which numbers therefore are the logarithms of the ratios of 1 to 10, 1 to 2, and 1 to 3. But for the ſake of *his* mode of conſtructing logarithms, he changes this idea of *equal* ratiunculæ for that of other ratiunculæ, ſo conſtituted as that the *ſame* infinite number of them ſhall be contained in the ratio of 1 to every other number whatever; and that therefore theſe latter ratiunculæ will be of *unequal* or different magnitudes in all the different ratios, and in ſuch ſort that, in any one ratio, the *magnitude* of each of the ratiunculæ in this latter caſe, will be as the *number* of them in the former. And therefore, if between 1 and any number propoſed, there be taken any infinity of mean proportionals, the infinitely ſmall ratio of the firſt term 1 to the ſaid firſt term, together with the infinitely ſmall augment or decrement of the firſt of thoſe means from the firſt term 1, will be a *ratiuncula* of the ratio of 1 to the ſaid number; and as the number of all the *ratiunculæ* in theſe continued proportionals is the ſame, their ſum, or the whole ratio, will be directly proportional to the magnitude of one of the ſaid *ratiunculæ* in each ratio. But it is alſo evident that the firſt of any number of means between 1 and any number, is always equal to ſuch root of that number whoſe index is expreſſed by the number of thoſe proportionals from 1; ſo if m denote the number of proportionals from 1, then the firſt term after 1 will be the mth root of that number. Hence the indefinite root of any number being extracted, the *differentiola* of the ſaid root from unity, ſhall be as the logarithm of that number. So if there be required the logarithm of the ratio of 1 to $1 + q$; the firſt term after 1 will be $\overline{1 + q}|^{\frac{1}{m}}$, and therefore the required logarithm will be as $\overline{1 + q}|^{\frac{1}{m}} - 1$. But by the binomial theorem, $\overline{1 + q}|^{\frac{1}{m}}$ is $= 1 + \frac{1}{m} q + \frac{1}{m} \cdot \frac{1 - m}{2m} q^2 + \frac{1}{m} \cdot \frac{1 - m}{2m} \cdot \frac{1 - 2m}{3m} q^3$ &c; or, by omitting the 1 in the compound numerators, as infinitely ſmall in reſpect of the infinite number m, the ſame ſeries

will become $1 + \frac{1}{m} q + \frac{1}{m} . \frac{-m}{2m} q^2 + \frac{1}{m} . \frac{-m}{2m} . \frac{-2m}{3m} q^3$ &c, or, by abbreviation, it is $1 + \frac{1}{m} q - \frac{1}{2m} q^2 + \frac{1}{3m} q^3 - \frac{1}{4m} q^4$ &c; and hence, finding the differentiola by ſubtracting 1, the logarithm of the ratio of 1 to $1 + q$ will be as $\frac{1}{m}$ into $q - \frac{1}{2} q^2 + \frac{1}{3} q^3 - \frac{1}{4} q^4 + \frac{1}{5} q^5 - \frac{1}{6} q^6$ &c. Now the index m may be taken equal to any infinite number, and thus all the varieties of ſcales of logarithms may be produced: ſo, if m be taken 1000000 &c, the theorem will give Napier's logarithms; but if m be taken equal to 2302585 &c, there will ariſe Briggs's logarithms.

This theorem being for the increaſing ratio of 1 to $1 + q$; if that for the decreaſing ratio of 1 to $1 - q$ be alſo ſought, it will be obtained by a proper change of the ſigns, by which the decrement of the firſt of the infinite number of proportionals will be found to be $\frac{1}{m}$ into $q + \frac{1}{2} q^2 + \frac{1}{3} q^3 + \frac{1}{4} q^4$ &c, which therefore is as the logarithm of the ratio of 1 to $1 - q$.

Hence the terms of any ratio being a and b, q becomes $\frac{b-a}{a}$, or the difference divided by the leſs term, when it is an increaſing ratio; or $q = \frac{b-a}{b}$ when the ratio is decreaſing, or as b *to* a. Wherefore the logarithm of the ſame ratio may be doubly expreſſed; for putting x for the difference $b - a$ of the terms, it will be

$$\text{either } \frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c,}$$
$$\text{or } \frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} \text{ \&c.}$$

But if the ratio of a to b be ſuppoſed divided into two parts, namely, into the ratio of a to $\frac{1}{2}a + \frac{1}{2}b$ or $\frac{1}{2}z$, and the ratio of $\frac{1}{2}z$ to b, then will the ſum of the logarithms of thoſe two ratios be the logarithm of the ratio of a to b. Now by ſubſtituting in the foregoing ſeries, the logarithms of thoſe two ratios will be

$$\frac{1}{m} \text{ into } \frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \text{ \&c.}$$
$$\text{and } \frac{1}{m} \text{ into } \frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \text{ \&c; and ſo the ſum}$$
$$\frac{1}{m} \text{ into } \frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} \text{ \&c will be the log. of the ratio of } a \text{ to } b.$$

Moreover, if from the logarithm of the ratio of a to $\frac{1}{2}z$ be taken that of $\frac{1}{2}z$ to b, we ſhall have the logarithm of the ratio of ab to $\frac{1}{4}z^2$; and the half of this gives that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical mean. And conſequently the logarithm of this ratio will be equal to half the difference of that of the above two ratios, and will therefore be $\frac{1}{m}$ into $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8}$ &c.

The above ſeries are ſimilar to ſome that were before given by Newton and Gregory

Gregory for the ſame purpoſe, deduced from the conſideration of the hyperbola. But the rule which is properly our author's own, is that which follows, and is derived from the ſeries above given for the logarithm of the ſum of two ratios. For the ratio of ab to $\frac{1}{4}z^2$ or $\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2$, having the difference of its terms $\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2$ or $\overline{\tfrac{1}{2}b - \tfrac{1}{2}a}|^2$ or $\frac{1}{4}x^2$, which in the caſe of finding the logarithms of prime numbers is always 1, if we put the ſum of the terms $\frac{1}{4}z^2 + ab = y^2$, the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$, or $\frac{1}{2}z$, will be found to be

$$\frac{1}{m} \text{ into } \frac{1}{y^2} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} \text{ \&c.}$$

And theſe rules our learned author exemplifies by ſome caſes in numbers, to ſhew the eaſieſt mode of application in practice.

Again by means of the ſame binomial theorem he reſolves with equal facility the reverſe of the problem, namely, from the logarithm given, to find its number or ratio: For as the logarithm of the ratio of 1 to $1 + q$ was proved to be $\overline{1+q}|^{\frac{1}{m}} - 1$, and that of the ratio of 1 to $1 - q$ to be . . . $1 - \overline{1-q}|^{\frac{1}{m}}$; hence calling the given logarithm L, in the former caſe

it will be $\overline{1 + q}|^{\frac{1}{m}} = 1 + L$,

and in the latter $\overline{1 - q}|^{\frac{1}{m}} = 1 - L$;

$$\left.\begin{array}{ll} \text{and therefore } 1 + q & = \overline{1 + L}|^m \\ \text{and } 1 - q & = \overline{1 - L}|^m \end{array}\right\}, \text{ that is by the binomial theorem}$$

$$1 + q = 1 + mL + \tfrac{1}{2}m^2L^2 + \tfrac{1}{6}m^3L^3 + \tfrac{1}{24}m^4L^4 + \tfrac{1}{120}m^5L^5 \text{ \&c,}$$
$$\text{and } 1 - q = 1 - mL + \tfrac{1}{2}m^2L^2 - \tfrac{1}{6}m^3L^3 + \tfrac{1}{24}m^4L^4 - \tfrac{1}{120}m^5L^5 \text{ \&c.}$$

m being any infinite index whatever, differing according to the ſcale of logarithms, being 1000,000 &c in Napier's or the hyperbolic logarithms, and 2302,585 &c. in Briggs's.

If one term of the ratio, of which L is the logarithm, be given, the other term will be eaſily obtained by the ſame rule. For, if L be Napier's logarithm of the ratio of a, the leſs term, to b, the greater, then, according as a or b is given, we ſhall have

$$b = a \text{ into } 1 + L + \tfrac{1}{2}L^2 + \tfrac{1}{6}L^3 + \tfrac{1}{24}L^4 \text{ \&c,}$$
$$\text{or } a = b \text{ into } 1 - L + \tfrac{1}{2}L^2 - \tfrac{1}{6}L^3 + \tfrac{1}{24}L^4 \text{ \&c.}$$

Whence, by the help of the logarithms contained in the tables, may eaſily be found the number to any given logarithm to a great extent; For if the ſmall difference between the given logarithm L and the neareſt tabular logarithm, either greater or leſs, be called l, and the number anſwering to the tabular logarithm a when it is leſs than the given logarithm, but b when greater; it will follow that the number anſwering to the logarithm L will be either

$$a \text{ into } 1 + l + \tfrac{1}{2}l^2 + \tfrac{1}{6}l^3 + \tfrac{1}{24}l^4 + \tfrac{1}{120}l^5 \text{ \&c,}$$
$$\text{or } b \text{ into } 1 - l + \tfrac{1}{2}l^2 - \tfrac{1}{6}l^3 + \tfrac{1}{24}l^4 - \tfrac{1}{120}l^5 \text{ \&c.}$$

 which

which ſeries converge ſo quick, l being always very ſmall, that the firſt two terms $1 \pm l$ are generally ſufficient to find the number to 10 places of figures.

Dr. Halley ſubjoins alſo an eaſy approximation for theſe ſeries, by which it appears that the number anſwering to the log. is nearly

$$\frac{1+\frac{1}{2}l}{1-\frac{1}{2}l} \times a, \text{ or } \frac{1-\frac{1}{2}l}{1+\frac{1}{2}l} \times b, \left\{\begin{matrix}\text{in Napier's}\\ \text{logs.}\end{matrix}\right. \text{ and } \left.\right\} \frac{n+\frac{1}{2}l}{n-\frac{1}{2}l} \times a, \text{ or } \frac{n-\frac{1}{2}l}{n+\frac{1}{2}l} \times b, \left\{\begin{matrix}\text{in Briggs's}\\ \text{logarithms;}\end{matrix}\right.$$

where n is $= 0.434,294,481,903$, &c $= \frac{1}{m}$.

Of Mr. Sharp's Methods.

The labours of Mr. Abraham Sharp of Little-Horton, near Bradford in Yorkſhire, in this branch of mathematics, were very great and meritorious. His merit, however, conſiſted rather in the improvement and illuſtration of the methods of former writers, than in the invention of any new ones of his own. In this way he greatly extended and improved Dr. Halley's method above-deſcribed, as alſo thoſe of Mercator and Wallis; illuſtrating theſe improvements by extenſive calculations, and by them computing our table 5, conſiſting of the logarithms of all numbers to 100, and of all prime numbers to 1100, each to 61 places. He alſo compoſed a neat compendium of the beſt methods for computing the natural ſines, tangents, and ſecants, chiefly from the rules before given by Newton; and by Newton's, or Gregory's, ſeries $a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7$ &c, for the arc in terms of the tangent, he computed the circumference of the circle to 72 places, namely, from the arc of 30 degrees, whoſe tangent t is $= \sqrt{\frac{1}{3}}$ to the radius 1. Other aſtoniſhing inſtances of his induſtry and labour appear in his *Geometry Improved*, printed in 1717, and ſigned *A. S. Philomath*, from whence the ſaid table of logarithms was extracted. This ingenious man was ſome time aſſiſtant at the Royal Obſervatory to Mr. Flamſteed, the firſt Aſtronomer Royal; and being one of the moſt accurate and indefatigable computers that ever exiſted, he was for many years the common reſource for Mr. Flamſteed, Sir Jonas Moore, Dr. Halley, &c. in all intricate and troubleſome calculations. He afterwards retired to his native place at Little-Horton, where, after a life ſpent in intenſe ſtudy and calculations, he died the 18th of July 1742, in the 91ſt year of his age.

Of the Conſtruction of Logarithms by Fluxions.

It appears by the very definition and deſcription given by Napier of his logarithms, as ſtated in page 46 of this introduction, that the fluxion of his, or the hyperbolic logarithm, of any number, is a fourth proportional to that number, its logarithm, and unity; or which is the ſame, that it is equal to the fluxion of the number divided by the number: For the deſcription ſhews that $z1 : za$ or $1 : : \dot{z}1$ the fluxion of $z1 : \dot{z}a$, which therefore is $= \frac{\dot{z}1}{z1}$; but $\dot{z}a$ is alſo equal to

 the

the fluxion of the logarithm A &c by the defcription; therefore the fluxion of the logarithm is equal to $\frac{\dot{z1}}{z1}$, the fluxion of the quantity divided by the quantity itfelf. The fame thing appears again at art. 2 of that little piece in the appendix to his *Conftructio Logarithmorum*, intituled *Habitudines Logarithmorum & fucrum naturalium numerorum invicem*, where he obferves that, as any greater quantity is to a lefs, fo is the velocity of the increment or decrement of the logarithms at the place of the lefs quantity, to that at the greater. Now this velocity of the increment or decrement of the logarithms being the fame thing as their fluxions, that proportion is this, $x : a : :$ flux. log. a : flux. log. x; hence if a be $= 1$, as at the beginning of the table of numbers, where the fluxion of the logs. is the index or characteriftic c, which is alfo 1 in Napier's, or the hyperbolic, logarithms, and 43429 &c, in Briggs's, the fame proportion becomes $x : 1 : : c$: flux. log. x; but the conftant fluxion of the numbers is alfo 1, and therefore that proportion is alfo this, $x : \dot{x} : : c : \frac{c\dot{x}}{x} =$ the fluxion of the logarithm of x; and in the hyperbolic logarithms, where c is $= 1$, it becomes $\frac{\dot{x}}{x} =$ the fluxion of Napier's, or the hyperbolic, logarithm of x. This fame property has alfo been noticed by many other authors fince Napier's time. And the fame, or a fimilar, property is evidently true in all the fyftems of logarithms whatever, namely, that the modulus of the fyftem is to any number as the fluxion of its logarithm is to the fluxion of the number.

Now from this property, by means of the doctrine of fluxions, are derived other ways for making logarithms, which have been illuftrated by many writers on this branch, as Craig, Jo. Bernoulli, and almoft all the writers on fluxions. And this method chiefly confifts in expanding the reciprocal of the given quantity in an infinite feries, then multiplying each term by the fluxion of the faid quantity, and laftly taking the fluents of the terms; by which there arifes an infinite feries of terms for the logarithm fought. So, to find the logarithm of any number N; put any compound quantity for N, as fuppofe $\frac{n+x}{n}$;

then the fluxion of the logarithm, or $\frac{\dot{N}}{N}$, being $\frac{\dot{x}}{n+x} = \frac{\dot{x}}{n} - \frac{x\dot{x}}{nn} + \frac{x^2\dot{x}}{n^3} - \frac{x^3\dot{x}}{n^4}$ &c,

the fluents give log. of N, or log. of $\frac{n+x}{n}, = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ &c.

And writing $-x$ for x gives log. $\frac{n-x}{n} = -\frac{x}{n} - \frac{x^2}{2n^2} - \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ &c.

Alfo becaufe $\frac{n}{n\pm x} = 1 \div \frac{n\pm x}{n}$, or log. $\frac{n}{n\pm x} = 0 -$ log. $\frac{n\mp x}{n}$

we have log. $\frac{n}{n+x} = -\frac{x}{n} + \frac{x^2}{2n^2} - \frac{x^3}{3n^3} + \frac{x^4}{4n^4}$ &c,

and log. $\frac{n}{n-x} = +\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4}$ &c.

And

And by adding and ſubtracting any of theſe ſeries to or from one another, and multiplying or dividing their correſponding numbers, various other ſeries for logarithms may be found, converging much quicker than theſe do.

In like manner by aſſuming quantities otherwiſe compounded for the value of N, various other forms of logarithmic ſeries may be found by the ſame means.

Of Mr. Cotes's Logometria.

Mr. Roger Cotes was elected the firſt Plumian profeſſor of aſtronomy and experimental philoſophy in the univerſity of Cambridge, January 1706, which appointment he filled with the greateſt credit, till he died the 5th of June 1716, in the prime of life, having not quite compleated the 34th year of his age. His early death was a great loſs to the mathematical world, as his genius and abilities were of the brighteſt order, as is manifeſt by the ſpecimens of his performance given to the public. Among theſe are his *Logometria*, firſt printed in Number 338 of the Philoſophical Tranſactions, and afterwards in his *Harmonia Menſurarum*, publiſhed in 1722 with his other works, by his couſin, and ſucceſſor in the Plumian profeſſorſhip, Dr. Robert Smith. In this piece he firſt treats in a general way of meaſures of ratios, which meaſures, he obſerves, are quantities of any kind whoſe magnitudes are analogous to the magnitudes of the ratios, theſe magnitudes mutually increaſing and decreaſing together in the ſame proportion. He remarks that the ratio of equality has no magnitude, becauſe it produces no change by adding and ſubtracting; that the ratios of greater and leſs inequality, are of different affections; and therefore, if the meaſure of the one of theſe be conſidered as poſitive, that of the other will be negative; and the meaſure of the ratio of equality nothing; that there are endleſs ſyſtems of theſe, which have all their meaſures of the ſame ratios proportional to certain given quantities, called *moduli*, which he defines afterwards; and the ratio of which they are the meaſures, each in its peculiar ſyſtem, is called the modular ratio, *ratio modularis*, which ratio is the ſame in all ſyſtems. He then adverts to logarithms, which he conſiders as the *numerical* meaſures of ratios; and he deſcribes the method of arranging them in tables, with the uſe of them in multiplication and diviſion, raiſing of powers and extracting of roots, by means of the correſponding operations of addition and ſubtraction, multiplication and diviſion.

After this introduction, which is only a ſlight abridgment of the doctrine long before very amply treated of by others, and particularly by Kepler and Mercator, we arrive at the firſt propoſition, which has juſtly been cenſured as obſcure and imperfect, ſeemingly through an affectation of brevity, intricacy, and originality without ſufficient room for a diſplay of this qualification. The reaſoning in this propoſition, ſuch as it is, ſeems to be ſomething between that of Kepler and the principles of fluxions, to which the quantities and expreſſions are nearly allied. However, as it is my duty rather to narrate than explain, I ſhall here exhibit it exactly as it ſtands. This propoſition is "to determine the meaſure of any ratio," as, for inſtance, that of AC to AB, and which is effected in this manner. Conceive the

the difference BC to be divided into innumerable very ſmall particles as PQ, and the ratio between AC and AB into as many ſuch very ſmall ratios, as between AQ and AP. Then, if the magnitude of the ratio between AQ and AP be given, by dividing there will alſo be given that of PQ to AP; and therefore, this being given, the magnitude of the ratio between AQ and AP may be expounded by the given quantity $\frac{PQ}{AP}$; for, AP remaining conſtant, conceive the particle PQ to be augmented or diminiſhed in any proportion, and in the ſame proportion will the magnitude of the ratio between AQ and AP be augmented or diminiſhed: Alſo, taking any determinate quantity M, the ſame may be expounded by $M \times \frac{PQ}{AP}$; and therefore the quantity $M \times \frac{PQ}{AP}$ will be the meaſure of the ratio betweeen AQ and AP. And this meaſure will have divers magnitudes, and be accommodated to divers ſyſtems, according to the divers magnitudes of the aſſumed quantity M, which therefore is called the *modulus* of the ſyſtem. Now like as the ſum of all the ratios AQ to AP is equal to the propoſed ratio AC to AB, ſo the ſum of all the meaſures $M \times \frac{PQ}{AP}$ (found by the known methods) will be equal to the required meaſure of the ſaid propoſed ratio.

The general ſolution being thus diſpatched, from the general expreſſion he next deduces other forms of the meaſure in ſeveral corollaries and ſcholia; as 1ſt, The terms AP, AQ, approach the nearer to equality as the ſmall difference PQ is leſs; ſo that either $M \times \frac{PQ}{AP}$ or $M \times \frac{PQ}{AQ}$ will be the meaſure of the ratio between AQ and AP to the modulus M. 2d, That hence the modulus M is to the meaſure of the ratio between AQ and AP, as either AP or AQ is to their difference PQ. 3d, The ratio between AC and AB being given, the ſum of all the $\frac{PQ}{AP}$ will be given; and the ſum of all the $M \times \frac{PQ}{AP}$ is as M: therefore the meaſure of any given ratio is as the modulus of the ſyſtem from which it is taken. 4th, Therefore, in every ſyſtem of meaſures, the modulus will always be equal to the meaſure of a certain determinate and immutable ratio; which therefore he calls *the modular ratio*. 5th, To illuſtrate the ſolution by an example: let z be any determinate and permanent quantity, x a variable or indeterminate quantity, and $\dot{x}$ its fluxion; then to find the meaſure of the ratio between $z + x$ and $z - x$, put this ratio equal to the ratio between y and 1, expounding the number y by AP, its fluxion $\dot{y}$ by PQ, and 1 by AB: then the fluxion of the required meaſure of the ratio between y and 1 is $M \times \frac{\dot{y}}{y}$. Now for y reſtore its val. $\frac{z+x}{z-x}$, and for $\dot{y}$ the fluxion of that value $\frac{2z\dot{x}}{\overline{z-x}|^2}$, ſo ſhall the fluxion of the meaſure become $2M \times \frac{z\dot{z}}{zz-xx}$ or $2M$ into $\frac{\dot{x}}{z} + \frac{\dot{x}x^2}{z^3} + \frac{\dot{x}x^4}{z^5}$ &c. And therefore that meaſure will be $2M$ into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5}$ &c. In like manner the meaſure of the

the ratio between $1 + v$ and 1 will be found to be - - - M into $v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4$ &c. And hence, to find the number from the logarithm given, he reverts the series in this manner: If the last measure be called m, we

$$\text{shall have } \frac{m}{M} \text{ or } Q = v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 \text{ \&c,}$$
$$\text{and therefore } Q^2 = \ldots v^2 - v^3 + \tfrac{11}{12}v^4 - \tfrac{5}{6}v^5 \text{ \&c,}$$
$$\text{and } Q^3 = \ldots\ldots v^3 - \tfrac{3}{2}v^4 + \tfrac{7}{4}v^5 \text{ \&c,}$$
$$\text{and } Q^4 = \ldots\ldots\ldots v^4 - 2v^5 \text{ \&c,}$$
$$\text{and } Q^5 = \ldots\ldots\ldots\ldots v^5 \text{ \&c;}$$

then by adding continually we shall have

$$Q + \tfrac{1}{2}Q^2 = v - \tfrac{1}{6}v^3 + \tfrac{5}{24}v^4 - \tfrac{13}{60}v^5 \text{ \&c,}$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 = v - \tfrac{1}{24}v^4 + \tfrac{3}{40}v^5 \text{ \&c,}$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 + \tfrac{1}{24}Q^4 = v - \tfrac{1}{120}v^5 \text{ \&c,}$$
$$Q + \tfrac{1}{2}Q^2 + \tfrac{1}{6}Q^3 + \tfrac{1}{24}Q^4 + \tfrac{1}{120}Q^5 = v \text{ \&c,}$$

that is $v = Q + \frac{1}{2}Q^2 + \frac{1}{6}Q^3 + \frac{1}{24}Q^4 + \frac{1}{120}Q^5$ &c. And therefore the required ratio of $1 + v$ to 1 is equal to the ratio of $1 + Q + \frac{1}{2}Q^2$ &c to 1. Put now $m = M$, or $Q = 1$, and the above will become the ratio of $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$ &c. to 1 for the constant modular ratio. In like manner, if the ratio between 1 and $1 - v$ be proposed, the measure of this ratio will come out M into $v + \frac{1}{2}v^2 + \frac{1}{3}v^3 + \frac{1}{4}v^4$ &c; which being called m, and $\frac{m}{M} = Q$, that ratio will be the ratio of 1 to $1 - Q + \frac{1}{2}Q^2 - \frac{1}{6}Q^3 + \frac{1}{24}Q^4$ &c. And hence taking $m = M$, or $Q = 1$, the said modular ratio will also be the ratio of 1 to $1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}$ &c. And the former of these expressions for the modular ratio comes out the ratio of 2.718,281,828,459, &c. to 1, and the latter the ratio of 1 to 0.367,879,441,171, &c.

In the 2d proposition our learned author gives directions for constructing Briggs's canon of logarithms, namely, first by the general series 2M into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5}$ &c, finding the logarithms of a few such ratios as that of 126 to 125, 225 to 224, 2401 to 2400, 4375 to 4374, &c, from whence the logarithm of 10 will be found to be 2.302,585,092,994, &c, when M is 1; but since Briggs's log. of 10 is 1, therefore as 2.302,585, &c, is to the modulus 1, so is 1 (Briggs's log. of 10) to 0.434,294,481,903, &c, which therefore is the modulus of Brigg's logarithms. Hence he deduces the logarithms of 7, 5, 3, and 2. In like manner are the logarithms of other prime numbers to be found, and from them the logarithms of composite numbers by addition and subtraction only.

He then remarks that the first term of the general series 2 M into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5}$ &c will be sufficient for the logarithms of intermediate numbers between those in the table, or even for numbers beyond the limits of the table. Thus, to find the logarithm answering to any intermediate number; let a and e be two numbers, the one the given number, and the other the nearest tabular number, a

* The computation of these numbers may be seen in one of the tracts contained in this present volume, pages 358, 359, 360, and 371, 372, 373, 374, and 375.

being

being the greater, and e the lefs, of them; put $z = a + e$ their fum, $x = a - e$ their difference, $\lambda =$ the logarithm of the ratio of a to e, that is, the excefs of the logarithm of a above that of e: fo fhall the faid difference of their logarithms be $\lambda = 2M \times \frac{x}{z}$ very nearly.—And, if there be required the number anfwering to any given intermediate logarithm, becaufe λ is $= \frac{2Mx}{z} = \frac{2Mx}{2a-x}$ or $\frac{2Mx}{2e+x}$, therefore x will be $= \frac{\lambda a}{M+\frac{1}{2}\lambda}$, or $\frac{\lambda e}{M-\frac{1}{2}\lambda}$, very nearly.

In the 3d propofition our ingenious author teaches how to convert the canon of logarithms to logarithms of any other fyftem, by means of their *moduli*. And in feveral more propofitions he exemplifies the canon of logarithms in the folution of various important problems in geometry and phyfics; fuch as the quadrature of the hyperbola, the defcription of the logiftica, the equi-angular fpiral, the nautical meridian, &c; the defcent of bodies in refifting mediums, the denfity of the atmofphere at any altitude, &c, &c.

Of Dr. Taylor's Conftruction of Logarithms.

Dr. Brook Taylor (a very learned mathematician, and fecretary to the Royal Society, who died at Somerfet-Houfe, Nov. 1731) gave the following method of conftructing logarithms in number 352 of the Philofophical Tranfactions. His method is founded on thefe three confiderations: 1ft, That the fum of the logarithms of any two numbers is the logarithm of the product of thofe numbers; 2d. That the logarithm of 1 is nothing, and confequently that the nearer any number is to 1, the nearer will its logarithm be to 0; 3d, That the product of two numbers or factors, of which the one is greater and the other lefs than 1, is nearer to 1 than that factor is which is on the fame fide of 1 with itfelf; fo of the two numbers $\frac{2}{3}$ and $\frac{4}{3}$, the product $\frac{8}{9}$ is lefs than 1, but yet nearer to it than $\frac{2}{3}$ is, which is alfo lefs than 1. On thefe principles he founds the prefent approximation, which he explains by the following example. To find the relation between the logarithms of 2 and 10: In order to this he affumes two fractions as $\frac{128}{100}$ and $\frac{8}{10}$, or $\frac{2^7}{10^2}$ and $\frac{2^3}{10}$, whofe numerators are powers of 2, and their denominators powers of 10, the one fraction being greater and the other lefs than unity or 1. Having fet thefe two down, in the form of decimal fractions, below each other in the firft column of the following table, and in the fecond column A and B for their logarithms, expreffing by an equation how they are compofed of the logarithms of 2 and 10, the numbers in queftion, thofe logarithms being denoted thus, $l\,2$ and $l\,10$. Then

1,280,000,000,000	A = . . . =	$7l2-$	$2l10$	$l2$ △ 0,28
0,800,000,000,000	B = . . . =	$3l2-$	$l10$	▽ 0,33
1,024,000,000,000	C = A + B =	$10l2-$	$3l10$	△ 0,300
0,990,352,031,429	D = B + 9C =	$93l2-$	$28l10$	▽ 0,301,07
1,004,336,277,664	E = C + 2D =	$169l2-$	$59l10$	△ 0,301,020
0,998,959,536,107	F = D + 2E =	$485l2-$	$146l10$	▽ 0,301,030,9
1,000,162,894,165	G = E + 4F =	$2136l2-$	$643l10$	△ 0,301,029,96
0,999,936,281,874	H = F + 6G =	$13301l2-$	$4004l10$	▽ 0,301,029,997
1,000,035,441,215	I = G + 2H =	$28738l2-$	$8651l10$	△ 0,301,029,995,1
0,999,971,720,830	K = H + I =	$42039l2-$	$12655l10$	▽ 0,301,029,995,9
1,000,007,161,046	L = I + K =	$70777l2-$	$21306l10$	△ 0,301,029,995,62
0,999,993,203,514	M = K + 3L =	$254370l2-$	$76573l10$	▽ 0,301,029,995,67
1,000,000,364,511	N = L + M =	$325147l2-$	$97879l10$	△ 0,301,029,995,663,5
0,999,999,764,687	O = M + 18N =	$6107016l2-$	$1838335l10$	▽ 0,301,029,995,664,0
comp. ar. 235313				
$0 = 364511\,O + 235313\,N = 2302585825187\,l2 - 693147400972\,l10$				△ 0,301,029,995,663,987

multiplying the two numbers in the first column together, there is produced a third number 1,024, against which is written C, for its logarithm, expressing likewise by an equation in what manner C is formed of the foregoing logarithms A and B. And in the same manner the calculation is continued throughout; only observing this compendium, that before multiplying the two last numbers already entered in the table, to consider what power of one of them must be used to bring the product the nearest that can be to unity. Now, after having continued the table a little way, this is found by only dividing the differences of the numbers from unity one by the other, and taking the nearest quotient for the index of the power sought. Thus, the second and third numbers in the table being 0,8 and 1,024, their differences from unity are 0,200 and 0,024; hence 0,200 ÷ 0,024 gives 9 for the index; and therefore multiplying the 9th power of 1,024 by 0,8 produces the next number 0,990,352,031,429, whose logarithm is $D = B + 9C$.

When the calculation is continued in this manner till the numbers become small enough, or near enough to 1, the last logarithm is supposed equal to nothing, which gives an equation expressing the relation of the logarithms, and from whence the required logarithm is determined. Thus supposing $G = 0$, we have $2136l2 - 643l10 = 0$, and hence, because the logarithm of 10 is 1, we obtain $l2 = \frac{643}{2136} = 0{,}301{,}029{,}96$, too small in the last figure; which so happens because the number corresponding to G is greater than 1. And in this manner are all the numbers in the third or last column obtained, which are continual approximations to the logarithms of 2.

There is another expedient which renders this calculation still shorter, and it is founded on this consideration; that when x is small, $\overline{1+x}|^n$ is nearly $= 1 + nx$. Hence if $1 + x$ and $1 - z$ be the two last numbers already found in the first

first column of the table, the product of their powers $\overline{1+x}|^m \times \overline{1-z}|^n$ will be nearly $= 1$; and hence the relation of m and n may be thus found, $\overline{1+x}|^m \times \overline{1-z}|^n$ is nearly $= \overline{1+mx} \times \overline{1-nz} = 1 + mx - nz - mnxz = 1 + mx - nz$ nearly, which being also $= 1$ nearly, therefore $m : n :: z : x :: l.\,\overline{1-z} : l.\,\overline{1+x}$; whence $x\,l.\,\overline{1-z} + z\,l.\,\overline{1+x} = 0$. For example, let 1,024 and 0,990352 be the last numbers in the table, their logarithms being C and D: here we have $1,024 = 1 + x$, and $0,990352 = 1 - z$; consequently $x = 0,024$, and $z = 0,009648$; and hence the ratio $\frac{z}{x}$ in small numbers is $\frac{201}{500}$. So that for finding the logarithms proposed, we may take $500\,D + 201\,C = 48510\,l2 - 14603\,l\,10 = 0$; which gives $l2 = 0,3010307$. And in this manner are found the numbers in the last line of the table.

Of Mr. Long's Method.

In number 339 of the Philosophical Transactions, are given a brief table and method for finding the logarithm to any number, and the number to any logarithm, by Mr. John Long, B. D. Fellow of C. C. C. Oxon. This table and method are similar to those described in chap. 14 of Briggs's *Arithmetica Logarithmica*, differing only in this, that in this table by Mr. Long, the logarithms, in each class, are in arithmetical progression, the common difference being 1; but in Briggs's little table the column of natural numbers has the like common difference. The table consists of eight classes of logarithms and their corresponding numbers as follows:

Lo.	Nat. Numb.	Log.	Nat. Numb.	Log.	Nat. Numb.	Log.	Nat. Numb.
,9	7,943282347	,009	1,020939484	,00009	1,000207254	,0000009	1,000002072
,8	6,309573445	8	1,018591388	8	1,000184224	8	1,000001842
,7	5,011872336	7	1,016248694	7	1,000161194	7	1,000001611
,6	3,981071706	6	1,013911386	6	1,000138165	6	1,000001381
,5	3,162277660	5	1,011579454	5	1,000115136	5	1,000001151
,4	2,511886432	4	1,009252886	4	1,000092106	4	1,000000921
,3	1,995262315	3	1,006931669	3	1,000069080	3	1,000000690
,2	1,584893193	2	1,004615794	2	1,000046053	2	1,000000460
,1	1,258925412	1	1,002305238	1	1,000023026	1	1,000000230
,09	1,230268771	,0009	1,002074475	,000009	1,000020724	,00000009	1,000000207
8	1,202264435	8	1,001843766	8	1,000018421	8	1,000000184
7	1,174897555	7	1,001613109	7	1,000016118	7	1,000000161
6	1,148153621	6	1,001382506	6	1,000013816	6	1,000000138
5	1,122018454	5	1,001151956	5	1,000011513	5	1,000000115
4	1,096478196	4	1,000921459	4	1,000009210	4	1,000000092
3	1,071519305	3	1,000691015	3	1,000006908	3	1,000000069
2	1,047128548	2	1,000460623	2	1,000004605	2	1,000000046
1	1,023292992	1	1,000230285	1	1,000002302	1	1,000000023

 where,

where, because the logarithms in each clafs are the continual multiples 1, 2, 3, &c of the loweft, it is evident that the natural numbers are fo many fcales of geometrical proportionals, the loweft being the common ratio, or the afcending numbers are the 1, 2, 3, &c powers of the loweft, as expreffed by the figures 1, 2, 3, &c of their correfponding logarithms. Alfo the laft number in the firft, fecond, third, &c clafs, is the 10th, 100th, 1000th, &c root of 10; and any number in any clafs is the 10th power of the correfponding number in the next following clafs.

To find the logarithm of any number, as fuppofe of 2000, by this table: Look in the firft clafs for the number next lefs than the firft figure 2, and it is 1,995262315, againft which is 3 for the firft figure of the logarithm fought. Again, dividing 2 the number propofed by 1,995262315 the number found in the table, the quotient is 1,002374467; which being looked for in the fecond clafs of the table, and finding neither its equal nor a lefs, 0 is therefore to be taken for the fecond figure of the logarithm; and the fame quotient 1,002374467 being looked for in the third clafs, the next lefs is there found to be 1,002305238, againft which is 1 for the third figure of the logarithm; and dividing the quotient 1,002374467 by the faid next lefs number 1,002305238, the new quotient is 1,000069070; which being fought in the fourth clafs gives 0, but fought in the fifth clafs gives 2, which are the fourth and fifth figures of the logarithm fought: again, dividing the laft quotient by 1,000046053 the next lefs number in the table, the quotient is 1,000023015, which gives 9 in the 6th clafs for the 6th figure of the logarithm fought: and again dividing the laft quotient by 1,000020724 the next lefs number, the quotient is 1,000002291, the next lefs than which in the 7th clafs gives 9 for the 7th figure of the logarithm: and dividing the laft quotient by 1,000002072, the quotient is 1,000000219, which gives 9 in the 8th clafs for the 8th figure of the logarithm: and again, the laft quotient 1,000000219 being divided by 1,000000207 the next lefs, the quotient 1,000000012 gives 5 in the fame 8th clafs, when one figure is cut off, for the 9th figure of the logarithm fought. All which figures collected together give 3,301029995 for Briggs's logarithm of 2000, the index 3 being fupplied; which logarithm is true in the laft figure.

To find the number anfwering to any given logarithm, as, fuppofe, to 3.301,030,0: omitting the characteriftic, againft the other figures 3, 0, 1, 0, 3, 0, 0, as in the firft column in the margin, are the feveral numbers as in the 2d column, found from their refpective 1ft, 2d, 3d, &c claffes; the effective numbers of which multiplied continually together, the laft product is 2,000000019966, which, becaufe the characteriftic is 3, gives 2000,000019966, or 2000 only, for the required number anfwering to the given logarithm.

3	1,995262315
0	0
1	1,002305238
0	0
3	1,000069080
0	0
0	0

Of

Of Mr. Jones's Method.

In the 61ſt volume of the Philoſophical Tranſactions, is a ſmall paper on logarithms, which had been drawn up and left unpubliſhed, by the learned and ingenious William Jones, Eſq. The method contained in this memoir, depends on an application of the doctrine of fluxions, to ſome properties drawn from the nature of the exponents of powers. Here all numbers are conſidered as ſome certain powers of a conſtant determinate root: ſo any number x may be conſidered as the z power of any root r, or that $x = r^z$ is a general expreſſion of all numbers in terms of the conſtant root r and a variable exponent z. Now the index z being the logarithm of the number x, therefore to find this logarithm, is the ſame thing as to find what power of the radical r is equal to the number x.

From this principle, the relation between the fluxions of any number x and its logarithm z is thus determined: put $r = 1 + n$; then is $x = r\,z = \overline{1+n}|^{z}$, and $x + \dot{x} = \overline{1+n}|^{z+\dot{z}} = \overline{1+n}|^{z} \times \overline{1+n}|^{\dot{z}} = x \times \overline{1+n}|^{\dot{z}} =$ (by expanding $\overline{1+n}|^{\dot{z}}$, omitting the 2d, 3d, &c powers of $\dot{z}$, and writing q for $\frac{n}{1+n}$) $x + x\dot{z} \times : q + \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4$ &c; therefore $\dot{x} = ax\dot{z}$, putting a for the ſeries $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c, or $f\dot{x} = x\dot{z}$, putting $f = \frac{1}{a}$.

Now when $r = 1 + n = 10$, as in the common logarithms of Briggs's form; then $n = 9$, $q = 0.9$, and the ſeries $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c gives $a = 2.302,585$ &c, and therefore its reciprocal f is $= 0.434,294$ &c. But, if $a = 1 = f$, the form will be that of Napier's logarithms.

From the above form $x\dot{z} = f\dot{x}$ or $\dot{z} = \frac{f\dot{x}}{x}$, are then deduced many curious and general properties of logarithms, with the ſeveral ſeries heretofore given by Gregory, Mercator, Wallis, Newton, and Halley. But of all theſe ſeries, that one which our author ſelects for conſtructing the logarithms, is this; putting $N = \frac{r-p}{r+p}$, the logarithm of $\frac{r}{p}$ is $= 2f \times : N + \frac{1}{3}N^3 + \frac{1}{5}N^5 + \frac{1}{7}N^7$ &c in the caſe in which $r - p$ is $= 1$, and conſequently then $N = \frac{1}{2r-1}$ or $\frac{1}{2p+1}$; which ſeries will then converge very faſt.

Hence, having given any numbers, p, q, r, &c, and as many ratios a, b, c, &c, compoſed of them, the difference between the two terms of each ratio being 1; as alſo the logarithms A, B, C, &c of thoſe ratios given: to find the logarithms P, Q, R, &c of thoſe numbers; ſuppoſing $f = 1$. For inſtance, if $p = 2$, $q = 3$, $r = 5$; and $a = \frac{9}{8} = \frac{3^2}{2^3}$, $b = \frac{16}{15} = \frac{2^4}{3 \cdot 5}$, $c = \frac{25}{24} = \frac{5^2}{3 \cdot 2^3}$. Now the logarithms A, B, C, of theſe ratios a, b, c, being found by the above ſeries, from the nature of powers we have theſe three equations.

4 $A =$

$$\left.\begin{array}{l} A = 2Q - 3P \\ B = 4P - Q - R \\ C = 2R - Q - 3P \end{array}\right\} \text{which equations reduced give} \left\{\begin{array}{l} P = 3A + 4B + 2C = \text{log. of } 2, \\ Q = 5A + 6B + 3C = \text{log. of } 3, \\ R = 7A + 9B + 5C = \text{log. of } 5. \end{array}\right.$$

And hence $P + R = 10A + 13B + 7C$ is $=$ the logarithm of 2×5 or 10.

An elegant tract on logarithms, as a comment on Dr. Halley's method, was alſo given by Mr. Jones in his *Synopſis Palmariorum Matheſeos*, publiſhed in the year 1706. And in the Philoſophical Tranſactions he communicated various improvements in goniometrical properties, and the ſeries relating to the circle and to trigonometry.

The memoir above deſcribed was delivered to the Royal Society by their then librarian, Mr. John Robertſon, a worthy, ingenious, and induſtrious man; who alſo communicated to the Society ſeveral little tracts of his own relating to logarithmical ſubjects: he was alſo the author of an excellent Treatiſe on the Elements of Navigation, in two volumes; and he was ſucceſſively mathematical maſter to Chriſt's hoſpital in London; head maſter to the royal naval academy at Portſmouth; and librarian, clerk, and houſekeeper to the Royal Society; at whoſe houſe, in Crane-Court, Fleet-Street, he died in 1776, aged 64 years.

And among the papers of Mr. Robertſon, I have, ſince his death, found one containing the following particulars relating to Mr. Jones, which I here inſert, as I know of no other account of his life, &c. and as any true anecdotes of ſuch extraordinary men muſt always be acceptable to the learned. This paper is not in Mr. Robertſon's hand writing, but in a kind of running law-hand, and is ſigned R. M. 12 Sept. 1771.

"William Jones, Eſq. F. R. S. was born at the foot of Bodavon mountain [Myriadd Bodafon], in the pariſh of Llanfihangel tre'r Bardd, in the iſle of Angleſey, North Wales, in the year 1675. His father John George * was a farmer, of a good family, being deſcended from Hwfa ap Cynddelw, one of the 15 tribes of North Wales. He gave his two ſons the common ſchool education of the country, reading, writing and accounts, in Engliſh, and the Latin grammar. Harry his ſecond ſon took to the farming buſineſs; but William the eldeſt, having an extraordinary turn for mathematical ſtudies, determined to try his fortune abroad from a place where the ſame was but of little ſervice to him; he accordingly came to London, accompanied by a young man, Rowland Williams, afterwards an eminent perfumer in Wych Street. The report in the country is, that Mr. Jones ſoon got into a merchant's counting houſe, and ſo gained the eſteem of his maſter, that he gave him the command of a ſhip for a Weſt India voyage; and that upon his return he ſet up a mathematical ſchool, and publiſhed his book of navigation †; and that upon the death of the merchant he married his widow: that Lord Macclesfield's ſon being his pupil, he was made ſecretary

* "It is the cuſtom in ſeveral parts of Wales for the name of the father to become the ſurname of his children. John George the father was commonly called Sion Sors, of Llanbabo, to which pariſh he moved, and where his children were brought up."

† This tract on navigation, intituled, "A New Compendium of the whole Art of Practical Navigation," was publiſhed in 1702, and dedicated "to the reverend and learned Mr. John Harris, M. A. and F. R. S." the author (as I apprehend) of the "Univerſal Dictionary of Arts and Sciences," under whoſe roof Mr. Jones ſays he compoſed the ſaid treatiſe on Navigation.

to

to the chancellor, and one of the Deputy-tellers of the exchequer—and they have a ſtory of an Italian wedding which cauſed great diſturbance in Lord Macclesfield's family, but was compromiſed by Mr. Jones; which gave riſe to a ſaying, that Macclesfield was the making of Jones, and Jones the making of Macclesfield."

Mr. Jones died July 3, 1749, being a vice-preſident of the Royal Society; and left one daughter, and his widow big with another child, which proved a ſon, who is the preſent Sir William Jones, at this time, 1791, one of the judges in India, and highly eſteemed for his great abilities, extenſive learning, and eminent patriotiſm.

Of Mr. Andrew Reid and others.

Andrew Reid, Eſq. publiſhed in 1767 a quarto tract under the title of *An Eſſay on Logarithms*, in which he alſo ſhews the computation of logarithms from principles depending on the binomial theorem and the nature of the exponents of powers, the logarithms of numbers being here conſidered as the exponents of the powers of 10. He hence brings out the uſual ſeries for logarithms, and largely exemplifies Dr. Halley's moſt ſimple conſtruction.

Beſides the authors whoſe methods have been here particularly deſcribed, many others have treated on the ſubjects of logarithms, and of the ſines, tangents, ſecants, &c; among the principal of whom are Leibnitz, Euler, Maclaurin, Wolfius, and profeſſor Simſon in an accurate geometrical tract on logarithms, contained in his poſthumous works, which were elegantly printed in 4to. at Glaſgow, in the year 1776, at the expence of that very learned nobleman, the late Earl Stanhope, and by his lordſhip diſpoſed of in preſents among gentlemen moſt eminent for mathematical learning.

Of Mr. Dodſon's Anti-logarithmic Canon.

The only remaining conſiderable work of this kind publiſhed, that I know of, is the Anti logarithmic Canon of Mr. James Dodſon, a very ingenious mathematician, which work he publiſhed in folio in the year 1742; a very great performance, containing all logarithms under 100,000, and their correſponding natural numbers to 11 places of figures, with all their differences and the proportional parts; the whole arranged in the order contrary to that uſed in the common tables of numbers and logarithms, the exact logarithms being here placed firſt, and increaſing continually by 1, from 1 to 100,000, and their correſponding neareſt numbers in the columns oppoſite to them; and by means of the differences and proportional parts, the logarithm to any number, or the number to any logarithm, each to 11 places of figures, is readily found. This work contains alſo, beſides the conſtruction of the natural numbers to the given logarithms, "precepts and examples, ſhewing ſome of the uſes of logarithms, in facilitating the moſt difficult operations in common arithmetic, caſes of intereſt, annuities, menſuration, &c; to which is prefixed an introduction, containing a ſhort account of logarithms, and of the moſt conſiderable improvements made, ſince their invention, in the manner of conſtructing them."

The

The manner in which theſe numbers were conſtructed, conſiſts chiefly in imitations of ſome of the methods before deſcribed by Briggs, and is nothing more than generating a ſcale of 100000 geometrical proportionals from 1 the leaſt term to 10 the greateſt, each continued to 11 places of figures; and the means of effecting this are ſuch as eaſily flow from the nature of a ſeries of proportionals, and are briefly as follows. Firſt between 1 and 10 are interpoſed 9 mean proportionals; then between each of theſe 11 terms there are interpoſed 9 other means, making in all 101 terms; then between each of theſe a 3d ſet of 9 means, making in all 1001 terms; again between each of theſe a 4th ſet of 9 means, making in all 10001 terms; and laſtly between each two of theſe terms, a 5th ſet of 9 means, making in all 100001 terms, including both the 1 and the 10. The firſt four of theſe 5 ſets of means, are found each by one extraction of the 10th root of the greater of the two given terms, which root is the leaſt mean, and then multiplying it continually by itſelf according to the number of terms in the ſection or ſet; and the 5th or laſt ſection is made by interpoſing each of the 9 means by help of the method of differences before taught. Namely, putting 10 the greateſt term $= A$, $A^{\frac{1}{10}} = B$, $B^{\frac{1}{10}} = C$, $C^{\frac{1}{10}} = D$, $D^{\frac{1}{10}} = E$, and $E^{\frac{1}{10}} = F$; now extracting the 10th root of A or 10, it gives 1,2589254118 $= B = A^{\frac{1}{10}}$ for the leaſt of the 1ſt ſet of means; and then multiplying it continually by itſelf, we have B, B^2, B^3, B^4, &c to $B^{10} = A$ for all the 10 terms: 2dly, the 10th root of 1,2589254118 gives 1,0232929923 $= C = B^{\frac{1}{10}} = A^{\frac{1}{100}}$ for the leaſt of the 2d claſs of means, which being continually multiplied gives C, C^2 C^3, &c to $C^{100} = B^{10} = A$ for all the 2d claſs of 100 terms: 3dly, the 10th root of 1,0232929923 gives 1,0023052381 $= D = C^{\frac{1}{10}} = B^{\frac{1}{100}} = A^{\frac{1}{1000}}$ for the leaſt of the 3d claſs of means, which being continually multiplied gives D, D^2, D^3, &c to $D^{\frac{1}{1000}} = C^{\frac{1}{100}} = B^{\frac{1}{10}} = A$ for the 3d claſs of 1000 terms: 4thly, the 10th root 1,0023052381 gives 1,0002302850 $= E = D^{\frac{1}{10}} = C^{\frac{1}{100}} = B^{\frac{1}{1000}} = A^{\frac{1}{10000}}$ for the leaſt of the 4th claſs of means, which being continually multiplied gives E, E^2, E^3, &c to $E^{10000} = D^{1000} = C^{100} = B^{10} = A$ for the 4th claſs of 10000 terms. Now theſe 4 claſſes of terms thus produced, require no leſs than 11110 multiplications of the leaſt means by themſelves; which however are much facilitated by making a ſmall table of the firſt 10 or even 100 products of the conſtant multiplier, and from thence only taking out the proper lines and adding them together: and theſe 4 claſſes of numbers always prove themſelves at every 10th term, which muſt always agree with the correſponding ſucceſſive terms of the preceeding claſs. The remaining 5th claſs is conſtructed by means of differences, being much eaſier than the method of continual multiplication, the 1ſt and 2d differences only being uſed, as the 3d difference is too ſmall to enter the computation of the ſets of 9 means between each two terms of the 4th claſs. And the ſeveral 2d differences for each of theſe ſets of 9 means, are found from the properties of a ſet of proportionals 1, r, r^2, r^3, &c, as diſpoſed

poſed in the 1ſt column of the annexed table, and their ſeveral orders of differences as in the other columns of the table; where it is evident that each column, both that of the given terms of the progreſſion, and thoſe of their orders of differences, forms a ſcale of proportionals, having the ſame common ratio, and that each horizontal line, or row, forms a geometrical progreſſion, having all the ſame common ratio $r - 1$, which is alſo the 1ſt difference of each ſet of means; ſo $\overline{r-1}|^2$ is the 1ſt of the 2d differences, and which is conſtantly the ſame, as the 3d differences become too ſmall in the required terms of our progreſſion to be regarded, at leaſt near the beginning of the table: hence, like as 1, $\overline{r-1}$, and $\overline{r-1}|^2$ are the 1ſt term with its 1ſt and 2d differences; ſo r^n, $r^n.\overline{r-1}$, and $r^n.\overline{r-1}|^2$ are any other term with its 1ſt and 2d differences. And by this rule the 1ſt and 2d differences are to be found for every ſet of 9 means, viz. multiplying the 1ſt term of any claſs (which will be the ſeveral terms of the ſeries E, E^2, E^3, &c, or every 10th term of the ſeries F, F^2, F^3, &c) by $r - 1$ or $F - 1$ for the 1ſt difference, and this multiplied by $F - 1$ again for the true 2d difference at the beginning of that claſs. Thus the 10th root of 1,0002302850, or E, gives 1,000023026116 for F or the 1ſt mean of the loweſt claſs; therefore, $F - 1 = r - 1 =$,000023026116 is its 1ſt difference, and the ſquare of it is $\overline{r-1}|^2 =$,0000000005302 its 2d difference; then is ,000023026116 F^{10n} or ,000023026116 E^n the 1ſt difference, and ,0000000005302 F^{20n} or ,000000000 5302 E^{2n}, is the 2d difference at the beginning of the nth claſs of decades. And this 2d difference is uſed as the conſtant 2d difference through all the 10 terms, except towards the end of the table, where the differences increaſe faſt enough to require a ſmall correction of the 2d difference, and which Mr. Dodſon effects by taking a mean 2d difference among all the 2d differences in this manner; having found the ſeries of 1ſt differences $\overline{F-1}.\ E^n$, $\overline{F-1}.\ E^{n+1}$, $\overline{F-1}.\ E^{n+2}$ &c, take the differences of theſe, and $\frac{1}{10}$ of them will be the mean 2d differences to be uſed, namely, $\frac{F-1}{10}.\overline{E^{n+1}-E}$, $\frac{F-1}{10}.\overline{E^{n+2}-E^{n+1}}$, &c, are the mean 2d differences. And this is not only the more exact, but alſo the eaſier, way. The common 2d difference and the ſucceſſive 1ſt differences are then continually added through the whole decade, to give the ſucceſſive terms of the required progreſſion.

Terms	1ſt dif.	2d. dif.	3d dif.	&c
1 ×	$\overline{r-1}$ ×	$\overline{r-1}\vert^2$ ×	$\overline{r-1}\vert^3$ ×	
1	1	1	1	&c
r	r	r	r	
r^2	r^2	r^2	r^2	
r^3	r^3	r^3	r^3	
&c	&c	&c	&c	

JOANNIS KEPLERI

IMP. CÆS. FERDINANDI II. MATHEMATICI,

CHILIAS LOGARITHMORUM

AD TOTIDEM NUMEROS ROTUNDOS;

PRÆMISSA

DEMONSTRATIONE LEGITIMA

ORTUS LOGARITHMORUM EORUMQUE USUS,

QUIBUS

NOVA TRADITUR ARITHMETICA, SEU COMPENDIUM,

Quo, post numerorum notitiam, nullum nec admirabilius, nec utilius, solvendi pleraque Problemata Calculatoria, præsertim in Doctrina Triangulorum, citrà Multiplicationis, Divisionis, Radicúmque extractionis, in Numeris prolixis, labores molestissimos.

AD

ILLUSTRISSIMUM PRINCIPEM ET DOMINUM,

DN. PHILIPPUM, LANDGRAVIUM HASSIÆ, &c.

CUM PRIVILEGIO AUTHORIS CÆSAREO.

Prima hujus tractatûs editio impressa fuit Marpurgi, et excusa typis Casparis Chemlini, Anno Domini MDCXXIV.

AD

ILLUSTRISSIMUM PRINCIPEM ET DOMINUM,

DN. PHILIPPUM, LANDGRAVIUM HASSIÆ,

COMITEM CHATTIMELIBOCCI, SIGENÆ, DITII, ET NIDDÆ,

DOMINUM MEUM CLEMENTISSIMUM.

De Munere ejus Celsit. amplissimo, deque usitata Cels. suæ Allusione, & Alliteratione ad Nomen PHILIPS, BILLIEBS.

TRIGINTA expensis vendit φίλα πολλὰ Philippis
Princeps PHILIPPUS Hassiæ.
Sed facies rerum versa est; pauperculus emptor
Mercemque pretium retulit
Triginta acceptis, sed vendidit ille Magistrum
Mortalium immanissimus.
I nunc & dubita, Tharsensi interprete, num sit
Dare quàm accipere beatius.
Triginta tamen en! penso totidem αργυρα * verbis;
φίλα πολλὰ penso Chiliade.
Corde, manu, promis redolentia munera fontes:
Mentemque redolent quæ accipis.
Corde φιλεῖς, manibus πλουτεῖς, sed Mente θεωρεῖς;
Tibi cor, manus, mentem dico.

Illustr. Cels. T.

Subjectissimus Cultor Gratusque Hospes,

JOHAN. KEPLERUS.

* Propositionibus.

DEMONSTRATIO

STRUCTURÆ LOGARITHMORUM.

POSTULATUM I.

OMNES proportiones inter ſe æquales, quacunque varietate binorum unius, & binorum alterius terminorum, eadem quantitate metiri ſeu exprimere.

AXIOMA I.

Si fuerint quantitates quotcunque ejuſdem generis, quocunque ordine ſibi invicem ſuccedentes, ut ſi ordine magnitudinis ſibi invicem ſuccedant: proportio extremarum compoſita eſſe intelligitur ex omnibus proportionibus intermediis binarum, & binarum inter ſe vicinarum.

Seu, quod eodem redit, proportio minuitur aucto minori termino, vel diminuto majori; augetur rationibus contrariis.

I. PROPOSITIO.

Medium proportionale inter duos terminos dividit proportionem terminorum in duas proportiones inter ſe æquales.

Nam ſi ſunt duo termini, eorumque medium proportionale: eſt ergò inter tres quantitates Analogia ſeu Proportionalitas.

At Analogia definitur æqualitate τῶν λόγων, proportionum: quare proportiones ſectione conſtitutæ, utpote partes proportionis totius propoſitæ, ſunt inter ſe æquales.

AXIOMA SEU NOTITIA COMMUNIS II.

Si fuerint quantitates quotcunque creſcentes ordine: proportio extremarum diviſa eſt per intermedias in partes unâ plures quàm ſunt intermediæ, diviſionem facientes.

 Sic

Sic quatuor interſtitia digitorum arguunt quinque digitos. Sic quinque corpora regularia ſibi invicem inſerta ordine interpoſitis orbibus inſcriptis & circumſcriptis, arguunt orbium talium ſex eſſe.

POSTULATUM II.

Proportionem inter datos duos terminos quoſcunque dividere in partes quotcunque (ut in partes numero continuè multiplici progreſſionis binariæ) & eouſque donec partes oriantur minores quantitate propoſita.

Proportio enim eſt etiam una ex quantitatibus continuis in infinitum dividuis.

Vide hic typum diviſæ proportionis inter 10 & 7, per triginta proportionales medias.

EXEMPLUM SECTIONIS,

In quâ proportio, quæ eſt inter 10 & 7, triceſimo actu, in partes æquales 1073741824 ſecatur, per totidem (unâ minus) medias proportionales claſſis triceſimæ, ubi ex unaqualibet claſſe ſola maxima, & termino proportionis majori viciniſſima, hic exprimitur.

	Denominatio per Numeros ſectionum, Mediæ proportionales, quæ eſt omnium in qualibet ſectione maxima.		Numeri partium proportionis æqualium, quas unius cujuſlibet claſſis mediæ proportionales conſtituunt.
Major terminus.		100000000000000000000	
	30 æ.	99999999966782056900	1073741824
	29 æ.	99999999933564113801	536870912
	28 æ.	99999999867128227702	268435456
	27 æ.	99999999734256455589	134217728
	26 æ.	99999999468512912883	67108864
	25 æ.	99999998937025838590	33554432
	24 æ.	99999997874051688629	16777216
	23 æ.	99999995748103422452	8388608
	22 æ.	99999991496207025698	4194304
	21 æ.	99999982992414774542	2097152
	20 æ.	99999965984832451665	1048576
	19 æ.	99999931969676473647	524288
	18 æ.	99999863939399228474	262144
	17 æ.	99999727878983581819	131072
	16 æ.	99999455758707662114	65536
	15 æ.	99998911520377310068	32768
	14 æ.	99997823052602499026	16384
	13 æ.	99995646152595997766	8192
	12 æ.	99991292494751867706	4096
	11 æ.	99982585747710211873	2048
	10 æ.	99965174527982251100	1024
	9 æ.	99930361184098514780	512
	8 æ.	99860770863843831172	256
	7 æ.	99721735575211210274	128
	6 æ.	99444245461323450059	64
&c.	5 æ.	98891579553719496652	32
Quindecim 4tæ.	4 æ.	97795445066296320009	16
Septem Tertiæ.	3 æ.	95639490757149812386	8
Tres Secundæ.	2 æ.	91469121922869443920	4
Unica Prima.	1 a.	83666002653407554820	2
Minor terminus.		70000000000000000000	

Hæc unica prima in ſuâ ſeu primâ claſſe, fit etiam una, eaque media, trium ſecundarum; & tres ſecundæ iſtæ, ſunt etiam inter ſeptem tertias, mediæ ſcilicet inter alias quatuor accedentes: & hæ ſeptem tertiæ, inſertis aliis octo, fiunt quindecim quartæ, & ſic conſequenter.

Hic differentia maximæ ex triceſimis mediis proportionalibus à termino ſectæ proportionis majori, eſt iſta 00000, 00003, 32179, 43100. Hæc igitur differentia conſtituitur ex arbitrio, pro menſurâ hujus minimi elementi, ſectæ proportionis, ſeu pro Logarithmo dictæ Triceſimarum maximæ. Hic igitur Logarithmus multiplicatus in numerum partium, quas conſtituunt illæ triceſimæ, producit ſequentem Logarithmum 35667, 49481, 37222, 14400. Hic eſt Logarithmus termini minoris, ſcilicet 70000, 00000, 00000, 00000.

Hic numerus unitate ſuperat numerum mediarum proportionalium cujuſque claſſis.

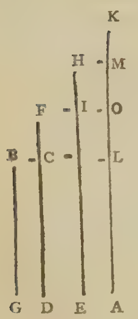
Hic Typus ſic intelligatur: Inter Terminos, majorem 100, &c. & minorem 70 &c. quæratur media proportionalis, hæc erit $83\frac{2}{3}$ &c. Sunt ergò proportionis inter dictos terminos conſtitutæ partes duæ, una inter 100 & $83\frac{2}{3}$, altera inter $83\frac{2}{3}$ & 70. Hæ per 1 prop. ſunt inter ſe æquales. Quæratur ſecundò, media proportionalis inter 100 & $83\frac{2}{3}$ &c. hæc erit $91\frac{1}{2}$ &c. rurſumque erunt partes inter ſe æquales, una inter 100 &c. & $91\frac{1}{2}$ &c. & altera inter $91\frac{1}{2}$ &c. & $83\frac{2}{3}$ &c. Ita prior ſemiſſis proportionis totius hîc eſt diviſus in duas quartas partes ejuſdem totius. Et intelligitur ſemiſſis alter, qui erat inter $83\frac{2}{3}$ & 70, per ſociam ipſius $91\frac{1}{2}$, ſeu ſecundarum trium minimam (quæ in hoc typo non exprimitur) ſimiliter diviſus eſſe in alias duas quartas totius. Quæratur tertiò media proportionalis inter 100, &c. & $91\frac{1}{2}$ &c. hæc erit $95\frac{2}{3}$ &c. determinans cum 100 partem totius octavam, quod indicat numerus illi ad dextram exterius reſpondens. Et ſic deinceps.

POSTULATUM III.

Minimum proportionis elementum quantulum pro minimo placuerit, metiri ſeu ſignare per quantitatem quamcunque; ut per exceſſum terminorum hujus Elementi.

II. PROPOSITIO.

Cùm fuerint tres continuè proportionales, quæ eſt proportio primæ ad ſecundam, vel ſecundæ ad tertiam, eadem eſt proportio differentiæ priorum, ad differentiam poſteriorum.



Sint continuè proportionales AK, EH, DF, differentia priorum KM, poſteriorum HI: Dico, ut eſt AK ad EH, ſic eſſe KM ad HI. Eſt enim HE ad FD, ut KA ad HE ex hypotheſi. Sed HE eſt æqualis ipſi MA, et FD æqualis ipſi IE, rurſum ex hypotheſi. Quare etiam MA erit ad IE, ut KA ad HE; quare per 17 quinti Eucl. etiam reſidua KM ad reſiduam HI, erit ut tota KA ad totam HE.

III. PROPOSITIO.

Cùm fuerint aliquot quantitates in proportione continuâ, minimarum minima erit differentia, maximarum maxima.

Nam per 2 prop. ſicut eſt maxima ad vicinam minorem, ſic eſt differentia inter maximas, ad differentiam ſuccedentium: Minor igitur eſt quælibet differentiarum ſuccedentium, quàm antecedens. Minima igitur eſt ultima differentia, quæ ſcilicet eſt inter minimas.

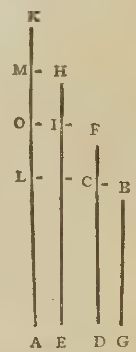
IV. PROPOSITIO.

Cùm fuerint aliquot quantitates in proportione continuâ: ſi differentia maximarum ſtatuitur menſura proportionis illarum: differentiæ quarumcunque duarum deinceps erunt minores menſurâ proportionis illarum juſtâ.

Nam quia ponuntur continuè proportionales, igitur æqualis eſt proportio inter duas maximas, & proportio inter quaſcunque duas minores deinceps ſitas per 1 Prop. Major verò eſt differentia inter maximas, differentiis inter quaſcunque alias deinceps ſitas per 3. Si ergò major differentia ſtatuitur menſura proportionis inter duas maximas per 3 poſtulatum: tunc eadem duarum maximarum differentia ſtatuitur etiam menſura proportionis inter duas minores deinceps ſequentes. At differentia inter duas minores, minor etiam eſt differentia inter maximas per 3 Prop. quare etiam minor eſt, quàm ut ſuorum terminorum proportionem metiatur.

V. PROPOSITIO.

In continuè proportionalibus ſi differentia maximarum ſtatuitur menſura proportionis illarum: omnes reliquæ proportiones, quæ ſunt inter maximam & unamquamlibet reliquarum minorum, ſortientur menſuras, majores differentiis ſuorum terminorum.



Nam proportio maximæ AK ad minimam GB componitur ex proportionibus binarum, & binarum deinceps uſque ad minimam, per Axioma 1. At omnes binarum deinceps ſitarum proportiones inter ſe ſunt æquales per 1 prop. menſuras igitur etiam æquales habent per poſtulat. 1. Quare quot ſunt elementa proportionis inter maximam AK & minimam GB, toties differentia KM, maximarum KA, HE, vel MA, continetur in menſurâ proportionis maximæ KA ad minimam BG.

Jam verò maximæ KA & minimæ BG, differentia KL componitur ex differentiis KM, MO, OL binarum, & binarum deinceps ſitarum omnibus.

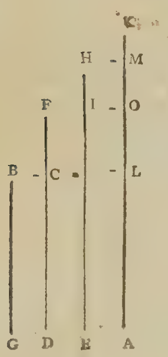
Sed quælibet differentia MO, OL, binarum deinceps ſeorſim minor eſt differentia KM maximarum per 3 Prop. quare etiam totidem junctæ differentiæ binarum deinceps, id eſt, differentia KL maximæ KA, & minimæ BG erit minor quàm multiplex ipſius KM differentiæ maximarum, ſecundum numerum elementorum proportionis ſectæ. Sed multiplex iſta eſt menſura proportionis inter maximam KA & minimam BG, ut jam oſtenſum. Ergò differentia KL maximæ & minimæ non æquat menſuram proportionis earum, poſitis quæ ſunt poſita.

In

In numeris ſic. Sint numeri 1000, 900, 810, 729, continuè proportionales. Maximi ſunt 1000, 900. Eorum differentia eſt 100. Sit hæc menſura arbitraria proportionis 1000, 900. Erit igitur etiam menſura hæc proportionis 900, 810, & proportionis 810, 729. Compoſita igitur proportionis inter 1000 & 729 menſura erit 300, quia elementa æqualia proportionis tria ſunt, per duos medios proportionales per axioma 2. Atqui terminorum hujus proportionis, ſcilicet 1000 & 729, differentia eſt 271, multò minor quàm 300, triplum ipſius 100.

VI. PROPOSITIO.

In continuè proportionalibus, ſi differentia maximæ & unius minorum non deinceps ſequentis ſtatuitur menſura proportionis illarum: reliquæ proportiones, quæ quidem ſunt inter maximam & unam prius aſcitâ majorem, ſortientur menſuram minorem differentiâ ſuorum terminorum: quæ verò proportiones ſunt inter quantitatem maximam & unamquamlibet, prius aſcitâ minorem, nanciſcentur menſuram majorem, quàm eſt differentia ſuorum terminorum.



Sint proportionales AK, EH, DF, GB, & aſſumantur maxima quidem AK, & una minorum non deinceps DF, ſitque illarum differentia KO, & KO menſuret quantitatem proportionis inter AK & DF. Et ſint aliæ quantitates, EH quidem major prius aſcitâ DF; GB verò illâ minor: & ſit MK differentia ipſarum AK, EH; ſic HI vel MO ſit differentia ipſarum EH, DF; denique, FC ſit differentia ipſarum DF, GB. Dico primò menſuram proportionis AK, EH fore minorem differentiâ terminorum MK.

Nam quia proportio AK, DF, menſuram accipit KO, eadem verò proportio habet partes æquales conſtitutas per mediam proportionalem EH, per 1 prop. Proportionis igitur AK, EH menſura erit dimidia ipſius KO per 1 poſtul. (vel pars alia aliquota, ſecundum numerum interjectarum EH.) At differentia MK major eſt quàm MO reſidua de KO per 3 prop. Major igitur quàm dimidium ipſius KO. Ergò & major, quàm ut poſſit eſſe menſura proportionis inter KA & HE majorem prius aſcitâ FD.

Dico ſecundò menſuram proportionis AK, GB fore majorem differentiâ LK. Rurſum enim proportionis EH, DF, quæ ſemiſſis eſt ipſius AK, DF, menſura erit ſemiſſis ipſius KO per 1 poſtul. vel pars alia aliquota, &c. At differentia HI vel MO minor eſt quàm MK, reſidua de KO per 3 prop. Minor igitur HI, quàm dimidium ipſius KO. Ergò minor quàm menſura proportionis EH, DF. Sed differentia FC vel OL rurſum minor eſt, quàm differentia MO per 3 prop. Plus igitur deficit OL menſura proportionis DF, GB, quàm MO, à menſura proportionis EH, DF. Eſt verò proportio DF, GB, æqualis proportioni EH, DF, quia hac quidem vice ipſi DF æquè propinqua eſt GB verſus minora, atque EH verſus majora, & ſi hæc non eſſet, alia ſumi poſſit, quippe inter continuè proportionales. Deficit igitur OL à menſurâ proportionis DF, GB: ſed KO ſtatuitur eſſe ipſa menſura proportionis AK, DF.

Cùm

Cùm ergò proportio AK, GB, compoſita ſit ex proportione AK, DF, & proportione DF, GB, per ax. I. Menſura etiam illius compoſita erit ex menſuris harum; ut ſi proportio DF, GB ſemiſſis eſt ipſius proportionis AK, DF, & proportio AK, GB ſeſquialtera ipſius AK, DF; erit etiam ipſius KO ſuſceptæ menſuræ, ſeſquialtera pro ipſius AK, GB proportionis menſura habenda.

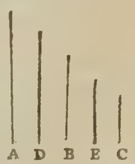
Similiter verò & KL componitur ex KO, OL; OL verò demonſtrata eſt eſſe minor dimidiâ ipſius KO: Tota igitur KL minor eſt quàm ſeſquialtera ipſius KO, minor igitur quàm menſura proportionis ſuæ inter AK, GB.

Per numeros ſic: Sint continuè proportionales 100000, 90000, 81000, 72900, 65610, 59049. Eligantur 100000, 72900, earumque differentia 27100 ſtatuatur menſura proportionis terminorum. Cùm igitur hæc proportio habeat partes tres æquales, quarum una 100000, 90000, cum differentiâ 10000: altera 90000, 81000, cum differentia 9000: tertia 81000, 72900, cum differentia 8100. Quælibet vero proportionum harum ſit pars tertia ſuæ totius, valebit etiam tertiam partem menſuræ illius, ſcilicet $9033\frac{1}{3}$, hæc vero menſura Elementorum proportionis minor quidem eſt, quàm primorum terminorum differentia 10000, major vero quam ſecunda differentia 9000, major etiam multo quam tertia 8100. Conſequenter etiam major quam differentia 7290, numerorum 72900, 65610: quam & differentia 6561 numerorum 65610, 59049. Ac proinde proportionis 100000, 9000 terminorum differentia 10000, excedit menſuram ſuſceptam $9033\frac{1}{3}$. Nec minus etiam proportionis 100000, 81000 differentia 19000 excedit menſuram ſuam, ſcilicet duplum ipſius $9033\frac{1}{3}$, ſcilicet $18066\frac{2}{3}$, quia termini minores 90000, & 81000 adhuc ſtant ante primò adjunctum 72900. E contrario proportionis 100000, 65610 differentia 34390 minor eſt quam illius menſura $36133\frac{1}{3}$, quadruplum ſcilicet ipſius $9033\frac{1}{3}$. Sic etiam proportionis 100000, 59049 differentia 40951 minor eſt quintuplo ipſius $9033\frac{1}{3}$, ſcilicet $45166\frac{2}{3}$, quia termini minores 65610 & 59049 ſtant poſt primo adjunctum 72900 utpote minores illo.

Proportionales.	Differentia.
100000	
90000	-10000
81000	-9000
72900	-8100
65610	-7290
59049	-6561

VII. PROPOSITIO.

Si quantitates aliquot ordine magnitudinis deinceps collocentur, binæ deinceps proportiones æquales facientes, ipſæ quantitates continuè proportionales erunt.



Collocentur deinceps A, D, B, E, C, hoc ordine minores à primâ, ſintque AB & BC proportiones æquales, dico B eſſe mediam proportionalem inter A, C. Si enim non: erit vel major vel minor mediâ proportionali; ſit major illâ, & ſit verbi cauſa E ipſa media proportionalis. Erit igitur AE proportio major quam AB, per I ax. quare EC minor quam AE. At ſi E media proportionalis, tunc AE & EC proportiones ſunt æquales per I prop. Non igitur minor erit media proportionalis quam B.

Sit major, & ſit D, rurſum idem abſurdum ſequetur.

Si ergo AB, BC æquales proportiones, ipſa B eſt media proportionis, &c.

VIII. PROPOSITIO.

Si quantitates quæcunque deinceps collocentur, ordine magnitudinis, quarum, quæ intermediæ, non ſint inter proportionales, medias, proportionis cujuſcunque, ſive actu continuatæ, ſive poteſtate continuandæ interpoſitione omiſſarum: intermediæ tales proportionem extremarum non dividunt in commenſurabilia.

Definitiones. 1. Commenſurabilia enim ex eo dicuntur, quod habeant unam communem menſuram, quam quodlibet contineat ſecundum certum numerum aliquoties exactè, ſic ut nihil, quod eâ menſurâ minus ſit, reſtet reſiduum. 2. Jam vero menſura proportionum communis, eſt & ipſa aliqua proportio, minor utrâque menſurandâ. 3. Omnis vero proportio eſt inter duos terminos. 4. Et Proportio repetitione ſui, menſurans aliam proportionem, incipit ab uno menſurandæ termino, eique ſociat alium, pro ratione quantitatis ſuæ minoris: tum illo jam pro antecedenti ſumpto, ſtatuit alium conſequentem, hoc identidem, quoaduſque permeatur proportionis menſurandæ quantitas: non aliter, quam cum intervallo pedum Circini metimur lineam, fixo pede Circini in unâ lineæ extremitate, pede altero punctum ſignamus, deinde pede priore in hoc punctum tranſlato, punctum aliud altero pede metamur, verſus ulteriora, donec emenſi fuerimus totam lineam. 5. Et Proportio proportionem exactè menſurare dicitur, quando in hâc continuâ terminorum interpoſitione & coaptatione tandem ultimus terminus proportionis menſurantis, cum ſecundo termino menſuratæ coincidit in quantitate. Igitur identitas illa proportionis menſurandis continuè repetitæ efficit, terminos continuè proportionales per 7 prop. Ergo ſi proportio aliqua duas proportiones exactè metitur, neceſſe eſt, ut termini quos ipſa menſurans interponit, ſint cum ipſius menſurandæ terminis continuè proportionales. Si ergò nulla unquam, quantumvis parva proportio poteſt inveniri, quæ repetitione ſui, terminos ultimos aſſequatur proportionum menſurandarum, ſic ut tam major communis terminus, quàm duo minores proportionum menſurandarum ſint cum menſurantis terminis interpoſitis continuè proportionales: proportiones illæ ſunt inter ſe incommenſurabiles.

IX. PROPOSITIO.

Cum duæ longitudines effabiles non fuerint ad invicem, ut duo numeri ejuſdem ſpeciei figurativæ, verbi cauſâ, duo quadrati, aut duo cubi: non cadent inter illas, longitudines aliæ effabiles, mediæ proportionales, numero tot quot ipſa ſpecies poſtulat, verbi causâ, quadrati unam, cubi duas, biquadrati tres, &c.


A B C D

Sint enim duæ longitudines AD habentes, ſe quidem ad invicem, ut numerus ad numerum, at non ut numerus cubicus ad cubicum: & quia de cubo agimus, de duabus igitur mediis proportionalibus erit dicendum, ſint eæ B & C. Dico B & C non eſſe longitudines effabiles.

Si enim quis contendat esse effabiles, esto hoc positum. Sunt igitur ut Numeri. Sunt autem simul mediæ proportionales inter A, D, ex hypothesi. Et quia etiam A, D sunt ut numeri, quippe effabiles, & ipsæ supponuntur, habent verò duas medias B & C, ut numeros, quare per 21 octavi Eucl. A & D similes erunt solidi: quare per 27 ejusdem erunt ad invicem, ut Numerus ad numerum cubicum. Hoc verò est contra primam propositionis hypothesin. Falsum igitur positum fuit, B & C esse longitudine effabiles. Vera igitur est negatio in propositione comprehensa.

Eodem modo etiam de quadratis, & de unâ mediâ proportionali ratiocinari possumus, deductione ad impossibile: nec minus & de cæteris speciebus, post quadratum & cubum sequentibus.

X. PROPOSITIO.

Si ex aliquot quantitatibus effabilibus ordine magnitudinis invicem sequentibus duæ extremæ non fuerint ad invicem, ut duo numeri quadrati, aut duo cubi, aut duo alii ejusdem speciei; intermediarum nulla dividet proportionem in commensurabilia.

Nam nisi duæ quantitates effabiles recipiant media proportionalia effabilia, earum proportio non dividetur per effabilem intermediam in commensurabilia per 8 prop. At si duæ quantitates non fuerint inter se, ut duo numeri ejusdem speciei figurativæ, non recipiunt media proportionalia effabilia, per 9 prop. Quare illæ intermediæ quas propositio admittit, cum sint effabiles, non erunt ex proportionalibus mediis. Non igitur dividunt proportionem extremarum in commensurabilia.

XI. PROPOSITIO.

Omnes proportiones deinceps ordinatæ, quæ sunt inter terminos effabiles æqualitate Arithmetica se invicem excedentes, inter se sunt incommensurabiles.

Nam termini extremi effabiles vel recipiunt effabilem mediam proportionalem quantitatem unam pluresve, vel non recipiunt. Si non recipiunt, nulla igitur effabili, & sic neque medio arithmetico dividitur eorum proportio in commensurabilia per 8 prop. Recipiant verò effabile medium proportionale, ut termini 8 & 18, recipiunt enim 12 effabilem, cum sint ut 4 ad 9, quadratus ad quadratum. Est verò inter 8, 18, medium Arithmeticum 13, ideo proportio 8, 13 major est quàm 8, 12 & 13, 18 minor, quàm 12, 18, quantitate utrinque parvæ proportionis inter terminos 12, 13, sed proportio 12, 13 nulli reliquarum est commensurabilis. Nam termini 8, 13, quia non sunt ad invicem, ut numerus figuratus ad alium ejusdem figurationis per 10 prop. non

capiunt mediam vel medias proportionales effabiles, quare numerus 12 non eſt unus ex iis numeris, qui inter 8 & 13 intercidunt in continua proportione: non eſt igitur commenſurabilis proportio 8, 13 proportioni 12, 13 vel 8, 12. Sic ex iiſdem fundamentis, quia termini 12, 18 non capiunt effabilem mediam proportionalem: ergo 12, 13 & 13, 18 ſunt incommenſurabiles; ad commenſurabilem igitur 8, 12 ipſi 8, 18 eſt appoſita incommenſurabilis 12, 13. Tota ergo 8, 13 eſt ipſi 8, 18 incommenſurabilis. Sic in 13, 18 de commenſurabili 12, 18 dempta eſt pars incommenſurabilis 12, 13, reſidua ergo 13, 18 eſt incommenſurabilis ipſi 12, 18. At neque proportio inter 8, 13 eſt commenſurabilis ipſi proportioni inter 13, 18, quia tota inter 8, 18 alterutri parti inter 8, 13 eſt incommenſurabilis, ergo & partes invicem per 16 decimi Eucl. Item ſi eſſent partes commenſurabiles, menſuram utraque communem haberet, atque illa etiam compoſitam inter 8, 18 emetiretur, & ſic partes cum tota commenſurabiles eſſent, atqui jam demonſtratum eſt, neutram partem toti eſſe commenſurabilem. Non ſunt igitur partes, quas hîc facit medium arithmeticum inter ſe commenſurabiles.

XII. PROPOSITIO.

Si quantitates quæcunque deinceps collocentur ordine magnitudinis, proportionis verò inter maximas menſura ſtatuatur differentia inter eas, differentia inter quaſcunque alias, ex poſitis, minor erit menſurâ ſuæ proportionis; & ſi proportionis inter minimas menſura ſtatuatur differentia minimarum: differentiæ reliquæ erunt majores menſurâ proportionis ſuorum terminorum.

Aut enim continuè proportionales ſunt quantitates collocatæ actu, vel poteſtate ſupplendi omiſſas, & tunc patet propoſitum per VI & per III. Aut non ſunt in proportione continua, ſic ut partes conſtituant incommenſurabiles: & tum conceptione mentis in infinitas particulas æquales ſecari intelligeretur per interpoſitas infinitas medias proportionales: ita redigentur cum iis, quæ actu ſunt continuè proportionales ad eandem vim demonſtrationis.

COROLLARIUM.

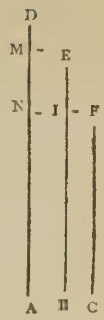
Quod ſi ſuperet menſura proportionis inter maximas, differentiam earum: hujus menſuræ ad hanc differentiam proportio minor erit quàm ſequentis menſuræ ad differentiam ſuam: cum proportionalium eadem ſit ratio.

XIII. PROPOSITIO.

Si quantitates tres ordine magnitudinis ſe inſequantur, proportio minimarum duarum in proportione extremarum continebitur rarius, quàm differentia minimarum in differentiâ extremarum: & viciſſim proportio maximarum in proportione extremarum continebitur ſæpius, quàm differentia illarum in differentiâ harum.

Con-

Contineatur enim in adjecto diagrammate differentia minimarum EI, vel MN, in differentia DN extremarum aliquoties licet non exactè: & capiat proportio minimarum BE, CF menſuram differentiam MN, erit igitur proportionis AD, CF menſura minor quàm differentia DN per 12; rarius igitur continebitur MN, in menſura ipſius AD, CF proportionis, quàm in ND longiore; rarius igitur & ipſa proportio in proportione. Viciſſim contineatur DM, differentia in DN aliquoties: & ſit DM menſura proportionis AD, BE. Erit proportionis AD, CF menſura major quàm DN, ſæpius igitur erit DM, in menſura ipſius AD, CF, quàm in DN. Ergo, &c.



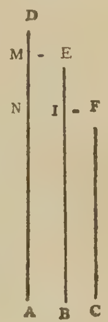
COROLLARIUM.

Differentia eadem inter terminos binos hinc majores, inde minores exiſtente: proportio inter majores minor erit, inter minores major.

XIV. PROPOSITIO.

Si quantitates tres ordinentur deinceps, æqualibus differentiis invicem excedentes: proportio inter extremas eſt major quàm dupla proportionis maximarum.

Sint tres quantitates, AD maxima, BE media, CF minima, & ſit exceſſus primæ ſuper ſecundam DM, æqualis exceſſui ſecundæ ſuper tertiam EI, vel MN. Dico proportionem inter AD, CF, majorem eſſe quàm duplam ipſius inter AD ad BE. Menſuretur enim proportio AD, BE per differentiam DM per 3 poſtul. Erit igitur menſura proportionis BE, CF major quàm differentia IE, vel NM per 12. Sed MN æquat DM menſuram proportionis inter AD, BE. Ergo menſura proportionis BE, CF eſt major menſura proportionis AD, BE; Et ſic ipſa proportio BE, CF major eſt proportione AD, BE. Sed proportio AD, CF componitur ex proportione AD, BE & ex proportione BE, CF per axioma 1. Ergo proportio AD, CF partes habet AD, BE, & eam majorem BE, CF, major igitur dupla ipſius AD, BE.



COROLLARIUM.

Hinc ſequitur, ſemiſſem proportionis extremarum eſſe majorem proportione maximarum, minorem proportione minimarum.

XV. PROPOSITIO.

Si duæ quantitates proportionem conſtituerint, dimidium verò quantitatis majoris dematur de quantitate utraque, reſiduæ quantitates proportionem conſtituent majorem duplâ prioris.

Sint

Sint quantitates 10, 9 & ablato dimidio ipſius 10, hoc eſt 5, ex utraque relinquantur 5, 4. Dico proportionem inter 5, 4 dupla ipſius 10, 9 majorem eſſe. Duplicentur enim 5, 4 fient 10, 8, eritque proportio eadem 5, 4 quæ & 10, 8. Differentia vero 10, 8, hoc eſt 2, dupla erit differentiæ 5, 4, hoc eſt differentiæ 10, 9, ſçilicet 1.

Si vero tres quantitates ordinentur 10, 9, 8, quarum prima 10 excedat tertiam 8, duplo ejus, quod excedit ſecundam 9, ſeu in quibus æquales ſunt exceſſus 10, 9 & 9, 8, proportio extremarum 10, 8 (id eſt 5, 4) maior eſt dupla ipſius 10, 9 maximarum per 14. Ergo.

XVI. PROPOSITIO.

Incommenſurabilium proportionum partes aliquotæ ſunt inter ſe incommenſurabiles.

Nam pars aliquota eſt ſui toti commenſurabilis, at tota illa perhibetur toti ſociæ incommenſurabilis, ergo & pars unius toti alteri erit incommenſurabilis per 14 decimi Eucl. & pro eadem, & parti aliquotæ alterius.

XVII. PROPOSITIO.

Si mille numeri invicem ſuccedant ordine naturali, differentes bini unitate, initio facto à maximo 1000; deinde proportio inter maximos 1000, 999, biſectione continuâ, ſecetur in partes minutiores, quàm eſt exceſſus proportionis inter proximos 999, 998, ſuper proportionem inter maximos 1000, 999: minimum verò illud elementum proportionis inter 1000, 999, capiat menſuram differentiam inter 1000 & proportionalem illam mediam, quæ alter elementi terminus eſt: ulterius ſi proportio inter 1000, 998 ſeorſim ſecetur in partes duplo plures quàm prior proportio inter 1000, 999, & hujus ſeparatæ diviſionis minimum elementum ſeorſim capiat menſuram, ſuorum terminorum (quorum alter ſit 1000) differentiam, eodemque modo quælibet proportio ipſius 1000 ad ſequentes numeros, ut 997, &c. biſectione continuâ ſecetur in particulas tantæ magnitudinis, ut verſentur inter ſeſquiplum & dodrantem elementi, quod emerſerat ex ſectione proportionis primæ inter 1000 & 999, ſinguliſque elementis menſura detur à ſuorum terminorum differentiâ, maximo exiſtente 1000, & ſi hoc facto, cuicunque ex mille proportionibus, menſura conſtituatur ex tanto numero menſurarum elementi ſui, in quot elementa ipſa diviſa fuit: proportiones omnes, ad omnem calculi ſubtilitatem, emendatas exactaſque habebunt menſuras.

Nam

Nam ſuccedant invicem numeri 1000, 999, 998, &c. ordine naturali, differentes unitate: erit inter maximas 1000, 999, minima proportio: major inter proximos 999, 998: hæc iterum major proximâ inter 998 & 997 ſic ſemper & hoc per 14 prop. Donec 500, 499 fiat major quàm dupla ipſius 1000, 999 per prop. 15. Dico ſecundò exceſſum ſecundæ ſuper primam fore minimum: ſic ut ſemper exceſſus ſequentis ſuper præcedentem ſit major priore exceſſu: ut quoties ſequens proportio duplo longius diſtiterit à prima, quàm aliqua præcedentium, toties exceſſus ſequentis ſuper primam amplius quàm duplo major ſit exceſſu præcedentis. Fiat enim ut 1000 ad 999, ſic 999 ad $998\frac{001}{1000}$. Igitur proportio 999 ad $998\frac{1}{1000}$ eſt eadem quæ 1000 ad 999. Aufer illam à proportione 999, 998, relinquitur exceſſus 998001, 998000.

Fiat etiam ut 1000 ad 999, ſic 998 ad $997\frac{002}{1000}$. Igitur proportio 998 ad $997\frac{2}{1000}$ eſt eadem quæ 1000 ad 999. Aufer illam à proportione 998, 997, relinquitur exceſſus, proportio inter 997002 & 997000. At in priori exceſſu proportionis 998001, 998000, termini differebant per 1, in hoc verò exceſſu termini 997002, 997000 differunt per 2. Atqui ſi æquales fuiſſent majores termini hîc & illîc, proportio ſequens, ubi dupla differentia prioris, fuiſſet major dupla ipſius per 14. Multò igitur major erit proportio, ubi etiam minor fuerit terminus, qui proportioni ſequenti eſt loco majoris ſcilicet 997002.

Lucis cauſa ſint numeri minores & pauciores, & qui etiam unitate differant, ut 10, 9, 8, 7, 6, 5, 4, 3, 2, 1. Dico exceſſum proportionis 8, 7 ſuper proport. 10, 9, amplius quàm duplo majorem eſſe quàm exceſſum proportionis 9, 8 ſuper eandem 10, 9. Reducatur enim ut prius prima proportio inter 10, 9 cum ſingulis ſequentium, quas bini deinceps numeri conſtituunt, reducantur, inquam, ad communem terminum maximum. Hic etiam exceſſuum termini differunt magis, magiſque, ut quia 72, 70 eſt loco ſecundo, differentia 2 (ut prioris 1, inter 81, 80, dupla) indicat exceſſum 72, 70 eſſe majorem duplo ipſius 81, 80. Sic 54, 50 loco quarto major duplo eſt ipſius 72, 70 loco ſecundo. Sic 18, 10 loco octavo major duplo ipſius 54, 50, indice differentia 8, dupla ipſius 4, inter terminos minores.

	Eſt	At proportio inter 10, 9 valet	Exceſſus illarum.
10, 9	100, 90	100, 90	
9, 8	90, 80	90, 81	81, 80
8, 7	80, 70	80, 72	72, 70
7, 6	70, 60	70, 63	63, 60
6, 5	60, 50	60, 54	54, 50
5, 4	50, 40	50, 45	45, 40
4, 3	40, 30	40, 36	36, 30
3, 2	30, 20	30, 27	27, 20
2, 1	20, 10	20, 18	18, 10

Cùm

Cùm igitur in priori exemplo mille numerorum, minimus exceſſus ſit 998001, 998000, facilè ſecamus proportionem primam 1000, 999 in partes æquales, minores, poſito exceſſu numero. Reducatur enim ille exceſſus ad terminum minorem dividendæ, fiat ſcilicet ut 998000 ad 998001, ſic 99900000000 ad 999001002004, &c. unde apparet differentiam terminorum fieri 1002004, &c. quæ eſt paulò major quàm milleſima pars de differentia terminorum proportionis primæ. Ergò per 13 non totupla pars proportionis 1000, 999 eſt in exceſſu proportionis ſequentis 999, 998.

Quæratur ergò medium proportionale inter 1000, 999. Id ſecabit proportionem terminorum in æquales duas partes, per 1 prop. quæratur ſecundò inter hanc inventam mediam, & 1000 alia media proportionalis, ut ita prior media intelligatur, circumdata duabus mediis proportionalibus aliis, quarum tamen unius ſolum, verſus terminum 1000 inveſtigatione opus eſt. Per has igitur tres medias (ut eſt in ax. 2.) ſecabitur proportio in partes 4. Sic proportionalis tertia inter prius inventam, & 1000 ſecabit in partes 8 quarta in 16, & ſic conſequenter (per poſtul. 2) in 32, 64, 128, 256, 512, 1024, quod fit actu decimo. Hic itaque numerus 1024 certò conſtituit particulas proportionis 1000, 999 minores ſupra inveſtigato exceſſu: quia proportio capit hujus exceſſus minus quàm mille; hic verò conſtituuntur mille viginti quatuor particulæ. Hic igitur particula milleſima viceſima quarta capiat loco menſuræ, differentiam termini ſui minoris, ſeu mediæ proportionalis decimæ à termino majore 1000 per 3 poſtulat.

Pergimus ad ſimilem ſectionem proportionis inter 1000, 998. Hæc igitur eſt incommenſurabilis priori 1000, 999. Nam per 11 prop. proportio 999, 998 eſt incommenſurabilis proportioni 1000, 999, qui termini bini differunt unitate æqualiter, ſed proportio 1000, 998 componitur ex illis inter ſe incommenſurabilibus per axiom. 1. Cum verò totum in incommenſurabilia ſecatur, ipſum ſingulis eſt incommenſurabile per 17 decimi Eucl.

Eſt verò hæc proportio 1000, 998 major quàm dupla prioris 1000, 999 exceſſu incommenſurabili, ut ſupra allegatione præmiſſæ 14 indicatum. Pars igitur ejus milleſima vigeſima quarta plus quàm duplo major eſt minimo elemento prioris; biſecetur igitur, ut totius fiant partes 2048 per inquiſitionem undecimæ proportionalis. Tunc ſanè elementum hoc ejus erit proximè æquale elemento minimo prioris, majus tamen illo etiamnum, & illi incommenſurabile per 16 præmiſſam.

Si ergò illud accepit menſuram, differentiam ſuorum terminorum, hoc jam elementum proportionis, quippe majus illo, menſuram habebit majorem differentiâ ſuorum terminorum, per 12 prop. Ac proinde ſi prioris elementi terminorum differentia multiplicetur 1024ies pro menſurâ proportionis inter 1000, 999 tunc poſterioris elementi terminorum differentia, multiplicata 2048ies, adhuc minor erit menſurâ proportionis inter 1000, 998.

Verumtamen ſi attendamus ad quantitatem hujus defectus, illa eſt omninò ſubtiliſſima, & nullâ calculi diligentiâ obſervabilis. Quia enim proportio inter 1000 & 999 ſecta eſt in particulas plus quàm mille, & elementi tam parvi menſura conſtituta eſt differentia terminorum 1000000, & 999999, ſcilicet 1, & minor adhuc certè proportio proxima inter 999999, 999998 major priori per 12 prop. menſuram habebit, quæ excedat terminorum differentiam primam vix

vix millies millesimâ sui; ac proinde composita proportio 1000000, 999998 major est dupla prioris vix millies millesimâ prioris particulâ. At jam elementum secundæ proportionis inter 1000, 998 in superioribus constitutum, nequaquam est majus duplo prioris proportionis elemento; sed ob idipsum factæ sunt partes non 1024, sed 2048, ut esset pars ista proximè æqualis priori, quomodo excedit illam rursum vix millesima parte illius, si ergò totum superadderet differentiæ suæ millies millesimam, ad constituendam proportionis mensuram, pars utique millesima totius non plus addet, quàm millies millesimæ prioris elementi millesimam.

Transeamus ad sectionem sequentis proportionis inter 1000, 997. Hæc verò est paulò major quàm tripla primæ inter 1000, 999. Quare si secetur in æquales 1024 particulas, particula talis etiam erit paulò major quàm tripla elementi primi. Sin ulterius illæ bisecentur per undecimam proportionalem, existent numero 2048, eritque unaquælibet major quàm sesquialtera primi elementi. Nam triplæ dimidia est ipsa sesquialtera totius. Quia ergò superant primum elementum plus quàm dimidio ipsius, bisecentur denuò constitutione duodecimæ proportionalis, ut fiant partes 4096: quælibet major dodrante prioris elementi, quia sesquialteri dimidium est dodrans. Satis igitur appropinquat elemento priori. Si igitur huic elemento proportionis tertiæ mensura seorsim ponatur, differentia terminorum ipsius peccabit quidem illa excessu, per 12 præmissam, at peccato nunquam æstimabili, ob causas in secundæ sectione dictas.

Hoc pacto, proportio inter 1000, 996, per sectionem in 4096, paulò majus sortitur elementum quàm est elementum primæ, & proportio inter 1000, 995 paulò majus quinque quantis primi elementi. At proportio inter 1000, 994 jam per tredecimam proportionalem, capere debet partes 8192 ut sint rursum paulò majores dodrante primi elementi. Et proportio inter 1000, 993 paulò majores habebit septem octavis primi, & proportio inter 1000, 992 rursum paulò majores primo elemento.

Hoc igitur tantisper obtinet, quoad numeri ordinis millenarii proximi invicem appropinquaverint magis ipsi 500 dimidio primi, quàm ipsi primo 1000.

Nam quia proportio inter 500, 499 major est quàm dupla proportionis inter 1000, 999 per 15 præmissam: Jam igitur excessus ipsius 500, 499 super ipsam 1000, 999, superat ipsam 1000, 999, quare in sectione proportionum, quæ proximè antecedunt proportionem inter 500 & 499 per unam superabundantem, bisectionem, vel etiam quadrisectionem, redigendum est elementum, emergens ad propinquitatem elementi prioris in propositione præfinitam. At in iis quæ sequuntur proportionem inter 501 & 500 sectione porrò non est opus. Nam quia proportio inter 500, 499 eadem est quæ inter 1000, 998, idcirco si fuerit notificata 1000, 998 & 1000, 500, noscetur etiam composita ex utraque 1000, 499 sine sectione laboriosa. Igitur inquisitio proportionalium desinit in proportione dupla, scilicet inter 1000 & 500.

XVIII. PROPOSITIO.

Cognitâ proportione numeri cujuscunque ad primum 1000: simul cognoscitur etiam numerorum reliquorum continuæ ejusdem proportionis, ad eundem primum 1000 proportio.

Nota sit mensura proportionis inter A & B. Et sit ut A ad B, sic B ad C, & C ad D, & D ad E. Erunt igitur æquales mensuræ proportionum harum singularum ei, quæ est primò notæ A ad B per I postul. Jam verò proportio A ad C componitur ex proportionibus A ad B, & B ad C, per I ax. quare & mensura proportionis AC componetur ex duarum proportionum AB & BC mensuris. Id est, mensura ipsius AB duplicata, dat mensuram ipsius AC, triplicata ipsius AD, quadruplicata ipsius AE.

Hoc pacto cognitâ proportione inter 1000, 900, cognoscitur etiam ipsius 1000 ad 810, & ad 729.

Et ex 1000 ad 800, etiam 1000 ad 640 & ad 512.
Et ex 1000 ad 700, etiam 1000 ad 490 & ad 343.
Et ex 1000 ad 600, etiam 1000 ad 360 & ad 216.
Et ex 1000 ad 500, etiam 1000 ad 250 & ad 125.

COROLLARIUM.

Hinc oritur præceptum quadrandi, cubicè multiplicandi, &c. & vicissim radicem quadratam, cubicam, &c. extrahendi in primis numerorum figuris. Est enim ut maximus chiliadis tanquam denominator ad numerum propositum tanquam numeratorem, sic hic ad fractionis quadratum, & hoc ad cubum.

XIX. PROPOSITIO.

Cognitâ proportione numeri ad primum 1000; si duo alii in eâdem inter se proportione fuerint; eorum unius proportione ad 1000 cognitâ, noscetur etiam reliqui proportio ad eundem 1000.

Sit A 1000, & nota mensura proportionis A ad B. Sit verò ut A ad B, sic C ad D, & sit nota mensura proportionis A ad C. Dico etiam innotescere mensuram proportionis A ad D. Quia enim nota est mensura ipsius AB proportionis, nota etiam erit ipsius CD proportionis, ut quæ illi ponitur æqualis, per I postul. Nota verò est etiam AC, & AD est composita ex AC & CD, per I ax. quare etiam mensura ipsius AD componetur ex mensurâ ipsius AC, ut ex mensurâ ipsius CD, id est, ipsius AB.

COROL-

COROLLARIUM I.

Hoc pacto, ex notificatis proportionibus quindecim, prop. 18 præmiſſis, noſcentur aliæ centum viginti numerorum intra millenarium, ad ipſum millenarium.

Quoties enim datæ fuerint proportiones ad 1000 duorum numerorum talium, in quibus vel ambobus, duos ultimos locos habuerint cyphræ, ut 900, 800, vel in eorum altero quidem duos, in reliquo verò unum, ut 700, 810, velut 700, 10. Vel ſi unum ſolum, eumque ultimum in utroque numero locum ex tribus obtinuerit cyphra: quæ tamen hanc cyphram antecedit figura in altero numero, ex paribus in reliquo quinarius fuerit, ut 620, 950 ſic 620, 50 ſic 20, 950, vel ſi alter quidem cyphrâ in ultimo loco caruerit, ex paribus tamen fuerit ut 512, 12 vel 2 reliquus fuerit 500. Omnibus hiſce caſibus inſtitutâ multiplicatione numerorum, proveniunt in fine tres cyphræ, quibus abjectis formatur numerus, unus ex mille ordinis naturalis, ſeu progreſſionis arithmeticæ.

COROLLARIUM II.

Hinc oritur præceptum tractandi regulam trium, quando uno loco occurrit rotundus 1000.

Nam ſi ille occurrit primo loco in tali ſitu:
A 1000 dat B, quid C?

Tunc additur menſura proportionis AB ad menſuram proportionis AC, ita fit menſura proportionis AD.

Sin autem 1000 occurrat loco ſecundo vel tertio, in tali ſitu B dat A 1000, quid C? vel tali B dat C, quid A 1000?

Tunc aufertur menſura proportionis AB à menſura proportionis AC, vel ejus multiplicis proximè majoris, ità relinquitur menſura vel ipſius proportionis AD, vel ejus æquè multiplicis.

XX. PROPOSITIO.

Quando fuerint ut primus ad ſecundum, ſic tertius ad quartum, notæ verò fuerint proportiones ipſius 1000 ad tres priores, innoteſcet etiam proportio ejuſdem 1000 ad quartum.

Sit enim A 1000, & ſit ut B ad C, ſic D ad E. Notæ verò ſint proportiones AB, AC, AD; Dico innoteſcere etiam proportionem AE. Nam quia ut B ad C, ſic D ad E, æqualis eſt igitur menſura proportionis BC, menſuræ proportionis DE. Sed BE eſt compoſita ex BD, DE per 1 axiom. Æqualis igitur erit proportio BE, proportionibus BD, BC ſimul ſumptis. Sed & AE eſt compoſita ex AB, BE: ſic AD ex AB, BD, ſic AC ex AB, BC per 1 axioma. Si ergò pro BD, BC, ſumantur proportiones notæ AD, AC, tunc AB bis acceſſit. Viciſſim ſi pro BE ſumatur AE proportio quæſita, tunc AB ſemel tantum acceſſit. Si ergò à junctis AD, AC notis, abſtuleris AB notam ſemel, relinquitur AE proportio quæſita.

COROLLARIUM I.

Hâc methodo præterquam quod ſuperiorum chiliadis multæ iteratò exquiruntur, accedunt illis inſuper aliquot aliæ.

COROLLARIUM II.

Hinc oritur præceptum tractandi regulam trium, quando nuſpiam occurrit rotundus 1000: ut ſi ſic collocentur tres, B dat C, quid D?

Nam additur menſura proportionis AC, menſuræ proportionis AD, & à ſummâ aufertur menſura proportionis AB, vel ejus aliqua pars aliquota; ita relinquitur menſura proportionis AE, vel ejus æquè multiplex.

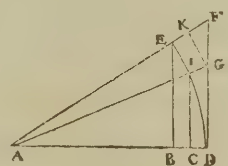
DEFINITIO.

Menſura cujuſlibet proportionis inter 1000 & numerum eo minorem, ut eſt definita in ſuperioribus, expreſſa numero, apponatur ad hunc numerum minorem in Chiliade, dicaturque LOGARITHMUS ejus, hoc eſt, numerus (ἀριθμὸς) indicans proportionem (λόγον) quam habet ad 1000 numerus ille cui logarithmus apponitur.

XXI. PROPOSITIO.

Si primus numerus ſit ſemidiameter circuli ſeu ſinus totus: omnis numerus minor, ut ſinus complementi alicujus arcus, logarithmum habet majorem ſagittâ arcus, minorem verò exceſſu ſecantis arcus ſupra radium ſeu ſemidiametrum, excepto unico proximo poſt ſemidiametrum, quia illius logarithmus ex hypotheſi eſt æqualis ſagittæ.

Sit A centrum circuli, AD ſemidiameter, DI, DE arcus, eorumque ſinus IC, EB, ſinus verò complementorum ſint CA, BA, ſagittæ CD, BD. Sit autem ut AD ad AC, ſic AC ad AB. Amplius ſint eorundem arcuum ſecantes AG, AF, per terminos I, E, in tangentes DG, DF educti, & ipſi AG æqualis abſcindatur ab AF, quæ ſit AK; denique ſit CD menſura proportionis CA, AD, ut minimi elementi arbitrarii. Dico menſuram proportionis BA ad AD, hoc eſt, logarithmum ipſius BA, majorem eſſe quàm BD, minorem verò quàm EF. Quod major ſit quàm BD, demonſtratum eſt ſuprà propoſitione duodecima. Quod verò minores ſint menſuræ proportionum harum IG, CF ſic probatur. Primum de CA, cum ipſa CD ſagitta, utpote



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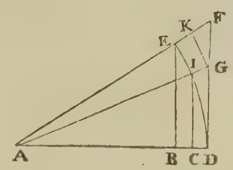
in minimo proportionum elemento ponatur esse logarithmus ipsius CA : CD sanè est minor quàm IG. Ut enim CA ad AD, sic IA, hoc est DA, ad AG, quia DG & CI parallelæ. Est igitur AD media proportionalis inter CA & AG. Ut igitur CA ad AD, sic differentia CA, AD, hoc est, CD, ad differentiam sequentem DA, AG, hoc est, ad IG. Sed CA est minor quàm AD, ergò & CD est minor quàm IG. Sed CD est logarithmus ipsius CA sinus complementi arcus ID, & IG est excessus secantis ejusdem arcus. Ergò logarithmus minor est hoc excessu.

Transeamus ad BA : cujus logarithmus major est quàm BD, demonstrandum est, illum non esse tantò majorem ipsâ BD, quin interim maneat minor ipso EF. Rursum igitur AD est media proportionalis inter BA & AF. Et quia posita est ut BA ad AC, sic CA ad AD : quare etiam ut EA ad AG, vel AK, sic KA vel GA ad AF. Sunt igitur continuè proportionales istæ AB, AC, AD, AK vel AG & AF. In eâdem igitur proportione sunt etiam BC, CD, IG, vel EK & KF. Minor verò est CD quàm IG, ut prius ostensum, minor igitur erit etiam IG vel EK quàm KE. Tota igitur EF major est quàm dupla ipsius IG, multò magis igitur EF major erit quàm dupla ipsius CD minoris. At proportionis inter BA, AD, ut quæ dupla est ipsius BA, AC per I propositionem, mensura seu logarithmus ipsius BA, est præcisè duplus ipsius CD per I postul. Minor est ergo logarithmus ipsius BA excessu secantis EF. Erat autem major sagittâ BD. Patet ergo propositum.

XXII. PROPOSITIO.

Iisdem positis, sagitta arcus cum excessu secantis superat duplum logarithmi, ad sinum complementi apponendi.

Sit enim primò Sinus complementi longissimus, aut longissima mediarum proportionalium, quibus aliqua proportio dividitur in partes arbitrario numero multas ; sic ut ejus, verbi causa, AC residuum CD, seu sagitta arcus ID sit ipsissima mensura arbitraria proportionis CD, logarithmus igitur ipsius CA est CD. At IG est major ipsâ CD. Juncti igitur excessus secantis IG & sagitta CD, plus efficiunt quàm duplum ipsius CD.



Sit deinde alia quæcunque minor linea proportionis continuæ, ut AB, & educta ex B perpendiculari in circumferentiam E, connexisque AE, & DG continuatis in F, sit EF excessus secantis, & BD sagitta ejusdem scilicet arcus ED. Dico junctos EF & BD, facere plus quàm duplum logarithmi ad BA apponendi, seu mensuræ ipsius BA, AD proportionis.

Quia est ut BA ad AD, sic DA ad AF, & CA media proportionalis inter BA, AD, ut igitur BA ad AC, sic GA ad AF, quare per 25 quinti Eucl. BA, AF junctæ sunt longiores junctis CA, AG, sive quia ut BA ad AD, sic DA ad AF, quare

BA,

BA, AF junctæ, ſuperant DA, AD duas medias. Ut verò BA ad AC, ſic etiam BC ad CD, & IG ad KF per 2 prop. quare etiam BC & KF junctæ, ſuperant CD & IG junctas. At CD, IG plus ſunt quàm duplum ipſius CD. Ergò BC KF junctæ, multò plus ſunt quàm duplum ipſius CD. Et ſic BD, EF plus ſunt quàm quadruplum ipſius CD. Sed duplum ipſius CD eſt logarithmus ſeu menſura proportionis BA, AD per 1 poſtul. Ergò BD, EF junctæ, plus ſunt quàm duplum logarithmi ipſius BA, ſeu menſuræ proportionis BA, AD.

Sit tertiò proportio BA, AC minor vel major quàm CA, AD, demonſtrabitur nihilominus quod junctæ BC, KF, ſuperent duplum tantæ partitionis de CD, quanta portio eſt proportio BA, AC, proportionis CA, AD.

COROLLARIUM.

Logarithmus Sinus complementi eſt minor medio arithmetico inter ſagittam & exceſſum ſecantis.

PRÆCEPTUM.

Sinus inventi in Canone Sinuum reſiduum ad totum adde exceſſui ſecantis complementi, ſummæ dimidium ſuperat logarithmum, ſagitta ipſa proximè minor eſt logarithmo.

Eſto Sinus	99970,1490	arcus
Ejus reſiduum ad ſin. totum	29,8510	ſagitta arcus complementi minor logarithmo
	29,8599	exceſſus ſecantis
	59,7109	ſumma
	29,8555	dimidium majus logarithmo.

Ergò logarithmus eſt inter $\begin{cases} 29{,}8510 \\ 29{,}8555. \end{cases}$

PRÆCEPTUM ALIUD.

Invento ſinus logarithmo, invenies etiam proximè logarithmum numeri rotundi, qui ſinu tuo ſcrupulato proximè minor eſt, ſi ſinus ſcrupuloſi exceſſum ſupra numerum rotundum adjeceris logarithmo ſinus invento.

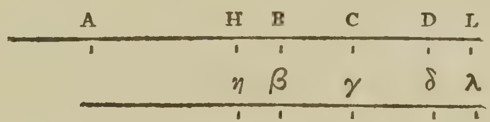
Ut quia ſinus 99970,149, logarithmus inventus eſt 29,854 circiter, ſi jam vis ſcire logarithmum rotundi 99970,000, vides exceſſum tui ſinus

ſcrupuloſi eſſe 149, hunc adde ad inventum ſinus logarithmum, obſervato puncto. Sic 29,854
149

30,003. Hic eſt logarithmus rotundi numeri 99970,000 proximè.

XXIII. PROPOSITIO.

Si tres quantitates invicem ſucceſſerint, æqualibus exceſſibus differentes, menſura proportionis inter maximam & mediam, cum menſurâ alterius inter mediam & minimam conſtituet proportionem, majorem quidem proportione majorum, minorem verò proportione minorum.



Sint tres quantitates AD, AC, AH, æqualibus exceſſibus DC, CH; ſit autem menſura proportionis DA, AC linea $\delta\gamma$, menſura verò proportionis CA, AH, linea $\gamma\eta$. Dico proportionem ipſius $\delta\gamma$ ad $\gamma\eta$ majorem quidem eſſe proportione ipſius CA ad AD, minorem verò proportione ipſius HA ad AC. Fiat enim ut DA ad AC, ſic CA ad AB. Ut igitur DA ad AC, ſic DC ad CB per 2 prop. Sed longior eſt DA quàm AC, longior igitur DC quam CB, longior igitur & CH quàm CB, differentia BH.

Cum igitur æquales ſint proportiones DA, AC & CA, AB, per 1 prop. menſura verò ipſius DA, AC ſit linea $\delta\gamma$, habebit & CA, AB menſuram æqualem ipſi $\delta\gamma$, per 1 poſtul. Et cum proportio CA, AH ſit compoſita ex proportione CA, AB & proportione BA, AH. Major igitur erit proportio CA, AH, quàm proportio DA, AC, æqualis ipſi CA, AB. Major igitur etiam menſura ejus $\gamma\eta$ quàm $\gamma\delta$. Abſcindatur à $\gamma\eta$ æqualis ipſi $\gamma\delta$, quæ ſit $\gamma\beta$, reſidua igitur $\beta\eta$, menſura erit reſiduæ proportionis BA, AH. Erit autem proportio $\gamma\beta$ ad CB minor quam $\beta\eta$ ad BH per 12 coroll. id eſt, major eſt $\beta\eta$ reſpectu $\beta\gamma$ vel $\gamma\delta$, quam HB reſpectu BC. Compoſitis igitur terminis, illic $\gamma\beta$ & $\beta\eta$ in $\gamma\eta$, hic CB & BH in CH, major erit $\gamma\eta$ reſpectu $\gamma\beta$, vel ejus æqualis $\gamma\delta$, quam CH reſpectu CB. Major igitur eſt proportio inter $\delta\gamma$, $\gamma\eta$, quam inter BC, CH, id eſt, CD.

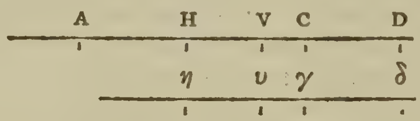
Ut verò BC ad CD, ſic CA ad AD per 2 prop. Major igitur proportio inter $\delta\gamma$, $\gamma\eta$, quam inter CA, AD. Sed $\delta\gamma$ & $\gamma\eta$ ſunt menſuræ, illa quidem proportionis inter DA, AC majores, hæc verò proportionis inter CA, AH minores. Ergò proportio menſurarum major eſt proportione terminorum minorum.

Rurſum

Rurſum fiat ut HA ad AC, ſic CA ad AL. Ut igitur HA ad AC, ſic HC ad CL, per 2. Sed brevior eſt HA quam AC, brevior igitur HC, hoc eſt DC, quam CL, differentia DL. Cum igitur æquales ſint proportiones HA, AC, & CA, AL, per 1 prop. Menſura verò ipſius HA, AC, ſic linea $\eta\gamma$, habebit & CA, AL menſuram æqualem ipſi $\eta\gamma$, per 1 poſtul. Eſto $\gamma\lambda$. Minor verò erat proportionis CA, AD menſura, puta $\gamma\delta$, minor igitur eſt $\gamma\delta$ quam $\gamma\lambda$. Exceſſus igitur $\delta\lambda$ erit menſura proportionis inter DA, AL, appoſitæ ad proportionem inter CA, AD. Et quia termini DA, AL, ſunt longiores quam CA, AD, quare per 12 coroll. minor eſt $\delta\lambda$ reſpectu $\lambda\gamma$, vel $\gamma\eta$, quam DL reſpectu LC. Major igitur reſidua $\gamma\delta$ reſpectu $\gamma\lambda$, vel $\gamma\eta$, quam CD vel HC reſpectu CL. Et quia proportio minuitur cuncto minori termino, per ax. 1. Minor igitur eſt proportio inter $\delta\gamma$, $\gamma\eta$, quam inter HC, CL. Ut vero HC ad CL, ſic termini trium minores HA ad AC, per 2 converſam. Minor eſt igitur proportio inter $\delta\gamma$, $\gamma\eta$ menſuras proportionum, quarum unam facit major terminus cum medio, alteram medius cum minimo, quam inter terminos minores.

XXIV. PROPOSITIO.

Dicta proportio inter duas menſuras, eſt minor dimidiâ proportione inter terminos extremos.



Sint enim termini extremi, ut prius AH, AD, media duo, arithmeticum AC, geometricum AV, ſit proportionis DA, AC menſura $\delta\gamma$, vel æqualis ipſi DC, vel etiam major per poſtul. 3. & applicetur ipſi $\delta\gamma$, alia $\gamma\upsilon$, quæ ſit menſura juſta proportionis CA, AV, ut ſic tota $\delta\upsilon$ menſuret proportionem DA, AV, reſiduæ igitur proportionis VA, AH menſura, erit priori $\delta\upsilon$ æqualis, per 1 poſtul. & 1 prop. ſit ea $\upsilon\eta$: Quia igitur $\delta\gamma$ eſt vel æqualis, vel major quam DC, ſequentis $\gamma\upsilon$ proportio ad CV, erit major quàm proportio ipſius $\gamma\delta$ ad CD, & tertiæ $\upsilon\eta$ proportio ad VH rurſum erit major quàm ſecundæ $\gamma\upsilon$ ad ſecundam CV. Et per compoſitionem totius $\eta\delta$ ad totam HD major erit proportio quàm partis $\upsilon\gamma$ ad partem VC. Permutatim igitur major erit proportio totius $\eta\delta$ termini majoris in priori, ad $\upsilon\gamma$ majorem, in poſteriori, quàm totius HD termini minoris, in priori proportione ad VC terminum minorem in poſteriori. Sed tota $\eta\delta$ conſtat ex terminis $\eta\gamma$, $\gamma\delta$, quorum differentia $\gamma\upsilon$, ſimiliter DH conſtat ex terminis DV, VH, quorum differentia VC: Major igitur eſt proportio terminorum $\eta\gamma$, $\gamma\delta$, junctarum, ad ſuam differentiam $\gamma\upsilon$, quàm terminorum DV, VH, ad differentiam VC. At verò auctâ proportione ſummæ terminorum, ad ſuam differentiam, minuitur ipſorum inter ſe terminorum ſeorſim poſitorum proportio, per coroll. 4, 13 & communem notitiam, quod proportionalium eadem ſit ratio: Minor eſt itaque proportio inter $\eta\gamma$, $\gamma\delta$, quàm inter DV, VH.

Sed

Sed proportio DV ad VH eſt æqualis proportioni DA ad AV per 2. Hæc verò DA ad AV proportio eſt dimidia ipſius DA ad AH per 1 prop. Minor eſt ergò proportio inter $\eta\gamma$, $\gamma\delta$, menſuras proportionum HA, AC & CA, AD, quàm dimidia inter terminos HA, AD.

EXEMPLUM HIC EST.

Numeri.	Numerorum differentiæ.	Logarithmi.	Logarithmorum differentia, ſeu menſuræ proportio.
1000		00	
750	---250	28768,21	28768,21
500	---250	69314,72	40546,51

Si verò fiat ut 1000 ad medium proportionale inter terminos extremos 1000 & 500 (id eſt, ad 70710,68), ſic menſura proportionis prioris, quæ eſt 28768,21 ad aliquem; is prodibit 40684,40, major quàm menſura proportionis poſterioris.

Si verò fiat ut medium proportionale inter 1000 & 360 (id eſt, ut 600) ad 1000, ſic menſura proportionis poſterioris 63598,86 ad aliquem, is prodibit 38159,32, minor quàm menſura proportionis prioris inter 1000 & 600.

ALIUD EXEMPLUM.

1000		0	
680	320	38566,25	38566,25
360	320	102165,11	63598,86

COROLLARIUM.

Cum igitur medium arithmeticum diſpeſcat proportionem in partes inæquales, quarum una major eſt ſemiſſe totius, altera minor, ſi quæratur, quæ ergò ſit ipſarum proportionum proportio inter ſe, reſpondetur, quod ea ſit paulò minor ſemiſſe dicto.

Exemplum inquirendi proxime majus, & proximè minus aliquid, menſurâ proportionis propoſitæ.

Nota ſit menſura proportionis inter 1000, 900, quæ ſit 10536,05, quæritur menſura proportionis 900, 800, ut ſint æquales differentiæ inter 1000, 900 & inter 900, 800.

Eſt igitur ut 9 ad 8 ſic 10536,05 ad 11706,72. Menſura verò proportionis 9, 8 eſt major. Rurſum ut medium proportionale inter 8, 10 (quod eſt 89,442719) ad 10, ſic 10536,05, ad 11779,66. At menſura proportionis inter 9 & 8 eſt minor, ſcilicet 11778,30.

NOTITIA COMMUNIS.

Omnis Numerus quantitatem exprimit effabilem.

XXV. PROPOSITIO.

Si numeri mille ſuccedant invicem, ordine naturali, bini differentes unitate; ſuſcipiantur verò bini quicunque, deinceps ordinati, tanquam termini proportionis alicujus; erit hujus proportionis menſura, ad menſuram proportionis inter duos maximos Chiliadis, in proportione majore quidem quàm quantam habet maximus ipſe 1000 ad majorem ex terminis ſuſceptis; minore verò quàm quantam habet idem 1000 ad minorem ex ſuſceptis. Minore etiam, quàm quantam habet 1000 ad medium proportionale inter ſuſceptos.

Suſcipiantur enim ex mille duo quicunque deinceps, puta 501 & 500. Et ſit logarithmus prioris 69114,92, poſterioris 69314,72. Horum logarithmorum differentia eſt 199,80. Quare per definitionem proportionis inter 501 & 500 menſura eſt 199,80. Eodem modo, quia maximi 1000 logarithmus eſt 0; proximi verò 999 logarithmus eſt 100,05, & horum duorum logarithmorum differentia itidem 100,05, menſura igitur proportionis inter 1000, 900 (vel inter 100000,00 & 99900,00) eſt 100,05; copuletur jam maximus 1000 cum utroque ſuſceptorum, ſcilicet & cum 501, & cum 500, copuletur etiam menſura 199,80 cum menſurâ 100,05, dico proportione inter 1000 & 501 majorem eſſe proportionem inter 199,80 & 100,05, minorem verò eandem proportione inter 1000, 500.

Demonſtratum eſt enim in prop. 23, menſuras duarum proportionum deinceps ordinatarum, comprehendere proportionem majorem minore earum, quæ ſint deinceps. Ut ſi ſint tres termini deinceps 1000, 999, 998 quorum priores quidem 1000, 999 faciant proportionem præcedentem, poſteriores 999, 998 ſequentem, præcedentis quidem proportionis menſura, ut prius erit 100,05, ſequentis verò menſura erit 100,15. Hæ duæ menſuræ 100,15 & 100,05, conſtituunt proportionem majorem, quàm termini 1000 & 999.

Jam verò duæ menſuræ, altera quidem proportionis inter 1000 & 999 hoc eſt, 100,05 altera verò proportionis inter 501 & 500, hoc eſt, 199,80; hæ, inquam,

inquam, duæ menſuræ, ut termini conſideratæ proportionem inter ſe conſtituunt compoſitam ex proportionibus omnibus, omnium binarum, & binarum deinceps menſurarum, intercedentium per axiom. 1. Similiter verò etiam termini ipſi 1000 & 501 proportionem inter ſe conſtituunt compoſitam ex totidem, hoc eſt, omnibus proportionibus omnium binorum, & binorum deinceps numerorum inter 1000 & 501 intercedentem per idem ax. 1.

Compoſitorum verò ex partibus æquè multiplicibus proportionalibus, eadem eſt proportio, quæ partium inter ſe ſingularum, hinc & inde combinatarum per 1. quinti Eucl. Ergò etiam hæ non deinceps ſitæ, ſed longè diſtantes menſuræ 100,05 & 199,80 proportionem facient majorem quàm termini non deinceps ſiti, ſed diſtantes longè, ſcilicet 1000 & 501. Eodem tenore rurſum incipiendo ſyllogiſmum eundem inferemus duas menſuras deinceps 00,15, & 100,05, conſtituere proportionem minorem quàm duos terminos deinceps 999 & 998. Menſurarum verò 100,05 & 199,80 proportionem componi ex omnibus interjectis, & ſimiliter terminorum 999 & 500 proportionem componi ex totidem interjectis, quare etiam proportionem inter 100 & 999,80 minorem eſſe quàm inter 999 & 500. Multò igitur minorem quàm inter 1000 & 500, ut quæ ad proportionem 999, 500, addit proportionem 1000, 999.

Tertio per eadem ſyllogiſmi veſtigia euntes, ſic colligemus ex prop. 24 præmiſsâ.

Menſurarum deinceps ſitarum, ſcilicet 1000,15 & 1000,05, proportio eſt minor quàm ea quæ eſt inter 1000 & medium proportionale terminorum 1000, 998, vel quàm ea quæ eſt inter hoc medium proportionale terminorum 1000, 998 & terminum 998. Menſurarum vero non deinceps, ut 100,05 & 199,80, proportio componitur ex interjectis omnium binarum, & binarum deinceps proportionalibus; & terminorum, quorum unus eſt medium proportionale inter 1000 & 998, id eſt, $\sqrt{998000}$ alter 500: vel quod idem eſt unus 1000 & alter medium proportionale inter 501 & 500, ſeu $\sqrt{250500}$; hæc, inquam, proportio componitur ex proportionibus, quas conſtituunt totidem interjectæ binorum, & binorum numerorum mediæ proportionales lineæ, accenſito termino 1000. Quare etiam menſuræ non deinceps ſitæ, ſcilicet 100,05 & 199,80 proportionem, exhibent minorem quàm 1000, cum medio proportionali inter ſuſceptos 501 & 500.

COROLLARIUM I.

Propoſito numero quocunque infra 1000, ejuſque logarithmo, quæcunque differentiæ logarithmorum antecedunt propoſitum, verſus initium chiliadis, ſunt ad primum logarithmum (qui ſcilicet ad 999 apponitur) in proportione majore, quàm 1000 ad propoſitum quæcunque ſequuntur verſus ultimum logarithmum, ſunt ad eum in proportione minore.

COROLLARIUM II.

Hoc adjumento facilè implentur loca chiliadis, quæ per ſuperiores propoſitiones nondum ſunt ſortita ſuos logarithmos.

XXVI. PROPOSITIO.

Differentia binorum logarithmorum, qui ſunt adſcripti ad numeros deinceps, eſt ad eorundem numerorum differentiam, in proportione majori quidem quàm eſt 1000 ad numerorum majorem; minore verò quàm idem 1000 eſt ad numerorum minorem.

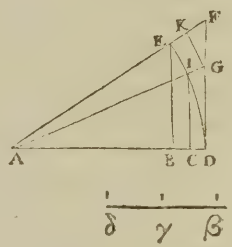
Facilè demonſtratur per antecedentem ejuſque corollarium. Nam ex unâ parte differentia duorum deinceps numerorum perpetuò eſt unitas (ſeu in prolongatis perpetuò 100,00.) Jam verò ultimus logarithmus eſt 100,05, idem, qui & menſuræ ultimæ & minimæ proportionis. Nulla igitur differentia numerorum deinceps differt à menſurâ ultimæ proportionis plus quàm quinque unitatibus numerorum prolongatorum. Ex alterâ verò parte menſura proportionis duorum deinceps numerorum nihil eſt aliud, quàm differentia duorum ad illos numeros poſitorum logarithmorum, per definitionem. Si igitur inter menſuram ſuſceptæ proportionis, & inter menſuram proportionis ultimæ eſt major proportio, quàm inter 1000 & terminum ſuſceptæ majorem, erit etiam inter differentiam logarithmorum deinceps, & inter differentiam numerorum, ad quos ſunt logarithmi, proportio major quàm inter 1000 & terminum ſuſceptæ majorem. Nam quod majori majus eſt, ipſius quoque multo eſt majus. Sed proportio inter differentiam logarithmorum, deinceps (verbi cauſa) inter 199,80 & 1000,00 differentiam perpetuam numerorum deinceps, eſt major quàm proportio inter 199,80 & 100,05; exceſſus enim eſt proportio inter 100,05, & 100,00. Et inter 199,80 & 100,05 proportio major fuit demonſtrata in præcedenti. Ergò multò eſt major proportio inter 199,80 & 100,00 quàm inter 1000 & (verbi cauſa) 501 terminum ſuſceptæ majorem. Eſt verò eadem proportio inter 199,80 differentiam logarithmorum, & 100,00 differentiam numerorum, etiam minor quàm proportio inter 1000 & 500, terminum ſuſceptæ minorem, quod ſic probo.

Proportio inter 199,80 & 100,05, eſt minor quàm proportio inter 999 & 500, per demonſtrata 25 præcedente; ſimiliter verò etiam proportio inter 100,05 & 100,00, ſeu inter 20,01 & 20,00, eſt minor quàm proportio inter 1000 & 999 per coroll. ad 13, quia ſcilicet eadem eſt differentia numerorum majorum 2001 & 2000, quæ minorum 1000 & 999, utrobique ſcilicet unitas.

Componitur verò proportio inter 199,80 & 100,00 ex utrâque ſuæ ſocietatis minore, ſcilicet ex prop. 199,80 ad 100,05 & ex prop. 100,05, ad 100,00. Sic etiam proportio 1000 ad 500 componitur ex utrâque ſuæ ſocietatis majore, ſcilicet ex prop. 999 ad 500, & ex prop. 1000 ad 999. Ergò etiam ipſa compoſita prior erit minor, & compoſita poſterior erit major.

XXVII. PROPOSITIO.

Si numeri ſuccedant invicem ordine naturali, bini deinceps differentes unitate : ad ſingulos verò apponantur logarithmi indices, ſeu menſuræ proportionum, quas conſtituunt abſoluti illi & rotundi numeri cum eorum maximo 1000; incrementa ſeu differentiæ horum logarithmorum, ſe habent ad logarithmum elementi minimi proportionum, ſicut ſecantes ipſi toti arcuum, quorum complementis abſoluti bini numeri ut ſinus competunt, ſeſe habent ad numerum maximum, ſeu radium circuli : ſic ut ex duobus ſecantibus duorum numerorum, inter quorum logarithmos differentia proponitur, minor quidem minorem conſtituat proportionem cum radio, quàm differentia propoſita cum omnium primâ, major majorem, atque etiam medium proportionale inter ſecantes majorem itidem.



In ſchemate prop. 21 ſint æquales DC, CB differentiæ numerorum abſolutorum CA, BA, quorum maximus DA. Et quia DC, CB æquales, major igitur eſt proportio BA, AC terminorum minorum, minor CA, AD majorum per corollar. ad 13. Major igitur & menſura proportionis BA, AC, quàm proportionis CA, AD, hoc eſt differentia logarithmorum ipſis CA & BA abſolutus reſpondentium, eſt major primo logarithmo per CD repræſentato. Sit $\delta\gamma$ logarithmus ipſius CA, & $\delta\beta$ in eaſdem lineæ partes, ſit logarithmus ipſius BA, & reſpondeat ipſi CA, ſecans GA, & ipſi BA, ſecans FA. Dico proportionem $\beta\gamma$ ad $\gamma\delta$, majorem eſſe quàm proportionem GA ad AD, minorem verò quàm FA ad AD, minorem etiam quàm medium proportionale inter FA & GA. Nam per 25 præced. major eſt proportio DA ad AB quàm $\beta\gamma$ ad $\gamma\delta$, major etiam DA ad medium inter BA, DA : ſed FA ad AD proportio æqualis eſt proportioni DA ad AB, quia DA eſt medium proportionale inter BA, AF : ſic etiam medii geometrici inter FA, GA ad DA proportio æqualis eſt proportioni DA, ad medium geometricum inter DA, AB. Major igitur etiam FA ad AD, major etiam medii proportionalis inter FA, AG ad AD quàm $\beta\gamma$ ad $\gamma\delta$. Sic per eandem minor eſt DA ad AC, & ſic etiam GA ad AD quàm $\beta\gamma$ ad $\gamma\delta$.

COROLLARIUM I.

Idem obtinet etiam tunc, ſi duo termini differunt, non ſolâ unitate elementi minimi, ſed aliâ unitate, quæ ſit illius decupla, centupla, millecupla.

COROLLARIUM II.

Hinc differentiæ ſatis juſtæ, præſertim ubi abſoluti numeri ſatis magni ſunt, extrui poſſunt, ſumpto medio arithmetico inter duos ſecantes parvos, vel etiam

(ſi

(si placet labor) medio geometrico inter secantes majores, exque differentiis continuè additis, accumulari logarithmi.

COROLLARIUM III. PRÆCEPTUM.

Divide sinum totum per utrumque proportionis susceptæ terminum, quotientis utriusque medium arithmeticum est quæsitum incrementum, hoc adde ad logarithmum termini majoris, prodit logarithmus termini minoris.

EXEMPLUM.

Propius & omninò proximum vero est medium geometricum, at non est pretium operæ tam operosæ, quando unus solus vel alter logarithmus est constituendus.

Datus esto logarithmus ad 700, scilicet 35667,4948, quæritur logarithmus ad 699. Divide ergò radium per 700, prodit 142857,142857,142857,142857, &c. divide etiam per 699, prodit 1430672, &c.

Medium arithmeticum est 142,962
Adde hoc ad 35667,4948

Prodit logarithmus ad 699.....35810,4568.

COROLLARIUM IV. PRÆCEPTUM DE LOGARITHMIS SINUUM.

Incrementum logarithmorum inter duos sinus sic inquires: inter secantes complementorum constituatur medium geometricum, dividaturque per differentiam sinuum, prodit differentia logarithmorum.

EXEMPLUM.

Sit sin. gr. 0 1 29 09, sec. compl. 343774682
0 2 58 18, sec. compl. 171887348

Diff. 29 09, medium geom. 2428, circiter
2909.
Quotiens 80000, est major quæsito incremento logarithmorum, quia secantes admodum magni sunt.

APPENDIX.

Eodem ferè modo posset etiam demonstrari, differentias secundas esse in duplâ proportione primarum, tertias in duplâ secundarum *. Verbi causâ, cum in ipso principio logarithmorum differentia prima sit 100,00000, æqualis scilicet ipsi differentiæ numerorum 100000,00000, & 99900,00000, secunda seu differentiarum differentia 10000; tertia

* Hæc falsa est. Vide Huttoni Mathematical Tables, pa. 58. EDIT.

20. Post-

20. Poſtquam ad numerum 50000,00000 ventum fuerit, logarithmi quidem proximi differentiam faciunt 200,00000 quæ ſic habet ad differentiam primam, ſicut numerus 50000,00000, ad maximum 100000,00000. Secunda verò differentia eſt 40000, in qua 10000 continetur 4ter. Tertia 328, in qua 20 continetur 16ies. At cùm in re inſolitâ laboremus penuriâ vocabulorum: quare ne nimium obſcura proponamus, demonſtratio dimittatur intentata.

XXVIII. PROPOSITIO.

Nullus numerus exactè exprimit menſuram proportionis inter binos unius millenarii numeros, methodo ſuperiori conſtitutam.

Nam quia termini uniuſcujuſque proportionis extremi non ſunt ab invicem, ut duo numeri ejuſdem ſpeciei figurativæ; tam multorum graduum, quot vices arbitrariæ ſunt aſſumptæ ad ſecundam proportionem in minima elementa arbitraria: mediæ ergò proportionales elementa conſtituentes ſunt ineffabiles per 9 prop. Differentia igitur inter mediarum proportionalium maximam, & terminum per 1000 ſignificatum eſt & ipſa ineffabilis. Sed menſura proportionis inter 1000 & terminum minorem effabilem in chiliade eſt multiplex hujus differentiolæ, id eſt, & commenſurabilis eſt illi. Ergò menſura hæc eſt terminis incommenſurabilis, hoc eſt, ineffabilis. At nullus ergò numerus, & ſic neque logarithmus, exacte exprimit hanc menſuram.

ADMONITIO.

Intereſt igitur obſervare, quouſque ſeſe proferat hoc vitium. Nam ſi proportio 999, 1000 ſecatur in particulas 16777216 per 24tas proportionales medias, & in particulæ unius menſurâ numero expreſsâ peccetur ſemiſſe unitatis: multiplicatus hic error cum ipsâ menſurâ elementi in numerum elementorum proportionis, efficiet 8000000 unitates.

XXIX. PROPOSITIO.

Si menſuræ proportionum omnium exprimantur numeris, ſeu logarithmis: non omnes proportiones ſortientur legitimam ſuam menſuræ portionem ad omnem minutiarum ſcrupuloſitatem.

Nam per 11 prop. proportiones numerorum chiliadis inter ſe ſunt incommenſurabiles; omnes verò eorum logarithmi ſunt effabiles per ax. 3, & ſic indicant menſuras inter ſe commenſurabiles: injuſtâ igitur partitione, ſi ad minima veniatur.

NOTA.

Locum idem habet etiam in ſinubus, inque proportione circuli ad circumferentiam, & paſſim; & intereſt animadvertere, à quâ figurâ numeri, vitium incipiat, ne in numeris eâ poſterioribus fruſtrà impendamus operam.

XXX. PROPOSITIO.

Si ad numerum 1000 chiliadis maximum referantur aliqui majores, ipſi verò 1000, ſit applicatus logarithmus 0, logarithmi majoribus competentes erunt privativi.

Referatur ad 1000 major 1024, fiatque ut 1024 ad 1000, ſic hic ad 97656,25; ſit etiam menſura proportionis inter 100000,00 & 97656,25, logarithmus 2371,6526. Cùm ergò proportio inter 1024 & 1000 ſit æqualis proportioni inter 100000,00 & 97656,25. Erit eadem menſura ejus. Et ſi ad 1024 apponeretur logarithmus 0, tunc ad 1000 apponendus eſſet logarithmus 2371,6526, ad numerum verò 97656,25 duplum hujus logarithmi, quia proportio 10240000 ad 97656,25 eſt dupla proportionis 1024 ad 1000; ſed quia ad 1000 in chiliade apponitur log. 0, deterſo logarithmo, & 2371,6526, & à duplo hujus etiam apud 97656,25, ſimplum eſt deterſum in canone chiliadis; ergò etiam à logarithmo ipſius 1024, cui applicaveramus logarithmum 0, detergendum eſt tantundem. Si verò à 0 auferas 2371,6526, relinquitur 2371,6526 privativum cum ſigno Coſſico.

METHO-

METHODUS COMPENDIOSISSIMA

CONSTRUENDI

CHILIADA

LOGARITHMORUM.

PRINCIPIO inquiratur logarithmus, qui metitur proportionem inter 100000,00 & 97656,25 quæsitâ mediâ proportionali, maximâ vicesimarum quartarum inter hos terminos, ejusque & numeri totalis prolongati differentiâ toties duplicatâ : emerget autem logarithmus 2371,6526, qui idem est etiam numeri 1024 defectivus per 30. Secundò idem fiat etiam cum proportione inter 1000 & 500: emerget autem logarithmus ad 500, iste 69314,7193, qui idem etiam duplicationis logarithmus dicitur.

Jam quia ut 1000 ad 500 sic 1024 ad 512, & hic ad 256, & hic ad 128, & hic ad 64, & hic ad 32, & hic ad 16, & hic ad 8, & hic ad 4, & hic ad 2, et hic ad 1.

Decupla est igitur proportio 1024 ad 1, proportionis 1000 ad 500. Quare logarithmus ad 1, tunc quidem erit decuplus logarithmi ad 500, cum numerus 1024 acceperit logarithmum 0, sed ubi ei privativus 2371,6526, fuerit applicatus, etiam ipsius 1 logarithmus erit diminuendus tanto. Diminuatur decuplum duplicantis. 693147,1928

2371,6526

Hic est logarithmus unitatis in chiliade nostrâ seu 100,00 690775,5422

Et igitur 10,00. 921034,0563.

Et quia ut 1 ad 10, sic hic ad 100, & hic ad 1000, tripla est itaque proportio 1000 ad 1 proportionis 1000 ad 100, tertia igitur pars logarithmi ad 1 est apponenda pro logarithmo ad 100, puta 230258,5141, & hic etiam est logarithmus decuplicationis, duæ verò tertiæ sunt logarithmus ad 10, scilicet 460517,0282. Quod si duplicantem abstuleris a logarithmo ad 1, emergit logarithmus ad 2, si ab hoc abstuleris logarithmum ad 10 restat logarithmus quinduplicationis.

Logar. ad 1, 690775,5422
Duplicans 69314,7193

Logar. ad 2, 621460,8229
Logar. ad 10, 460517,0281

Quinduplicans 160943,7948.

Hiſce ſic præparatis, jam conſtruantur logarithmi centum maximorum, initio facto à 999, hâc methodo. Totalis 1000 prolongatus 7, cyphris dividatur per ſingulos ordine; quotientes referantur in tabellam, ſunt enim ſecantes illorum arcuum, quorum complementa habent diviſores iſtos pro ſinubus; ut ſi diviſor ſeu ſinus complementi 999, quotiens ſeu ſecans erit 100,10010, ſin dividat 901, quotiens erit 110,98779; ſin 900 quotiens erit 111,11111. Diviſio autem continuatur propterea uſque ad octavam figuram, ut conſtat nobis quantum differat medium arithmeticum à geometrico, ubi maximè. Ut ſi multiplicatis in ſe duobus ultimis quotientibus, radix quæratur, ea erit 111,04942. At medium arithmeticum inter duos quotientes eſt 111,04945.

In ſtructurâ igitur centum minimorum logarithmorum media hæc duo inter ſe ſunt æqualia, uſque ad ſeptimam figuram incluſivè: in octavâ oritur differentia ternarii. Pone in omnibus centum mediis eſſe tantam. Si ergò centum media arithmetica ordine accumules, peccarent illa exceſſu non majore quàm 00300, ternarii ſcilicet in tertiâ figurâ poſt punctum. At verò non eſt in omnibus centum tanta differentia: in initialibus enim penitus evaneſcit, ut inter 100,00000 & 100,10010. Hic enim utrumque medium eſt 100,05005 & differentia occultatur in figuris ulterioribus, ſi quis illas erueret. Quare centum minimos logarithmos tutiſſimè conſtituimus per acumulationem mediarum proportionalium inter quotientes per 27 prop. Semper enim additis duobus ſecantibus ad logarithmi prioris duplum conflatur duplum logarithmi poſterioris.

Conſtructis his logarithmis; eligatur logar. ad 960, qui erit 4082,2001 ferè. Huic ſi continuè adjeceris duplicantem, emergent logarithmi ad 480, 240, 120, 60, 30, 15. Quia proportio 960, ad 15 eſt ſextupla proportionis 1000 ad 500. Ita emerget logarithmus ad 15, qui eſt 419970,5159,

& ad 30 log. 350655,7965.

Hunc ſi abſtuleris à logar. ad 10 ſcilicet 460517,0281

Reſtabit 109861,2316 triplicans logarithmus.

Hunc ergò aufer à logar. ad 1,690775,5422

Reſtat logar. ad 3,580914,3106.

Idem etiam ex log. ad 900 elici poteſt, probationis cauſa. Nam hujus logar. eſt 10536,0535. Et ſi jam hic eſt nimius ob cauſam dictam. Nam ut 1000 ad 900, ſic 9000 ad 8100, & ut 1000 ad 8100 ſic hic ad 6561. Quadruplum ergò logarithmi ad 900 reſpondet numero 6561 (ſeu 90000,00 numero 65610,00) ſcilicet 42144,2140. At qui numerus 65610,00 eſt ternarii de 1000 con-

1000 continuè multiplex. Ecce 6561, 2187, 729, 243, 81, 27, 9, 3, 1. Ergò proportio 6561 ad 1, eſt octupla proportionis 3 ad 1.

Ergò logarithmo ad 1000 ſcilicet 921034,0563
Aufer log. ad 65610,00 ſcilicet 42144,2140

Reſidui 878889,8423
Pars octava 109861,2303 et paulò minor quàm prius.

Scilicet, quia logarithmus ad 900, & ſic etiam ejus quadruplum, juſto majus fuit, id verò ſubtractum à juſto logar. ad 1, relinquit juſto minus. Et ſanè etiam biſectio continua proportionis inter 10 & 9 oſtendit ultimas hujus logar. ad 900 figuras, non 0535 ſed 0513 per 22 minus, cujus quadruplum eſt 88 peccatum reſidui, & hujus pars octava 11 ad 2303 addita facit 2314, planè ut in priori proceſſu invenimus 2316.

Idem etiam ex priori logar. ad 960, aliâ viâ, quia 1000 ad 960 eſt ut 9600 ad 9216, & cum logarithmus 960 ſit 4082,2001, erit logar. ad 9216 duplus prioris ſcilicet 8164,4002. At verò proportio 9216 ad 9 eſt decupla proportionis 1000 ad 500. Ecce: 9216, 4608, 2304, 1152, 576, 288, 144, 72, 36, 18, 9, & proportio 9 ad 1 eſt dupla proportionis 3 ad 1. Ergò ad decuplum duplationis 693147,1928 adde logar. ad numerum 92160,00

ſcilicet 8164,4002

701311,5930 aufer à logar. ad 10,00 ſummam quæ eſt logar. ad 90,00 ſcilic. 921034,0563

Reſtat 219722,4633
Hujus dimid. 109861,2316 eſt triplicans ut prius.

Sic ex log. ad 990 (vel 99000,00) qui eſt 1005,0331, vel ſine decuplicante, vel per eum perveniemus ad logarithmum ad 11 (vel 1100,00). Nam ad 98010,00 erit logarithmus duplus, ſcilicet 2010,0675. Hic additur ad quadruplum triplicantis, 439444,9256.

	facit 441454,9931 log. ad 1210,00
Hoc aufer à log. ad 10,00	921034,0563
Reſtat	479579,0632
Hujus dimid.	239789,5316 Undecuplat. aufer
à logar. ad 1.	690775,5422
Reſtat	450986,0106 log. ad 1100,00.

Sic quia ut 1000 ad 980, ſic hic eſt ad 9604. Duplus igitur logarithmi ad 980 eſt logarithmus ad 9604, ſcil. 4040,5422. Et verò proportio 9604 ad 2401 eſt dupla proportionis 1000 ad 500, & proportio 24010,00 ad 10,00 eſt

 qua-

quadrupla proportionis 70,00 ad 10,00. Adde igitur duplum duplicantis,

ſcilicet	138629,4386
ad	4040,5422
Summam	142669,9808 log. ad 2401
Aufer à log. 10,00000	921034,0563
Reſidui	778364,0775
Pars quarta	194591,0194 ſeptupl.
Hoc igitur aufer à log. ad 100	230258,5141
Reſtat logar. ad 700 vel 70000,00	35667,4947

Atque hic idem ad 700 per continuam biſectionem proportionis 10,7, per maximam triceſimarum proportionalium exactiſſimi tantus prodiit. Vide hanc ſuprà in tabellâ. Sic quia ut 1000 ad 950 ſic 9500 eſt ad 9025. Ergò logarithmus ad 9025 eſt illius duplus, ſcilicet 10258,6606,

Adde duplum quinduplicantis ſc.	321887,5896
Conſurgit	332146,2502 log. ad 3610,00
Hunc aufer à log. ad 10,00	921034,0563
Reſidui	588887,8061
Dimidium	294443,9030 novemdecuplat.
Aufer id à logar. ad 1 vel ad 100,00	690775,5422
Reſtat	396331,6392 log. ad 1900,00.

Eundem derivabimus etiam ex 912, cujus logar. 9211,5306, & quia proportio 912 ad 57 eſt quadrupla proportionis 1000 ad 500. Duplicantis adde quadruplum, ſcilicet 277258,8771. Conſurgit 286470,4077. Huic adde triplicantem

	109861,2316
Reſtat	396331,6393 log. ad 1900,00

Eundem ex 950 derivabimus aliâ viâ :

Logar. ad 95000,00	5129,3303
Decuplans	230258,5141
Logar. ad 9500,00	235387,8444
Quinduplicans	160943,7948
Logar. ad 1900,00	396331,6392.

Sic

Sic ad logar. ad 988 1207,2583
Adde duplicantis duplum 138629,4386

Venit logar. ad 247 --------- 139836,6969
Novemdecuplicans addatur 294443,9030

Prodit logar. ad 13 -------- 434280,5999
Auferatur logarith. ad 1 690775,5422

Reſtat 256494,9423 tridecuplans.

Sic ad logar. ad 969 3149,0672
Adde triplicantem 109861,2316

Venit log. ad 323 --------- 113010,2988
Adde novemdecuplum 294443,9030

Venit log. ad 17 ---------- 407454,2018 aufer hunc à logarith.
ad 1 690775,5422

Reſtat ſeptemdecuplans 283321,3404

Sic ad logar. 986 1409,8927
Adde duplicantem 69314,7193
Et ſeptemdecuplum 283321,3404

Venit logar. ad 29--------- 354045,9524 auferatur
à logar. 1 690775,5422

Reſtat 336729,5898 undetriginduplans.

Et ad log. ad 966 3459,1450
Adde duplicantem 69314,7193
Et triplicantem 109861,2316
Et ſeptuplicantem 194591,0194

Venit log. ad 23 ---------- 377226,1153 qui ablatus
à logar. ad 10,00 690775,5422

Relinquit 313549,4269 viginticuplans.

Eundem ex log. 920. Quia ut 1000 ad 920 ſic 9200 ad 8464, ergò
Logar. ad 920 duplum 16676,3247
Adde ad quadruplum duplicantis 277258,8771
Conſurgit 293935,2018 aufer
à logar. 1 de 100,00 921034,0563

Reſtat 627098,8545
Dimidium 313549,4272

Eundem ex logar. ad 920 jam abundanti ſic extruo:
Ad logar. ad 920 8338,1624
Adde duplicantis duplum 138629,4386
Et decuplicantem 230258,5141
Venit logar. ad 23 377226,1151

 Sic

Sic ad logar. ad 930	7257,0706	majuſculum
Adde triplicantem	109861,2316	
Et decuplicantem	230258,5141	
Venit logar. ad 31 - - - -	347376,8163	majuſculus autem
	8149	verior
à logar. ad 1	690775,5422	
	343398,7259	undetrigintuplicans minuſculus.

Eundem ex log. ad 961 elicio ſic. Nam quia ut 961 ad 31 ſic hoc ad 1. Ergò,

Aufer logarithmum ad 961	3978,0876	
à logar. ad 1	69077,5422	
Reſtat	686797,4546	
Hujus dimid.	343398,7273	undetrigintupla hic eſt juſtior.

Jam quia intra millenarium quadratus major non eſt quàm 961, non igitur opus nobis erit multiplicationibus aliis, ſecundum primos majores ipſo 31. Nam omnis primi majoris multiplex infra 1000 ſecundum aliquem numerum minorem, quàm 31 eſt multiplex: ut 979 eſt undecuplus primi 89.

Ubi notandum, quod logarithmus alicujus proportionis multiplicis ſit differentia inter logarithmum unitatis in chiliade, & inter logarithmum numeri, qui multiplicitate proportionis prodit. Is igitur logarithmus multiplicator additus logarithmo cujuſque numeri, conſtituit logarithmum partis, abſtractus logarithmum multiplicis. Idem verum eſt de logarithmis proportionum non multiplicium, qui ſunt nihil aliud quàm differentiæ, vel multiplicantium logarithmorum, vel appoſitorum ad numeros multiplicium denominatores: ut eadem eſt differentia inter ſeptuplicantem & triplicantem, quæ eſt inter logar. ad 7 & logar. ad 3. Hæc igitur addita logarithmo numeri tertii conſtituit logarithmum numeri proportionalis tertio minoris, ablata conſtituit logar. numeri proportionalis tertio majoris.

Hoc modo plerorumque primorum infra 500 logarithmi eruuntur ex uno centum poſitorum: nam in primo centenario nullus ſupereſt, in ſecundo ſoli quinque 127, 149, 167, 173, 179, in tertio undecim 211, 223, 251, 257, 263, 269, 271, 277, 281, 283, 293, in quarto undecim 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, in quinto novem 401, 409, 419, 421, 431, 433, 439, 443, 449. Summa 36.

Reſtant ii qui ſunt inter 500 & 900, primi 59 numero, prætereaque etiam multiplices jam expreſſorum triginta ſex totidem ſcilicet dupli; ex iis minorum ſedecim tripli, primorum ſeptem quadrupli, & ex his priorum quinque quintupli, duorum 127 & 149 ſextupli, unius 127 ſeptuplus.

Ut igitur etiam ad hos, & ad primos ſupra 500 numero 59 logarithmi habeantur, conſiderandæ ſunt differentiæ logarithmorum poſitorum ad interſperſos hiſce: & quâ methodo differentiæ prius ſunt conſtitutæ logarithmorum 100 minorum ſerie continua, eadem nunc etiam per prop. 27. coroll. 3, & maximè per ejus appendicem, ſi quis ea dextrè utatur, interrupta ſerie, & quidem longè facilius hæ differentiæ ſunt ſupplendæ, quia plerunque vel unus ſolus, vel bini deinceps, rarò tres deinceps logarithmis ſuis carebunt, ita ut in accumulatione differentiarum crebrò reverſio fiat, ad logarithmum jam antea certum.

CHILIAS

LOGARITHMORUM

JOH. KEPLERI, MATHEM. CÆSAREI.

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
	1		1611809,60	
			69314,72	
	2		1542494,88	
			40546,51	
	3		1501948,37 −	
			28768,21	
	4		1473180,16	
			22314,35	
	5		1450865,80 +	
			18232,16	
	6		1432633,65	
			15415,07	
	7		1417218,58	
			13353,14	
	8		1403865,44	
			11778,30	
	9		1392087,14 −	
			10536,05	
	10		1381551,08 +	
			69314,72	
	20		1312236,37	
			40546,51	
	30		1271689,85 +	
			28768,21	
	40		1242921,65 −	
			22314,35	
	50		1220607,29	
			18232,16	
	60		1202375,13	
			15415,07	
	70		1186960,07 −	
			13353,14	
	80		1173606,93 −	
			11778,30	
	90		1161828,62	
			10536,05	
	1,00		1151292,57	
			69314,72	
	2,00		1081977,85	
			40546,51	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
	3,00		1041431,34	
			28768,21	
	4,00		1012663,13	
			22314,35	
	5,00		990348,78 —	
			18232,16	
	6,00		972116,62	
			15415,08	
	7,00		956701,55	
			13353,14	
	8,00		943348,41 +	
			11778,30	
	9,00		931570,11 —	
			10536,05	
	10,00		921034,06 —	
			69314,72	
	20,00		851719,34 —	
			40546,51	
	30,00		811172,82 +	
			28768,21	
	40,00		782404,62	
			22314,35	
	50,00		760090,26	
			18232,16	
	60,00		741858,11 —	
			15415,07	
	70,00		726443,04 —	
			13353,14	
	80,00		713089,90	
			11778,30	
	90,00		701311,59 —	
3 26			10536,05	
0 3 26	100,00	0 1 26	690775,54	0 4
3 27			69314,72	
0 6 53	200,00	0 2 53	621460,82 +	0 7
3 26			40546,51	
0 10 19	300,00	0 4 19	580914,31	0 11
3 26			28768,21	
0 13 45	400,00	0 5 46	552146,10 +	0 14
3 27			22314,35	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 27			22314,35	
0 17 12	500,00	0 7 12	529831,75 −	0 18
3 26			18232,16	
0 20 38	600,00	0 8 38	511599,59	0 22
3 26			15415,07	
0 24 4	700,00	0 10 5	496184,52 +	0 25
3 26			13353,14	
0 27 30	800,00	0 11 31	482831,38 +	0 29
3 26			11778,30	
0 30 56	900,00	0 12 58	471053,08	0 32
3 27			10536,05	
0 34 23	1000,00	0 14 24	460517,03	0 36
3 26			9531,02	
0 37 49	1100,00	0 15 50	450986,01	0 40
3 26			8701,14	
0 41 15	1200,00	0 17 17	442284,87	0 43
3 27			8004,27	
0 44 42	1300,00	0 18 43	434280,60	0 47
3 26			7410,80	
0 48 8	1400,00	0 20 10	426869,80 +	0 50
3 26			6899,27	
0 51 34	1500,00	0 21 36	419970,52 −	0 54
3 26			6453,86	
0 55 0	1600,00	0 23 2	413516,67 −	0 58
3 27			6062,47	
0 58 27	1700,00	0 24 29	407454,20	1 1
3 26			5715,84	
1 1 53	1800,00	0 25 55	401738,36	1 5
3 27			5406,72	
1 5 20	1900,00	0 27 22	396331,64	1 8
3 26			5129,33	
1 8 46	2000,00	0 28 48	391202,31	1 12
3 26			4879,02	
1 12 12	2100,00	0 30 14	386323,29	1 16
3 26			4652,00	
1 15 38	2200,00	0 31 41	381679,29	1 19
3 27			4445,17	
1 19 5	2300,00	0 33 7	377226,12 −	1 23
3 26			4255,97	
1 22 31	2400,00	0 34 34	372970,15 +	1 26
3 26			4082,20	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 26			4082,20	
1 25 57	2500,00	0 36 0	368887,95 +	1 30
3 27			3922,07	
1 29 24	2600,00	0 37 26	364965,88	1 34
3 26			3774,03	
1 32 50	2700,00	0 38 53	361191,85 −	1 37
3 26			3636,77	
1 36 16	2800,00	0 40 19	357555,08 +	1 41
3 27			3509,13	
1 39 43	2900,00	0 41 46	354045,95	1 44
3 26			3390,15	
1 43 9	3000,00	0 43 12	350655,80 −	1 48
2 26			3278,99	
1 46 35	3100,00	0 44 38	347376,81 +	1 52
3 27			3174,86	
1 50 2	3200,00	0 46 5	344201,95 −	1 55
3 26			3077,17	
1 53 28	3300,00	0 47 31	341124,78	1 59
3 27			2985,29	
1 56 55	3400,00	0 48 58	338139,49 +	2 2
3 26			2898,76	
2 0 21	3500,00	0 50 24	335240,73	2 6
3 26			2817,08	
2 3 47	3600,00	0 51 50	332423,65 −	2 10
3 26			2739,90	
2 7 13	3700,00	0 53 17	329683,75 −	2 13
3 27			2666,83	
2 10 40	3800,00	0 54 43	327016,92	2 17
3 27			2597,55	
2 14 7	3900,00	0 56 10	324419,37 +	2 20
3 26			2531,78	
2 17 33	4000,00	0 57 36	321887,59	2 24
3 26			2469,26	
2 20 59	4100,00	0 59 2	319418,33	2 28
3 27			2409,75	
2 24 26	4200,00	1 0 29	317008,58 −	2 31
3 26			2353,05	
2 27 52	4300,00	1 1 55	314655,53 −	2 35
3 27			2298,95	
2 31 19	4400,00	1 3 22	312356,58 −	2 38
3 26			2247,29	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 26			2247,29	
2 34 45	4500,00	1 4 48	310109,29 —	2 42
3 27			2197,89	
2 38 12	4600,00	1 6 14	307911,40 —	2 46
3 26			2150,62	
2 41 38	4700,00	1 7 41	305760,78 —	2 49
3 26			2105,34	
2 45 4	4800,00	1 9 7	303655,44 —	2 53
3 27			2061,93	
2 48 31	4900,00	1 10 34	301593,51 —	2 56
3 27			2020,27	
2 51 58	5000,00	1 12 0	299573,24 —	3 0
3 27			1980,27	
2 55 25	5100,00	1 13 26	297592,97 +	3 4
3 26			1941,81	
2 58 51	5200,00	1 14 53	295651,16 +	3 7
3 27			1904,81	
3 2 18	5300,00	1 16 19	293746,35 —	3 11
3 27			1869,22	
3 5 45	5400,00	1 17 46	291877,13 +	3 14
3 26			1834,91	
3 9 11	5500,00	1 19 12	290042,22 —	3 18
3 27			1801,85	
3 12 38	5600,00	1 20 38	288240,37	3 22
3 26			1769,96	
3 16 4	5700,00	1 22 5	286470,41	3 25
3 27			1739,17	
3 19 31	5800,00	1 23 31	284731,24 —	3 29
3 27			1709,45	
3 22 58	5900,00	1 24 58	283021,79	3 32
3 26			1680,71	
3 26 24	6000,00	1 26 24	281341,08	3 36
3 27			1652,93	
3 29 51	6100,00	1 27 50	279688,15	3 40
3 26			1626,05	
3 33 17	6200,00	1 30 17	278062,10 —	3 43
3 27			1600,04	
3 36 44	6300,00	1 39 43	276462,06 +	3 47
3 27			1574,83	
3 40 11	6400,00	1 32 10	274887,23	3 50
3 26			1550,42	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 26			1550,42	
3 43 37	6500,00	1 33 36	273336,81 −	3 54
3 27			1526,75	
3 47 4	6600,00	1 35 2	271810,06	3 58
3 26			1503,79	
3 50 30	6700,00	1 36 29	270306,27 +	4 1
3 27			1481,50	
3 53 57	6800,00	1 37 55	268824,77 +	4 5
3 27			1459,88	
3 57 24	6900,00	1 39 22	267364,89	4 8
3 26			1438,88	
4 0 50	7000,00	1 40 48	265926,01	4 12
3 27			1418,46	
4 4 17	7100,00	1 42 14	264507,55	4 16
3 27			1398,62	
4 7 44	7200,00	1 43 41	263108,93 −	4 19
3 26			1379,33	
4 11 10	7300,00	1 45 7	261729,60 −	4 23
3 27			1360,57	
4 14 37	7400,00	1 46 34	260369,03 −	4 26
3 27			1342,31	
4 18 4	7500,00	1 48 0	259026,72 +	4 30
3 27			1324,52	
4 21 31	7600,00	1 49 26	257702,20	4 34
3 26			1307,21	
4 24 57	7700,00	1 50 53	256394,99 +	4 37
3 27			1290,34	
4 28 24	7800,00	1 52 19	255104,65 +	4 41
3 27			1273,90	
4 31 51	7900,00	1 53 46	253830,75	4 44
3 27			1257,88	
4 35 18	8000,00	1 55 12	252572,87	4 48
3 27			1242,25	
4 38 45	8100,00	1 56 38	251330,62	4 52
3 27			1227,01	
4 42 12	8200,00	1 58 5	250103,61	4 55
3 26			1212,14	
4 45 38	8300,00	1 59 31	248891,47 +	4 59
3 27			1197,61	
4 49 5	8400,00	2 0 58	247693,86 −	5 2
3 27			1183,45	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 27			1183,45	
4 52 32	8500,00	2 2 24	246510,41	5 6
3 27			1169,60	
4 55 59	8600,00	2 3 50	245340,81 −	5 10
3 27			1156,09	
4 59 26	8700,00	2 5 17	244184,72 +	5 13
3 27			1142,86	
5 2 53	8800,00	2 6 43	243041,86 −	5 17
3 27			1129,96	
5 6 20	8900,00	2 8 10	241911,90	5 20
3 28			1117,33	
5 9 48	9000,00	2 9 36	240794,57 −	5 24
2 27			1104,99	
5 13 15	9100,00	2 11 2	239689,58 +	5 28
3 27			1092,90	
5 16 42	9200,00	2 12 29	238596,68	5 31
3 27			1081,09	
5 20 9	9300,00	2 13 55	237515,59	5 35
3 27			1069,53	
5 23 36	9400,00	2 15 22	236446,06 −	5 38
3 27			1058,21	
5 27 3	9500,00	2 16 48	235387,85 −	5 42
3 28			1047,13	
5 30 31	9600,00	2 18 14	234340,72 −	5 46
3 27			1036,28	
5 33 58	9700,00	2 19 41	233304,44 −	5 49
3 27			1025,65	
5 37 25	9800,00	2 21 7	232278,79 −	5 53
3 27			1015,24	
5 40 52	9900,00	2 22 34	231263,55 −	5 56
3 28			1005,03	
5 44 20	10000,00	2 24 0	230258,52 −	6 0
3 27			995,04	
5 47 47	10100,00	2 25 26	229263,48 +	6 4
3 27			985,23	
5 51 14	10200,00	2 26 53	228278,25 +	6 7
3 28			976,61	
5 54 55	10300,00	2 28 19	227302,64	6 11
3 27			966,20	
5 58 22	10400,00	2 29 46	226336,44 +	6 14
3 28			956,94	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 28			956,94	
6 1 50	10500,00	2 31 12	225379,50	6 18
3 27			947,87	
6 5 17	10600,00	2 32 38	224431,63 —	6 22
3 28			938,98	
6 8 45	10700,00	2 34 5	223492,65 +	6 25
3 27			930,24	
6 12 12	10800,00	2 35 31	222562,41 +	6 29
3 28			921,66	
6 15 40	10900,00	2 36 58	221640,75	6 32
3 28			913,25	
6 18 55	11000,00	2 38 24	220727,50 —	6 36
3 27			904,99	
6 22 22	11100,00	2 39 50	219822,51 +	6 40
3 28			896,86	
6 25 50	11200,00	2 41 17	218925,65	6 43
3 28			888,90	
6 29 18	11300,00	2 42 43	218036,75	6 47
3 27			881,06	
6 32 45	11400,00	2 44 10	217155,69	6 50
3 28			873,37	
6 36 13	11500,00	2 45 36	216282,32	6 54
3 28			865,80	
6 39 41	11600,00	2 47 2	215416,52 —	6 58
3 27			858,38	
6 43 8	11700,00	2 48 29	214558,14	7 1
3 28			851,07	
6 46 36	11800,00	2 49 55	213707,07	7 5
3 28			843,88	
6 50 4	11900,00	2 51 22	212863,19 —	7 8
3 27			836,83	
6 53 31	12000,00	2 52 48	212026,36	7 12
3 28			829,88	
6 56 59	12100,00	2 54 14	211196,48	7 16
3 28			823,05	
7 0 27	12200,00	2 55 41	210373,43	7 19
3 28			816,33	
7 3 55	12300,00	2 57 7	209557,10	7 23
3 27			809,72	
7 7 22	12400,00	2 58 34	208747,38 —	7 26
3 28			803,22	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 28			803,22	
7 10 50	12500,00	3 0 0	207944,16	7 30
3 28			796,82	
7 14 18	12600,00	3 1 26	207147,34 +	7 34
3 28			790,51	
7 17 46	12700,00	3 2 53	206356,83 —	7 37
3 28			784,32	
7 21 14	12800,00	3 4 19	205572,51	7 41
3 28			778,22	
7 24 42	12900,00	3 5 46	204794,29 +	7 44
3 28			772,20	
7 28 10	13000,00	3 7 12	204022,09 —	7 48
3 29			766,29	
7 31 39	13100,00	3 8 38	203255,80	7 52
3 28			760,46	
7 35 7	13200,00	3 10 5	202495,34	7 55
3 28			754,72	
7 38 35	13300,00	3 11 31	201740,62	7 59
3 28			749,07	
7 42 3	13400,00	3 12 58	200991,55 +	8 2
3 28			743,50	
7 45 31	13500,00	3 14 24	200248,05 +	8 6
3 28			738,00	
7 48 59	13600,00	3 15 50	199510,05 +	8 10
3 29			732,61	
7 22 28	13700,00	3 17 17	198777,44	8 13
3 29			727,27	
7 55 56	13800,00	3 18 43	198050,17	8 17
3 28			722,03	
7 59 24	13900,00	3 20 10	197328,14	8 20
3 28			716,85	
8 2 52	14000,00	3 21 36	196611,29	8 24
3 28			711,74	
8 6 20	14100,00	3 23 2	195899,55 —	8 28
3 29			706,72	
8 9 49	14200,00	3 24 29	195192,83	8 31
3 28			701,76	
8 13 17	14300,00	3 25 55	194491,07 +	8 35
3 28			696,86	
8 16 45	14400,00	3 27 22	193794,21 —	8 38
3 29			692,05	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 29			692,05	
8 20 14	14500,00	3 28 48	193102,16	8 42
3 28			687,28	
8 23 42	14600,00	3 30 14	192414,88 —	8 46
3 28			682,61	
8 27 10	14700,00	3 31 41	191732,27 +	8 49
3 29			677,96	
8 30 39	14800,00	3 33 7	191054,31 —	8 53
3 28			673,41	
8 34 7	14900,00	3 34 34	190380,90 +	8 56
3 29			668,90	
8 37 36	15000,00	3 36 0	189712,00 +	9 0
3 29			664,45	
8 41 5	15100,00	3 37 26	189047,55	9 4
3 28			660,07	
8 44 33	15200,00	3 38 53	188387,48	9 7
3 29			655,74	
8 48 2	15300,00	3 40 19	187731,74	9 11
3 29			651,47	
8 51 31	15400,00	3 41 46	187080,27 +	9 14
3 28			647,25	
8 54 59	15500,00	3 43 12	186433,02 +	9 18
3 29			643,09	
8 58 28	15600,00	3 44 38	185789,93 +	9 22
3 29			638,98	
9 1 57	15700,00	3 46 5	185150,95 +	9 25
3 29			634,92	
9 5 26	15800,00	3 47 31	184516,03	9 29
3 29			630,92	
9 8 55	15900,00	3 48 58	183885,11 +	9 32
3 30			626,96	
9 12 25	16000,00	3 50 24	183258,15	9 36
3 29			623,06	
9 15 54	16100,00	3 51 50	182635,09 +	9 40
3 29			619,19	
9 19 23	16200,00	3 53 17	182015,90	9 43
3 29			615,38	
9 22 52	16300,00	3 54 43	181400,52	9 47
3 29			611,63	
9 26 21	16400,00	3 56 10	180788,89	9 50
3 30			607,90	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 30			607,90	
9 29 51	16500,00	3 57 36	180180,99 —	9 54
3 29			604,24	
9 33 20	16600,00	3 58 2	179576,75 +	9 58
3 29			600,60	
9 36 49	16700,00	4 0 29	178976,15 +	10 1
3 29			597,01	
9 40 18	16800,00	4 1 55	178379,14 —	10 5
3 29			593,48	
9 43 47	16900,00	4 3 22	177785,66	10 8
3 29			589,97	
9 47 16	17000,00	4 4 48	177195,69	10 12
3 30			586,51	
9 50 46	17100,00	4 6 14	176609,18 —	10 16
3 29			583,09	
9 54 15	17200,00	4 7 41	176026,09 —	10 19
3 29			579,72	
9 57 44	17300,00	4 9 7	175446,37	10 23
3 29			576,37	
10 1 13	17400,00	4 10 34	174870,00 +	10 26
3 30			573,07	
10 4 43	17500,00	4 12 0	174296,93 +	10 30
3 29			569,79	
10 8 12	17600,00	4 13 26	173727,14 —	10 34
3 30			566,58	
10 11 42	17700,00	4 14 53	173160,56	10 37
3 30			563,38	
10 15 12	17800,00	4 16 19	172597,18	10 41
3 30			560,23	
10 18 42	17900,00	4 17 46	172036,95	10 44
3 29			557,10	
10 22 11	18000,00	4 19 12	171479,85 —	10 48
3 30			554,02	
10 25 41	18100,00	4 20 38	170925,83	10 52
3 30			550,97	
10 29 11	18200,00	4 22 5	170374,86 +	10 55
3 30			547,94	
10 32 41	18300,00	4 23 31	169826,92 —	10 59
3 30			544,96	
10 36 11	18400,00	4 24 58	169281,96 —	11 2
3 29			542,01	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 29			542,01	
10 39 40	18500,00	4 26 24	168739,95	11 6
3 30			539,08	
10 43 10	18600,00	4 27 50	168200,87	11 10
3 30			536,20	
10 46 40	18700,00	4 29 17	167664,67	11 13
3 30			533,33	
10 50 10	18800,00	4 30 43	167131,34	11 17
3 30			530,51	
10 53 40	18900,00	4 32 10	166600,83	11 20
3 30			527,70	
10 57 10	19000,00	4 33 36	166073,13 —	11 24
3 31			524,94	
11 0 41	19100,00	4 35 2	165548,19	11 28
3 30			522,19	
11 4 11	19200,00	4 36 29	165026,00 —	11 31
3 30			519,49	
11 7 41	19300,00	4 37 55	164506,51 +	11 35
3 30			516,79	
11 11 11	19400,00	4 39 22	163989,72 —	11 38
3 30			514,15	
11 14 41	19500,00	4 40 48	163475,57 +	11 42
3 31			511,50	
11 18 12	19600,00	4 42 14	162964,07 —	11 46
3 30			508,91	
11 21 42	19700,00	4 43 41	162455,16	11 49
3 30			506,33	
11 25 12	19800,00	4 45 7	161948,83 —	11 53
3 30			503,78	
11 28 42	19900,00	4 46 34	161445,05	11 56
3 31			501,25	
11 32 13	20000,00	4 48 0	160943,80 —	12 0
3 30			498,76	
11 35 43	20100,00	4 49 26	160445,04 +	12 4
3 31			496,28	
11 39 14	20200,00	4 50 53	159948,76 +	12 7
3 31			493,83	
11 42 45	20300,00	4 52 19	159454,93 +	12 11
3 30			491,40	
11 46 15	20400,00	4 53 46	158963,53 +	12 14
3 31			489,00	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 31			489,00	
11 49 46	20500,00	4 55 12	158474,53 +	12 18
3 31			486,61	
11 53 17	20600,00	4 56 38	157987,92	12 22
3 31			484,26	
11 56 48	20700,00	4 58 5	157503,66 −	12 25
3 31			481,94	
12 0 19	20800,00	4 59 31	157021,72 +	12 29
3 30			479,61	
12 3 49	20900,00	5 0 58	156542,11 −	12 32
3 31			477,33	
12 7 20	21000,00	5 2 24	156064,78	12 36
3 31			475,06	
12 10 51	21100,00	5 3 50	155589,72 −	12 40
3 31			472,81	
12 14 22	21200,00	5 5 17	155116,91 −	12 43
3 31			470,59	
12 17 53	21300,00	5 6 43	154646,32 −	12 47
3 31			468,39	
12 21 24	21400,00	5 8 10	154177,93 +	12 50
3 31			466,20	
12 24 55	21500,00	5 9 36	153711,73	12 54
3 32			464,04	
12 28 27	21600,00	5 11 2	153247,69 +	12 58
3 31			461,89	
12 31 58	21700,00	5 12 29	152785,80	13 1
3 31			459,77	
12 35 29	21800,00	5 13 55	152326,03	13 5
3 31			457,67	
12 39 0	21900,00	5 15 22	151868,36	13 8
3 32			455,58	
12 42 32	22000,00	5 16 48	151412,78 −	13 12
3 31			453,52	
12 46 3	22100,00	5 18 14	150959,26	13 16
3 31			451,47	
12 49 34	22200,00	5 19 41	150507,79 +	13 19
3 32			449,43	
12 53 6	22300,00	5 21 7	150058,36 −	13 23
3 31			446,43	
12 56 37	22400,00	5 22 34	149610,93	13 26
3 32			445,44	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 32			445,44	
13 0 9	22500,00	5 24 0	149165,49	13 30
3 32			443,46	
13 3 41	22600,00	5 25 26	148722,03	13 34
3 32			441,50	
13 7 13	22700,00	5 26 53	148280,53	13 37
3 32			439,56	
13 10 45	22800,00	5 28 19	147840,97	13 41
3 32			437,64	
13 14 17	22900,00	5 29 46	147403,33 +	13 44
3 33			435,73	
13 17 50	23000,00	5 31 12	146967,60	13 48
3 32			433,84	
13 21 22	23100,00	5 32 38	146533,76	13 52
3 32			431,96	
13 24 54	23200,00	5 34 5	146101,80 —	13 55
3 32			430,11	
13 28 26	23300,00	5 35 31	145671,69	13 59
3 32			428,27	
13 31 58	23400,00	5 36 58	145243,42	14 2
3 33			426,44	
13 35 31	23500,00	5 38 24	144816,98	14 6
3 32			424,63	
13 39 3	23600,00	5 39 50	144392,35	14 10
3 32			422,83	
13 42 35	23700,00	5 41 17	143969,52	14 13
3 32			421,05	
13 46 7	23800,00	5 42 43	143548,47 —	14 17
3 33			419,29	
13 49 40	23900,00	5 44 10	143129,18 —	14 20
3 32			418,54	
13 53 12	24000,00	5 45 36	142711,64	14 24
3 32			415,80	
13 56 44	24100,00	5 47 2	142295,84	14 28
3 33			414,08	
14 0 17	24200,00	5 48 29	141881,76	14 31
3 33			412,37	
14 3 50	24300,00	5 49 55	141469,39	14 35
3 32			410,68	
14 7 22	24400,00	5 51 22	141058,71	14 38
3 33			409,00	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 33			409,00	
14 10 55	24500,00	5 52 48	140649,71	14 42
3 33			407,33	
14 14 28	24600,00	5 54 14	140242,38	14 46
3 33			405,68	
14 18 1	24700,00	5 55 41	139836,70	14 49
3 33			404,04	
14 21 34	24800,00	5 57 7	139432,66 −	14 53
3 32			402,42	
14 25 6	24900,00	5 58 34	139030,24	14 56
3 33			400,80	
14 28 39	25000,00	6 0 0	138629,44	15 0
3 33			399,20	
14 32 12	25100,00	6 1 26	138230,24 −	15 4
3 33			397,62	
14 35 45	25200,00	6 2 53	137832,62 +	15 7
3 33			396,04	
14 39 18	25300,00	6 4 19	137436,58 +	15 11
3 33			394,47	
14 42 51	25400,00	6 5 46	137042,11 −	15 14
3 34			392,93	
14 46 25	25500,00	6 7 12	136649,18 −	15 18
3 33			391,39	
14 49 58	25600,00	6 8 38	136257,79	15 22
3 33			389,87	
14 53 31	25700,00	6 10 5	135867,92 +	15 25
3 34			388,35	
14 57 5	25800,00	6 11 31	135479,57 +	15 29
3 33			386,84	
15 0 38	25900,00	6 12 58	135092,73 −	15 32
3 34			385,36	
15 4 12	26000,00	6 14 24	134707,37 −	15 36
3 33			383,88	
15 7 45	26100,00	6 15 50	134323,49	15 40
3 34			382,41	
15 11 19	26200,00	6 17 17	133941,08	15 43
3 34			380,95	
15 14 53	26300,00	6 18 43	133560,13	15 47
3 33			379,51	
15 18 26	26400,00	6 20 10	133180,62	15 50
3 34			378,07	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 34			378,07	
15 22 0	26500,00	6 21 36	132802,55	15 54
3 34			376,65	
15 25 34	26600,00	6 23 2	132425,90	15 58
3 35			375,23	
15 29 9	26700,00	6 24 29	132050,67 —	16 1
3 34			373,84	
15 32 43	26800,00	6 25 55	131676,83 +	16 5
3 34			372,43	
15 36 17	26900,00	6 27 22	131304,40	16 8
3 35			371,07	
15 39 52	27000,00	6 28 48	130933,33 +	16 12
3 34			369,67	
15 43 26	27100,00	6 30 14	130563,66 —	16 16
3 34			368,33	
15 47 0	27200,00	6 31 41	130195,33 —	16 19
3 35			366,98	
15 50 35	27300,00	6 33 7	129828,55	16 23
3 34			365,63	
15 54 9	27400,00	6 34 34	129462,72	16 26
3 35			364,30	
15 57 44	27500,00	6 36 0	129098,42 +	16 30
3 34			362,97	
16 1 18	27600,00	6 37 26	128735,45 —	16 34
3 35			361,67	
16 4 53	27700,00	6 38 53	128373,78	16 37
3 35			360,36	
16 8 28	27800,00	6 40 19	128013,42	16 41
3 34			359,07	
16 12 2	27900,00	6 41 46	127654,35	16 44
3 35			357,78	
16 15 37	28000,00	6 43 12	127296,57	16 48
3 35			356,51	
16 19 12	28100,00	6 44 38	126940,06 +	16 52
3 35			355,23	
16 22 47	28200,00	6 46 5	126584,83 —	16 55
3 35			353,98	
16 26 22	28300,00	6 47 31	126230,85 +	16 59
3 35			352,74	
16 29 57	28400,00	6 48 58	125878,11	17 2
3 36			351,50	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 36			351,50	
16 33 33	28500,00	6 50 24	125526,61 +	17 6
3 35			350,26	
16 37 8	28600,00	6 51 50	125176,35 +	17 10
3 35			349,04	
16 40 43	28700,00	6 53 17	124827,31	17 13
3 35			347,82	
16 44 18	28800,00	6 54 43	124479,49 —	17 17
3 36			346,63	
16 47 54	28900,00	6 56 10	124132,86	17 20
3 35			345,42	
16 51 29	29000,00	6 57 36	123787,44	17 24
3 35			344,24	
16 55 4	29100,00	6 59 2	123443,20 +	17 28
3 36			343,04	
16 58 40	29200,00	7 0 29	123100,16 —	17 31
3 35			341,89	
17 2 15	29300,00	7 1 55	122758,27	17 35
3 36			340,72	
17 5 51	29400,00	7 3 22	122417,55 +	17 38
3 36			339,55	
17 9 27	29500,00	7 4 48	122078,00 —	17 42
3 35			338,41	
17 13 2	29600,00	7 6 14	121739,59 —	17 46
3 36			337,27	
17 16 38	29700,00	7 7 41	121402,32	17 49
3 36			336,14	
17 20 14	29800,00	7 9 7	121066,18 +	17 53
3 36			335,01	
17 23 50	29900,00	7 10 34	120731,17 +	17 56
3 37			333,89	
17 27 27	30000,00	7 12 0	120397,28 +	18 0
3 36			332,78	
17 31 3	30100,00	7 13 26	120064,50 +	18 4
3 36			331,67	
17 34 39	30200,00	7 14 53	119732,83	18 7
3 37			330,58	
17 36 16	30300,00	7 16 19	119402,25	18 11
3 36			329,49	
17 41 52	30400,00	7 17 46	119072,76	18 14
3 37			328,41	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 37			328,41	
17 45 29	30500,00	7 19 12	118744,35 +	18 18
3 37			327,33	
17 49 6	30600,00	7 20 38	118417,02	18 22
3 36			326,26	
17 52 42	30700,00	7 22 5	118090,76 −	18 25
3 37			325,21	
17 56 19	30800,00	7 23 31	117765,55 +	18 29
3 37			324,15	
17 59 56	30900,00	7 24 58	117441,40 +	18 32
3 37			323,10	
18 3 33	31000,00	7 26 24	117118,30 +	18 36
3 37			322,06	
18 7 10	31100,00	7 27 50	116796,24	18 40
3 37			321,03	
18 10 47	31200,00	7 29 17	116475,21 +	18 43
3 37			320,00	
18 14 24	31300,00	7 30 43	116155,21 +	18 47
3 38			318,98	
18 18 2	31400,00	7 32 10	115836,23 +	18 50
3 37			317,96	
18 21 39	31500,00	7 33 36	115518,27	18 54
3 37			316,96	
18 25 16	31600,00	7 35 2	115201,31	18 58
3 37			315,96	
18 28 53	31700,00	7 36 29	114885,35 +	19 1
3 38			314,96	
18 32 31	31800,00	7 37 55	114570,39 +	19 5
3 37			313,97	
18 36 8	31900,00	7 39 22	114256,42 +	19 8
3 38			312,99	
18 39 46	32000,00	7 40 48	113943,43	19 12
3 38			312,01	
18 43 24	32100,00	7 42 14	113631,42	19 16
3 38			311,05	
18 47 2	32200,00	7 43 41	113320,37 +	19 19
3 38			310,07	
18 50 40	32300,00	7 45 7	113010,30	19 23
3 39			309,12	
18 54 19	32400,00	7 46 34	112701,18	19 26
3 38			308,17	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 38			308,17	
18 57 57	32500,00	7 48 0	112393,01	19 30
3 38			307,21	
19 1 35	32600,00	7 49 26	112085,80	19 34
3 38			306,29	
19 5 13	32700,00	7 50 53	111779,51 +	19 37
3 39			305,34	
19 8 52	32800,00	7 52 19	111474,17	19 41
3 38			304,41	
19 12 30	32900,00	7 53 46	111169,76 –	19 44
3 38			303,49	
19 16 8	33000,00	7 55 12	110866,27 –	19 48
3 39			302,58	
19 19 47	33100,00	7 56 38	110563,69 +	19 52
3 38			301,66	
19 23 25	33200,00	7 58 5	110262,03 +	19 55
3 39			300,75	
19 22 4	33300,00	7 59 31	109961,28	19 59
3 38			299,85	
19 30 42	33400,00	8 0 58	109661,43 +	20 2
3 39			298,95	
19 34 21	33500,00	8 2 24	109362,48	20 6
3 39			298,06	
19 38 0	33600,00	8 3 50	109064,42 –	20 10
3 39			297,18	
19 41 39	33700,00	8 5 17	108767,24 –	20 13
3 39			296,30	
19 45 18	33800,00	8 6 43	108470,94	20 17
3 40			295,42	
19 48 58	33900,00	8 8 10	108175,52	20 20
3 39			294,55	
19 52 37	34000,00	8 9 36	107880,97	20 24
3 39			293,69	
19 56 16	34100,00	8 11 2	107587,28 +	20 28
3 40			292,82	
19 59 56	34200,00	8 12 29	107294,46 –	20 31
3 39			291,97	
20 3 35	34300,00	8 13 55	107002,49 –	20 35
3 39			291,12	
20 7 14	34400,00	8 15 22	106711,37 –	20 38
3 40			290,28	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 40			290,28	
20 10 54	34500,00	8 10 48	106421,09	20 42
3 40			289,44	
20 14 34	34600,00	8 18 14	106131,65	20 46
3 39			288,60	
20 18 13	34700,00	8 19 41	105843,05 +	20 49
3 40			287,77	
20 21 53	34800,00	8 21 7	105555,28 +	20 53
3 40			286,94	
20 25 33	34900,00	8 22 34	105268,34	20 56
3 41			286,13	
20 29 14	35000,00	8 24 0	104982,21 +	21 0
3 40			285,30	
20 32 54	35100,00	8 25 26	104696,91	21 4
3 40			284,49	
20 36 34	35200,00	8 26 53	104412,42 —	21 7
3 40			283,70	
20 40 14	35300,00	8 28 19	104128,72 +	21 11
3 41			282,88	
20 43 55	35400,00	8 29 46	103845,84	21 14
3 40			282,09	
20 47 35	35500,00	8 31 12	103563,75	21 18
3 41			281,29	
20 51 16	35600,00	8 32 38	103282,46 —	21 22
3 41			280,50	
20 54 57	35700,00	8 34 5	103001,96 —	21 25
3 41			279,73	
20 58 38	35800,00	8 35 31	102722,23	21 29
3 42			278,94	
21 2 20	35900,00	8 36 58	102443,29	21 32
3 41			278,16	
21 6 1	36000,00	8 38 24	102165,13 —	21 36
3 41			277,39	
21 9 42	36100,00	8 39 50	101887,74 —	21 40
3 41			276,63	
21 13 23	36200,00	8 41 17	101611,11	21 43
3 42			275,86	
21 17 5	36300,00	8 42 43	101335,25	21 47
3 41			275,11	
21 20 46	36400,00	8 44 10	101060,14 +	21 50
3 42			274,34	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 42			274,34	
21 24 28	36500,00	8 45 36	100785,80 −	21 54
3 41			273,60	
21 28 9	36600,00	8 47 2	100512,20 −	21 58
3 42			272,86	
21 31 51	36700,00	8 48 29	100239,34	22 1
3 41			272,10	
21 35 32	36800,00	8 49 55	99967,24	22 5
3 42			271,38	
21 39 14	36900,00	8 51 22	99695,86 +	22 8
3 42			270,63	
21 42 56	37000,00	8 52 48	99425,23	22 12
3 41			269,90	
21 46 37	37100,00	8 54 14	99155,33 −	22 16
3 42			269,18	
21 50 19	37200,00	8 55 41	98886,15	22 19
3 42			268,46	
21 54 1	37300,00	8 57 7	98617,69	22 23
3 42			267,74	
21 57 43	37400,00	8 58 34	98349,95	22 26
3 43			267,02	
22 1 26	37500,00	9 0 0	98082,93 −	22 30
3 42			266,31	
22 5 8	37600,00	9 1 26	97816,62	22 34
3 43			265,61	
22 8 51	37700,00	9 2 53	97551,01	22 37
3 43			264,90	
22 12 34	37800,00	9 4 19	97286,11	22 41
3 43			264,20	
22 16 17	37900,00	9 5 46	97021,91	22 44
3 44			263,50	
22 20 1	38000,00	9 7 12	96758,41 −	22 48
3 43			262,82	
22 23 44	38100,00	9 8 38	96495,59 +	22 52
3 43			262,12	
22 27 27	38200,00	9 10 5	96233,47	22 55
3 44			261,44	
22 31 11	38300,00	9 11 31	95972,03	22 59
3 43			260,75	
22 34 54	38400,00	9 12 58	95711,28 −	23 2
3 44			260,08	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 44			260,08	
22 38 38	38500,00	9 14 24	95451,20	23 6
3 43			259,41	
22 42 21	38600,00	9 15 50	95191,79 +	23 10
3 44			258,73	
22 46 5	38700,00	9 17 17	94933,06	23 13
3 44			258,06	
22 49 49	38800,00	9 18 43	94675,00 −	23 17
3 43			257,40	
22 53 32	38900,00	9 20 10	94417,60	23 20
3 44			256,75	
22 57 16	39000,00	9 21 36	94160,85 +	23 24
3 44			256,07	
23 1 0	39100,00	9 23 2	93904,78 −	23 28
3 44			255,43	
23 4 44	39200,00	9 24 29	93649,35 −	23 31
3 45			254,78	
23 8 29	39300,00	9 25 55	93394,57	23 35
3 44			254,13	
23 12 13	39400,00	9 27 22	93140,44	23 38
3 45			253,49	
23 15 58	39500,00	9 28 48	92886,95 +	23 42
3 45			252,84	
23 19 43	39600,00	9 30 14	92634,11 −	23 46
3 45			252,21	
23 23 28	39700,00	9 31 41	92381,90	23 49
3 44			251,57	
23 27 12	39800,00	9 33 7	92130,33	23 53
3 45			250,94	
23 30 57	39900,00	9 34 34	91879,39	23 56
3 45			250,31	
23 34 42	40000,00	9 36 0	91629,08 −	24 0
3 45			249,69	
23 38 27	40100,00	9 37 26	91379,39	24 4
3 45			249,07	
23 42 12	40200,00	9 38 53	91130,32 +	24 7
3 46			248,45	
23 45 58	40300,00	9 40 19	90881,87	24 11
3 45			247,83	
23 49 43	40400,00	9 41 46	90634,04 +	24 14
3 46			247,22	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 46			247,22	
23 53 29	40500,00	9 43 12	90386,82 +	24 18
3 46			246,61	
23 57 15	40600,00	9 44 38	90140,21 +	24 22
3 46			246,00	
24 1 1	40700,00	9 46 5	89894,21 +	24 25
3 45			245,40	
24 4 46	40800,00	9 47 31	89648,81 +	24 29
3 46			244,79	
24 8 32	40900,00	9 48 58	89404,02 —	24 32
3 46			244,21	
24 12 18	41000,00	9 50 24	89159,81 +	24 36
3 46			243,60	
24 16 4	41100,00	9 51 50	88916,21	24 40
3 47			243,01	
24 19 51	41200,00	9 53 17	88673,20	24 43
3 47			242,43	
24 23 38	41300,00	9 54 43	88430,77	24 47
3 46			241,83	
24 27 24	41400,00	9 56 10	88188,94 —	24 50
3 47			241,26	
24 31 11	41500,00	9 57 36	87947,68	24 54
3 47			240,68	
24 34 58	41600,00	9 59 2	87707,00 +	24 58
3 47			240,09	
24 38 45	41700,00	10 0 29	87466,91	25 1
3 46			239,52	
24 42 31	41800,00	10 1 55	87227,39 —	25 5
3 47			238,95	
24 46 18	41900,00	10 3 22	86988,44 —	25 8
3 47			238,38	
24 50 5	42000,00	10 4 48	86750,06	25 12
3 47			237,81	
24 53 52	42100,00	10 6 14	86512,25 —	25 16
3 48			237,25	
24 57 40	42200,00	10 7 41	86275,00 —	25 19
3 47			236,69	
25 1 27	42300,00	10 9 7	86038,31 +	25 23
3 48			236,12	
25 5 15	42400,00	10 10 34	85802,19 —	25 26
3 48			235,57	

Arcus Circuli, cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 48			235,57	
25 8 3	42500,00	10 12 0	85566,62 —	25 30
3 48			235,02	
25 12 51	42600,00	10 13 26	85331,60 —	25 34
3 49			234,47	
25 16 40	42700,00	10 14 53	85097,13	25 37
3 48			233,92	
25 20 28	42800,00	10 16 19	84863,21 +	25 41
3 48			233,37	
25 24 16	42900,00	10 17 46	84629,84	25 44
3 48			232,83	
25 28 4	43000,00	10 19 12	84397,01	25 48
3 48			232,27	
25 31 52	43100,00	10 20 38	84164,74 —	25 52
3 49			231,77	
25 35 41	43200,00	10 22 5	83932,97 +	25 55
3 48			231,21	
25 39 29	43300,00	10 23 31	83701,76 —	25 59
3 49			230,68	
25 43 18	43400,00	10 24 58	83471,08	26 2
3 49			230,15	
25 47 7	43500,00	10 26 24	83240,93 —	26 6
3 49			229,62	
25 50 56	43600,00	10 27 50	83011,31	26 10
3 50			229,10	
25 54 46	43700,00	10 29 17	82782,21 +	26 13
3 49			228,57	
25 58 35	43800,00	10 30 43	82553,64	26 17
3 49			228,05	
26 2 24	43900,00	10 32 10	82325,59	26 20
3 50			227,53	
26 6 14	44000,00	10 33 36	82098,06 —	26 24
3 49			227,02	
26 10 3	44100,00	10 35 2	81871,04	26 28
3 50			226,50	
26 13 53	44200,00	10 36 29	81644,54	26 31
3 50			225,99	
26 17 43	44300,00	10 37 55	81418,55 +	26 35
3 50			225,48	
26 21 33	44400,00	10 39 22	81193,07 +	26 38
3 51			224,97	

Arcus circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 51			224,97	
26 25 24	44500,00	10 40 48	80968,10	26 42
3 50			224,46	
26 29 14	44600,00	10 42 14	80743,64 —	26 46
3 51			223,97	
26 33 5	44700,00	10 43 41	80519,67	26 49
3 51			223,46	
26 36 56	44800,00	10 45 7	80296,21	26 53
3 50			222,97	
26 40 46	44900,00	10 46 34	80073,24	26 56
3 51			222,47	
26 44 37	45000,00	10 48 0	79850,77	27 0
3 51			221,97	
26 48 28	45100,00	10 49 26	79628,80 —	27 4
3 51			221,49	
26 52 19	45200,00	10 50 53	79407,31	27 7
3 51			220,99	
26 56 10	45300,00	10 52 19	79186,32	27 11
3 52			220,51	
27 0 2	45400,00	10 53 46	78965,81	27 14
3 51			220,02	
27 3 53	45500,00	10 55 12	78745,79	27 18
3 52			219,54	
27 7 45	45600,00	10 56 38	78526,25	27 22
3 52			219,06	
27 11 37	45700,00	10 58 5	78307,19	27 25
3 52			218,58	
27 14 29	45800,00	10 59 31	78088,61 +	27 29
3 52			218,10	
27 18 21	45900,00	11 0 58	77870,51	27 32
3 53			217,63	
27 23 14	46000,00	11 2 24	77652,88	27 36
3 52			217,15	
27 27 6	46100,00	11 3 50	77435,73 —	27 40
3 53			216,69	
27 30 59	46200,00	11 5 17	77219,04	27 43
3 52			216,21	
27 34 51	46300,00	11 6 43	77002,83 —	27 47
3 53			215,75	
27 38 44	46400,00	11 8 10	76787,08 —	27 50
3 53			215,29	

Arcus circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 53			215,29	
27 42 37	46500,00	11 9 36	76571,79	27 54
3 53			214,82	
27 46 30	46600,00	11 11 2	76356,97	27 58
3 54			214,36	
27 50 24	46700,00	11 12 29	76142,61 —	28 1
3 53			213,91	
27 54 17	46800,00	11 13 55	75928,70	28 5
3 54			213,45	
27 58 11	46900,00	11 15 22	75715,25 +	28 8
3 53			212,99	
28 2 4	47000,00	11 16 48	75502,26	28 12
3 54			212,54	
28 5 58	47100,00	11 18 14	75289,72	28 16
3 54			212,09	
28 9 52	47200,00	11 19 41	75077,63	28 19
3 54			211,64	
28 13 46	47300,00	11 21 7	74865,99	28 23
3 54			211,19	
28 17 40	47400,00	11 22 34	74654,80	28 26
3 55			210,75	
28 21 35	47500,00	11 24 0	74444,05	28 30
3 54			210,30	
28 25 29	47600,00	11 25 26	74233,75 —	28 34
3 55			209,87	
28 29 24	47700,00	11 26 53	74023,88	28 37
3 54			209,42	
28 33 18	47800,00	11 28 19	73814,46 —	28 41
3 55			208,99	
28 37 13	47900,00	11 29 46	73605,47	28 44
3 55			208,55	
28 41 8	48000,00	11 31 12	73396,92	28 48
3 55			208,12	
28 45 3	48100,00	11 32 38	73188,80 +	28 52
3 56			207,68	
28 48 59	48200,00	11 34 5	72981,12	28 55
3 55			207,25	
28 52 54	48300,00	11 35 31	72773,87 -	28 59
3 56			206,83	
28 56 50	48400,00	11 36 58	72567,04	29 2
3 55			206,40	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 55			206,40	
29 0 45	48500,00	11 38 24	72360,64	29 6
3 56			205,97	
29 4 41	48600,00	11 39 50	72154,67	29 10
3 56			205,55	
29 8 37	48700,00	11 41 17	71949,12	29 13
3 56			205,13	
29 12 33	48800,00	11 42 43	71743,99	29 17
3 57			204,71	
29 16 30	48900,00	11 44 10	71539,28	29 20
3 56			204,29	
29 20 26	49000,00	11 45 36	71334,99	29 24
3 57			203,87	
29 24 23	49100,00	11 47 2	71131,12 —	29 28
3 57			203,46	
29 28 20	49200,00	11 48 29	70927,66	29 31
3 57			203,05	
29 32 17	49300,00	11 49 55	70724,61 +	29 35
3 58			202,63	
29 36 15	49400,00	11 51 22	70521,98	29 38
3 57			202,23	
29 40 12	49500,00	11 52 48	70319,75 +	29 42
3 57			201,81	
29 44 9	49600,00	11 54 14	70117,94 —	29 46
3 58			201,41	
29 48 7	49700,00	11 55 41	69916,53 —	29 49
3 57			201,01	
29 52 4	49800,00	11 57 7	69715,52	29 53
3 58			200,60	
29 56 2	49900,00	11 58 34	69514,92	29 56
3 58			200,20	
30 0 0	50000,00	12 0 0	69314,72	30 0
3 58			199,80	
30 3 58	50100,00	12 1 26	69114,92	30 4
3 59			199,40	
30 7 57	50200,00	12 2 53	68915,52 —	30 7
3 58			199,01	
30 11 55	50300,00	12 4 19	68716,51 +	30 11
3 59			198,61	
30 15 54	50400,00	12 5 46	68517,90 +	30 14
3 59			198,21	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
3 59			198,21	
30 19 53	50500,00	12 7 12	68319,69	30 18
3 59			197,83	
30 23 52	50600,00	12 8 38	68121,86 +	30 22
4 0			197,43	
30 27 52	50700,00	12 10 5	67924,43	30 25
3 59			197,04	
30 31 51	50800,00	12 11 31	67727,39 —	30 29
4 0			196,66	
30 35 51	50900,00	12 12 58	67530,73 —	30 32
4 0			196,27	
30 39 51	51000,00	12 14 24	67334,46 —	30 36
4 0			195,89	
30 43 51	51100,00	12 15 50	67138,57	30 40
4 1			195,50	
30 47 52	51200,00	12 17 17	66943,07	30 43
4 0			195,12	
30 51 52	51300,00	12 18 43	66747,95	30 47
4 0			194,75	
30 55 52	51400,00	12 20 10	66553,20 +	30 50
4 1			194,36	
30 59 53	51500,00	12 21 36	66358,84	30 54
4 0			193,99	
31 3 53	51600,00	12 23 2	66164,85 +	30 58
4 1			193,61	
31 7 54	51700,00	12 24 29	65971,24 +	31 1
4 1			193,23	
31 11 55	51800,00	12 25 55	65778,01 —	31 5
4 1			192,87	
31 15 56	51900,00	12 27 22	65585,14	31 8
4 1			192,49	
31 19 57	52000,00	12 28 48	65392,65 —	31 12
4 2			192,12	
31 23 59	52100,00	12 30 14	65200,53 —	31 16
4 1			191,76	
31 28 0	52200,00	12 31 41	65008,77	31 19
4 2			191,39	
31 32 2	52300,00	12 33 7	64817,38	31 23
4 2			191,02	
31 36 4	52400,00	12 34 34	64626,36	31 26
4 2			190,65	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 2			190,65	
31 40 6	52500,00	12 36 0	64435,71 −	31 30
4 3			190,30	
31 44 9	52600,00	12 37 26	64245,41	31 34
4 2			189,94	
31 48 11	52700,00	12 38 53	64055,47 +	31 37
4 3			189,57	
31 52 14	52800,00	12 40 19	63865,90	31 41
4 3			189,21	
31 56 17	52900,00	12 41 46	63676,69 −	31 44
4 3			188,86	
32 0 20	53000,00	12 43 12	63487,83	31 48
4 4			188,50	
32 4 24	53100,00	12 44 38	63299,33	31 52
4 3			188,15	
32 8 27	53200,00	12 46 5	63111,18	31 55
4 4			187,79	
32 12 31	53300,00	12 47 31	62923,39	31 59
4 4			187,44	
32 16 35	53400,00	12 48 58	62735,95 −	32 2
4 4			187,10	
32 20 39	53500,00	12 50 24	62548,85 +	32 6
4 4			186,74	
32 24 43	53600,00	12 51 50	62362,11 +	32 10
4 5			186,38	
32 28 48	53700,00	12 53 17	62175,73 +	32 13
4 4			186,05	
32 32 52	53800,00	12 54 43	61989,68	32 17
4 5			185,71	
32 36 57	53900,00	12 56 10	61803,97	32 20
4 5			185,36	
32 41 2	54000,00	12 57 36	61618,61 +	32 24
4 5			185,00	
32 45 7	54100,00	12 59 2	61433,61	32 28
4 5			184,67	
32 49 12	54200,00	13 0 29	61248,94 −	32 31
4 6			184,34	
32 53 18	54300,00	13 1 55	61064,60	32 35
4 5			183,99	
32 57 23	54400,00	13 3 22	60880,61 −	32 38
4 6			183,66	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 6			183,66	
33 1 29	54500,00	13 4 48	60696,95	32 42
4 6			183,32	
33 5 35	54600,00	13 6 14	60513,63	32 46
4 6			182,98	
33 9 41	54700,00	13 7 41	60330,65	32 49
4 7			182,65	
33 13 48	54800,00	13 9 7	60148,00	32 53
4 6			182,31	
33 17 54	54900,00	13 10 34	59965,69	32 56
4 7			181,99	
33 22 2	55000,00	13 12 0	59783,70 +	33 0
4 7			181,65	
33 26 8	55100,00	13 13 26	59602,05 +	33 4
4 7			181,32	
33 30 15	55200,00	13 14 53	59420,73	33 7
4 8			181,00	
33 34 23	55300,00	13 16 19	59239,73	33 11
4 7			180,67	
33 38 30	55400,00	13 17 46	59059,06	33 14
4 8			180,34	
33 42 38	55500,00	13 19 12	58878,72	33 18
4 8			180,02	
33 46 46	55600,00	13 20 38	58698,70	33 22
4 8			179,70	
33 50 54	55700,00	13 22 5	58519,00	33 25
4 9			179,37	
33 55 3	55800,00	13 23 31	58339,63	33 29
4 8			179,05	
33 59 11	55900,00	13 24 58	58160,58	33 32
4 9			178,73	
34 3 20	56000,00	13 26 24	57981,85	33 36
4 9			178,41	
34 7 29	56100,00	13 27 50	57803,44	33 40
4 9			178,10	
34 11 38	56200,00	13 29 17	57625,34 +	33 43
4 10			177,77	
34 15 48	56300,00	13 30 43	57447,57	33 47
4 9			177,46	
34 19 57	56400,00	13 32 10	57270,11 −	33 50
4 10			177,16	

Arcus Circuli cum differentiis.	Sinus feu Numeri abfoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes fexagenariæ.
4 10			177,16	
34 24 7	56500,00	13 33 36	57092,95 +	33 54
4 10			176,82	
34 28 17	56600,00	13 35 2	56916,13 +	33 58
4 11			176,53	
34 32 28	56700,00	13 36 29	56739,60 +	34 1
4 10			176,21	
34 36 38	56800,00	13 37 55	56563,39	34 5
4 11			175,90	
34 40 49	56900,00	13 39 22	56387,49 —	34 8
4 11			175,60	
34 45 0	57000,00	13 40 48	56211,89 +	34 12
4 11			175,28	
34 49 11	57100,00	13 42 14	56036,61	34 16
4 12			174,98	
34 53 23	57200,00	13 43 41	55861,63 +	34 19
4 11			174,67	
34 57 34	57300,00	13 45 7	55686,96 —	34 23
4 12			174,37	
35 1 46	57400,00	13 46 34	55512,59	34 26
4 12			174,06	
35 5 58	57500,00	13 48 0	55338,53 —	34 30
4 12			173,76	
35 10 11	57600,00	13 49 26	55164,77 —	34 34
4 12			173,47	
35 14 23	57700,00	13 50 53	54991,30 +	34 37
4 13			173,16	
35 18 36	57800,00	13 52 19	54818,14	34 41
4 13			172,86	
35 22 49	57900,00	13 53 46	54645,28	34 44
4 13			172,56	
35 27 2	58000,00	13 55 12	54472,72	34 48
4 14			172,27	
35 31 16	58100,00	13 56 38	54300,45 +	34 52
4 13			171,97	
35 35 29	58200,00	13 58 5	54128,48 +	34 55
4 14			171,67	
35 39 43	58300,00	13 59 31	53956,81 +	34 59
4 14			171,37	
35 43 57	58400,00	14 0 58	53785,44 —	35 2
4 14			171,09	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 14			171,09	
35 48 11	58500,00	14 2 24	53614,35 −	35 6
4 15			170,80	
35 52 26	58600,00	14 3 50	53443,55	35 10
4 14			170,50	
35 56 40	58700,00	14 5 17	53273,05 −	35 13
4 15			170,22	
36 0 55	58800,00	14 6 43	53102,83 +	35 17
4 15			169,92	
36 5 10	58900,00	14 8 10	52932,91	35 20
4 16			169,63	
36 9 26	59000,00	14 9 36	52763,28 −	35 24
4 15			169,35	
36 13 41	59100,00	14 11 2	52593,93 −	35 28
4 16			169,06	
36 17 57	59200,00	14 12 29	52424,87 −	35 31
4 16			168,78	
36 22 13	59300,00	14 13 55	52256,09	35 35
4 17			168,49	
36 26 30	59400,00	14 15 22	52087,60	35 38
4 16			168,21	
36 30 46	59500,00	14 16 48	51919,39	35 42
4 17			167,93	
36 35 3	59600,00	14 18 14	51751,46 +	35 46
4 17			167,64	
36 39 20	59700,00	14 19 41	51583,82	35 49
4 18			167,37	
36 43 38	59800,00	14 21 7	51416,45 +	35 53
4 17			167,08	
36 47 55	59900,00	14 22 34	51249,37	35 56
4 18			166,81	
36 52 13	60000,00	14 24 0	51082,56 +	36 0
4 18			166,52	
36 56 31	60100,00	14 25 26	50916,04 −	36 4
4 19			166,20	
37 0 50	60200,00	14 26 53	50749,78 +	36 7
4 19			165,97	
37 5 9	60300,00	14 28 19	50583,81	36 11
4 18			165,70	
37 9 27	60400,00	14 29 46	50418,11	36 14
4 19			165,43	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 19			165,43	
37 13 46	60500,00	14 31 12	50252,68 +	36 18
4 19			165,15	
37 18 5	60600,00	14 32 38	50087,53	36 22
4 20			164,88	
37 22 25	60700,00	14 34 5	49922,65	36 25
4 19			164,61	
37 26 44	60800,00	14 35 31	49758,04	36 29
4 20			164,34	
37 31 4	60900,00	14 36 58	49593,70 +	36 32
4 20			164,07	
37 35 24	61000,00	14 38 24	49429,63 +	36 36
4 21			163,80	
37 39 45	61100,00	14 39 50	49265,83 +	36 40
4 20			163,53	
37 44 5	61200,00	14 41 17	49102,30	36 43
4 21			163,26	
37 48 26	61300,00	14 42 43	48939,04 −	36 47
4 21			163,00	
37 52 47	61400,00	14 44 10	48776,04 −	36 50
4 22			162,74	
37 57 9	61500,00	14 45 36	48613,30 +	36 54
4 21			162,47	
38 1 30	61600,00	14 47 2	48450,83 +	36 58
4 22			162,20	
38 5 52	61700,00	14 48 29	48288,63	37 1
4 22			161,95	
38 10 4	61800,00	14 49 55	48126,68 +	37 5
4 23			161,68	
38 14 37	61900,00	14 51 22	47965,00 +	37 8
4 22			161,42	
38 18 59	62000,00	14 52 48	47803,58 +	37 12
4 23			161,16	
38 23 22	62100,00	14 54 14	47642,42	37 16
4 24			160,90	
38 27 46	62200,00	14 55 41	47481,52	37 19
4 23			160,64	
38 32 9	62300,00	14 57 7	47320,88	37 23
4 24			160,39	
38 36 33	62400,00	14 58 34	47160,49 +	37 26
4 24			160,12	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 24			160,12	
38 40 57	62500,00	15 0 0	47000,37 −	37 30
4 25			159,88	
38 45 22	62600,00	15 1 26	46840,49 +	37 34
4 25			159,61	
38 49 47	62700,00	15 2 53	46680,88 −	37 37
4 24			159,37	
38 54 11	62800,00	15 4 19	46521,51 +	37 41
4 25			159,11	
38 58 36	62900,00	15 5 46	46362,40 +	37 44
4 25			158,85	
39 3 1	63000,00	15 7 12	46203,55	37 48
4 26			158,61	
39 7 27	63100,00	15 8 38	46044,94 +	37 52
4 27			158,35	
39 11 54	63200,00	15 10 5	45886,59	37 55
4 27			158,10	
39 16 21	63300,00	15 11 31	45728,49	37 59
4 27			157,86	
39 20 48	63400,00	15 12 58	45570,63 +	38 2
4 26			157,60	
39 25 14	63500,00	15 14 24	45413,03	38 6
4 27			157,36	
39 29 41	63600,00	15 15 50	45255,67 +	38 10
4 27			157,10	
39 34 8	63700,00	15 17 17	45098,57 −	38 13
4 28			156,87	
39 38 36	63800,00	15 18 43	44941,70 +	38 17
4 27			156,62	
39 43 3	63900,00	15 20 10	44785,08 +	38 20
4 28			156,37	
39 47 31	64000,00	15 21 36	44628,71	38 24
4 28			156,12	
39 51 59	64100,00	15 23 2	44472,59 −	38 28
4 29			155,89	
39 56 28	64200,00	15 24 29	44316,70	38 31
4 29			155,65	
40 0 57	64300,00	15 25 55	44161,05 +	38 35
4 30			155,40	
40 5 27	64400,00	15 27 22	44005,65 +	38 38
4 29			155,15	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 29			155,15	
40 9 56	64500,00	15 28 48	43850,50 —	38 42
4 30			154,92	
40 14 26	64600,00	15 30 14	43695,58	38 46
4 30			154,68	
40 18 56	64700,00	15 31 41	43540,90	38 49
4 31			154,44	
40 23 27	64800,00	15 33 7	43386,46	38 53
4 31			154,20	
40 27 58	64900,00	15 34 34	43232,26	38 56
4 32			153,97	
40 32 30	65000,00	15 36 0	43078,29	39 0
4 32			153,73	
40 37 2	65100,00	15 37 26	42924,56 +	39 4
4 33			153,48	
40 41 35	65200,00	15 38 53	42771,08	39 7
4 32			153,26	
40 46 7	65300,00	15 40 19	42617,82	39 11
4 33			153,03	
40 50 40	65400,00	15 41 46	42464,79 +	39 14
4 33			152,79	
40 55 13	65500,00	15 43 12	42312,00 +	39 18
4 33			152,55	
40 59 46	65600,00	15 44 38	42159,45	39 22
4 33			152,32	
41 4 19	65700,00	15 46 5	42007,13 —	39 25
4 34			152,09	
41 8 53	65800,00	15 47 31	41855,04 —	39 29
4 33			151,87	
41 13 26	65900,00	15 48 58	41703,17 +	39 32
4 34			151,62	
41 18 0	66000,00	15 50 24	41551,55 —	39 36
4 35			151,40	
41 22 35	66100,00	15 51 50	41400,15 —	39 40
4 35			151,18	
41 27 10	66200,00	15 53 17	41248,97 +	39 43
4 36			150,94	
41 31 46	66300,00	15 54 43	41098,03	39 47
4 35			150,72	
41 36 21	66400,00	15 56 10	40947,31 +	39 50
4 36			150,49	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 36			150,49	
41 40 57	66500,00	15 57 36	40796,82	39 54
4 36			150,26	
41 45 33	66600,00	15 59 2	40646,56	39 58
4 37			150,03	
41 50 10	66700,00	16 0 29	40496,53 —	40 1
4 37			149,82	
41 54 47	66800,00	16 1 55	40346,71 +	40 5
4 38			149,58	
41 59 25	66900,00	16 3 22	40197,13 —	40 8
4 37			149,37	
42 4 2	67000,00	16 4 48	40047,76	40 12
4 38			149,14	
42 8 40	67100,00	16 6 14	39898,62	40 16
4 38			148,92	
42 13 18	67200,00	16 7 41	39749,70 —	40 19
4 39			148,70	
42 17 57	67300,00	16 9 7	39601,00 —	40 23
4 39			148,48	
42 22 36	67400,00	16 10 34	39452,52 —	40 26
4 39			148,26	
42 27 15	67500,00	16 12 0	39304,26	40 30
4 40			148,04	
42 31 55	67600,00	16 13 26	39156,22	40 34
4 40			147,76	
42 36 35	67700,00	16 14 53	39008,46	40 37
4 40			147,66	
42 41 15	67800,00	16 16 19	38860,80	40 41
4 41			147,38	
42 45 56	67900,00	16 17 46	38713,42 —	40 44
4 41			147,17	
42 50 37	68000,00	16 19 12	38566,25	40 48
4 42			146,95	
42 55 19	68100,00	16 20 38	38419,30	40 52
4 42			146,74	
43 0 1	68200,00	16 22 5	38272,56 +	40 55
4 43			146,52	
43 4 44	68300,00	16 23 31	38126,64 +	40 59
4 42			146,30	
43 9 26	68400,00	16 24 58	37979,74 —	41 2
4 43			146,09	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 43			146,09	
43 14 9	68500,00	16 26 24	37833,65 —	41 6
4 43			145,88	
43 18 52	68600,00	16 27 50	37687,77 —	41 10
4 43			145,67	
43 23 35	68700,00	16 29 17	37542,10	41 13
4 44			145,45	
43 28 19	68800,00	16 30 43	37396,65 —	41 17
4 44			145,25	
43 33 3	68900,00	16 32 10	37251,40 +	41 20
4 45			145,03	
43 37 48	69000,00	16 33 36	37106,37	41 24
4 45			144,83	
43 42 33	69100,00	16 35 2	36961,54 +	41 28
4 45			144,61	
43 47 18	69200,00	16 36 29	36816,93	41 31
4 46			144,40	
43 52 4	69300,00	16 37 55	36672,53 —	41 35
4 46			144,20	
43 56 50	69400,00	16 39 22	36528,33 +	41 38
4 47			143,99	
44 1 37	69500,00	16 40 48	36384,34 +	41 42
4 47			143,78	
44 6 24	69600,00	16 42 14	36240,56 +	41 46
4 48			143,57	
44 11 12	69700,00	16 43 41	36096,99	41 49
4 48			143,37	
44 16 0	69800,00	16 45 7	35953,62	41 53
4 48			143,17	
44 20 48	69900,00	16 46 34	35810,45 +	41 56
4 49			142,96	
44 25 37	70000,00	16 48 0	35667,49 +	42 0
4 49			142,75	
44 30 26	70100,00	16 49 26	35524,74	42 4
4 50			142,55	
44 35 15	70200,00	16 50 53	35382,19	42 7
4 50			142,35	
44 40 5	70300,00	16 52 19	35239,84 +	42 11
4 50			142,14	
44 44 55	70400,00	16 53 46	35097,70 —	42 14
4 51			141,95	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 51			141,95	
44 49 46	70500,00	16 55 12	34955,75 −	42 18
4 51			141,75	
44 54 37	70600,00	16 56 38	34814,00 +	42 22
4 52			141,54	
44 59 29	70700,00	16 58 5	34672,46	42 25
4 52			141,34	
45 4 21	70800,00	16 59 31	34531,12	42 29
4 52			141,14	
45 9 13	70900,00	17 0 58	34389,98 −	42 32
4 53			140,95	
45 14 6	71000,00	17 2 24	34249,03	42 36
4 53			140,74	
45 18 59	71100,00	17 3 50	34108,29 −	42 40
4 54			140,55	
45 23 53	71200,00	17 5 17	33967,74 −	42 43
4 54			140,35	
45 28 47	71300,00	17 6 43	33827,39	42 47
4 54			140,15	
45 33 41	71400,00	17 8 10	33687,24 −	42 50
4 55			139,96	
45 38 36	71500,00	17 9 36	33547,28 −	42 54
4 55			139,77	
45 43 31	71600,00	17 11 2	33407,51	42 58
4 56			139,57	
45 48 27	71700,00	17 12 29	33267,94 +	43 1
4 56			139,37	
45 53 23	71800,00	17 13 55	33128,57	43 5
4 57			139,18	
45 58 20	71900,00	17 15 22	32989,39	43 8
4 57			138,98	
46 3 17	72000,00	17 16 48	32850,41 −	43 12
4 58			138,79	
46 8 15	72100,00	17 18 14	32711,62 −	43 16
4 58			138,60	
46 13 13	72200,00	17 19 41	32573,02 −	43 19
4 59			138,41	
46 18 12	72300,00	17 21 7	32434,61 −	43 23
4 59			138,22	
46 23 11	72400,00	17 22 34	32296,39	43 26
4 59			138,03	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
4 59			138,03	
46 28 10	72500,00	17 24 0	32158,36 +	43 30
5 0			137,83	
46 33 10	72600,00	17 25 26	32020,53	43 34
5 0			137,65	
46 38 10	72700,00	17 26 53	31882,88	43 37
5 1			137,46	
46 43 11	72800,00	17 28 19	31745,42 +	43 41
5 1			137,27	
46 48 12	72900,00	17 29 46	31608,15 +	43 44
5 1			137,07	
46 53 13	73000,00	17 31 12	31471,08 −	43 48
5 2			136,90	
46 58 15	73100,00	17 32 38	31334,18	43 52
5 2			136,70	
47 3 17	73200,00	17 34 5	31197,48 −	43 55
5 3			136,53	
47 8 20	73300,00	17 35 31	31060,95	43 59
5 3			136,33	
47 13 23	73400,00	17 36 58	30924,62	44 2
5 4			136,14	
47 18 27	73500,00	17 38 24	30788,48	44 6
5 5			135,96	
47 23 32	73600,00	17 39 50	30652,52	44 10
5 5			135,78	
47 18 37	73700,00	17 41 17	30516,74	44 13
5 6			135,60	
47 33 43	73800,00	17 42 43	30381,14 +	44 17
5 6			135,40	
47 38 49	73900,00	17 44 10	30245,74 −	44 20
5 7			135,23	
47 43 56	74000,00	17 45 36	30110,51	44 24
5 7			135,04	
47 49 3	74100,00	17 47 2	29975,47	44 28
5 7			134,86	
47 54 10	74200,00	17 48 29	29840,61 −	44 31
5 8			134,68	
47 59 18	74300,00	17 49 55	29705,93	44 35
5 8			134,50	
48 4 26	74400,00	17 51 22	29571,43	44 38
5 9			134,32	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
5 9			134,32	
48 9 35	74500,00	17 52 48	29437,11	44 42
5 9			134,14	
48 14 44	74600,00	17 54 14	29302,97	44 46
5 10			133,96	
48 19 54	74700,00	17 55 41	29169,01	44 49
5 10			133,78	
48 25 4	74800,00	17 57 7	29035,23	44 53
5 11			133,60	
48 30 15	74900,00	17 58 34	28901,63	44 56
5 11			133,42	
48 35 26	75000,00	18 0 0	28768,21 −	45 0
5 12			133,25	
48 40 38	75100,00	18 1 26	28634,96 +	45 4
5 13			133,06	
48 45 51	75200,00	18 2 53	28501,90	45 7
5 13			132,90	
48 51 4	75300,00	18 4 19	28369,00 +	45 11
5 14			132,71	
48 56 18	75400,00	18 5 46	28236,29	45 14
5 14			132,54	
49 1 32	75500,00	18 7 12	28103,75 +	45 18
5 15			132,36	
49 6 47	75600,00	18 8 38	27971,39	45 22
5 15			132,19	
49 12 2	75700,00	18 10 5	27839,20 +	45 25
5 16			132,01	
49 17 18	75800,00	18 11 31	27707,19	45 29
5 16			131,84	
49 22 34	75900,00	18 12 58	27575,35 +	45 32
5 17			131,66	
49 27 51	76000,00	18 14 24	27443,69 −	45 36
5 18			131,50	
49 33 9	76100,00	18 15 50	27312,19	45 40
5 18			131,32	
49 38 27	76200,00	18 17 17	27180,87 +	45 43
5 19			131,14	
49 43 46	76300,00	18 18 43	27049,73 −	45 47
5 19			130,98	
49 49 5	76400,00	18 20 10	26918,75	45 50
5 20			130,81	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
5 20			130,81	
49 54 25	76500,00	18 21 36	26787,94 +	45 54
5 21			130,63	
49 59 46	76600,00	18 23 2	26657,31	45 58
5 21			130,46	
50 5 7	76700,00	18 24 29	26526,85	46 1
5 22			130,29	
50 10 29	76800,00	18 25 55	26396,56 —	46 5
5 22			130,13	
50 15 51	76900,00	18 27 22	26266,43 +	46 8
5 23			129,95	
50 21 14	77000,00	18 28 48	26136,48	46 12
5 24			129,79	
50 26 38	77100,00	18 30 14	26006,69 +	46 16
5 24			129,62	
50 32 2	77200,00	18 31 41	25877,07 +	46 19
5 25			129,45	
50 37 27	77300,00	18 33 7	25747,62 +	46 23
5 26			129,28	
50 42 53	77400,00	18 34 34	25618,34	46 26
5 26			129,11	
50 48 19	77500,00	18 36 0	25489,23 —	46 30
5 27			128,95	
50 53 46	77600,00	18 37 26	25360,28 —	46 34
5 27			128,79	
50 59 13	77700,00	18 38 35	25231,49 +	46 37
5 28			128,61	
51 4 41	77800,00	18 40 19	25102,88	46 41
5 28			128,46	
51 10 9	77900,00	18 41 46	24974,42 +	46 44
5 29			128,29	
51 15 38	78000,00	18 43 12	24846,13 +	46 48
5 30			128,12	
51 21 8	78100,00	18 44 38	24718,01 +	46 52
5 31			127,95	
51 26 39	78200,00	18 46 5	24590,06 —	46 55
5 31			127,80	
51 32 10	78300,00	18 47 31	24462,26 —	46 59
5 32			127,63	
51 37 42	78400,00	18 48 58	24334,63 —	47 2
5 33			127,47	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
5 33			127,47	
51 43 15	78500,00	18 50 24	24207,16	47 6
5 33			127,31	
51 48 48	78600,00	18 51 50	24079,85	47 10
5 34			127,15	
51 54 22	78700,00	18 53 17	23952,70 +	47 13
5 34			126,98	
51 59 56	78800,00	18 54 43	23825,72	47 17
5 35			126,82	
52 5 31	78900,00	18 56 10	23698,90 —	47 20
5 36			126,67	
52 11 7	79000,00	18 57 36	23572,23 +	47 24
5 37			126,50	
52 16 44	79100,00	18 59 2	23445,73	47 28
5 38			126,34	
52 22 22	79200,00	19 0 29	23319,39 —	47 31
5 38			126,19	
52 28 0	79300,00	19 1 55	23193,20 +	47 35
5 39			126,02	
52 33 39	79400,00	19 3 22	23067,18	47 38
5 40			125,86	
52 39 19	79500,00	19 4 48	22941,32	47 42
5 41			125,71	
52 45 0	79600,00	19 6 14	22815,61	47 46
5 41			125,55	
52 50 41	79700,00	19 7 41	22690,06	47 49
5 42			125,36	
52 56 23	79800,00	19 9 7	22564,67	47 53
5 42			125,24	
53 2 5	79900,00	19 10 34	22439,43 +	47 56
5 43			125,07	
53 7 48	80000,00	19 12 0	22314,36 —	48 0
5 44			124,93	
53 13 32	80100,00	19 13 26	22189,43 +	48 4
5 45			124,76	
53 19 17	80200,00	19 14 53	22064,67	48 7
5 46			124,61	
53 25 3	80300,00	19 16 19	21940,06	48 11
5 46			124,46	
53 30 49	80400,00	19 17 46	21815,60 +	48 14
5 47			124,30	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
5 47			124,30	
53 36 36	80500,00	19 19 12	21691,30	48 18
5 48			124,15	
53 42 24	80600,00	19 20 38	21567,15	48 22
5 49			123,98	
53 84 13	80700,00	19 22 5	21443,17 −	48 25
5 50			123,85	
53 54 3	80800,00	19 23 31	21319,32 +	48 29
5 51			123,68	
53 59 54	80900,00	19 24 58	21195,64 −	48 32
5 52			123,54	
54 5 46	81000,00	19 26 24	21072,10 +	48 36
5 52			123,38	
54 11 38	81100,00	19 27 50	20948,72 +	48 40
5 53			123,23	
54 17 31	81200,00	19 29 17	20825,49 +	48 43
5 54			123,07	
54 23 25	81300,00	19 30 43	20702,42	48 47
5 55			122,93	
54 29 20	81400,00	19 32 10	20579,49 +	48 50
5 56			122,77	
54 35 16	81500,00	19 33 36	20456,72 −	48 54
5 57			122,63	
54 41 13	81600,00	19 35 2	20334,09 +	48 58
5 58			122,47	
54 47 11	81700,00	19 36 29	20211,62 −	49 1
5 58			122,32	
54 53 9	81800,00	19 37 55	20089,30 −	49 5
5 59			122,18	
54 59 8	81900,00	19 39 22	19967,12	49 8
6 0			122,03	
55 5 8	82000,00	19 40 48	19845,09 +	49 12
6 1			121,87	
55 11 9	82100,00	19 42 14	19723,22 −	49 16
6 2			121,73	
55 17 11	82200,00	19 43 41	19601,49	49 19
6 3			121,58	
55 23 14	82300,00	19 45 7	19479,91	49 23
6 3			121,43	
55 29 17	82400,00	19 46 34	19358,48	49 26
6 4			121,29	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
6 4			121,29	
55 35 21	82500,00	19 48 0	19237,19	49 30
6 5			121,14	
55 41 26	82600,00	19 49 26	19116,05	49 34
6 6			120,99	
55 47 32	82700,00	19 50 53	18995,06	49 37
6 7			120,84	
55 53 39	82800,00	19 52 19	18874,22 —	49 41
6 8			120,71	
55 59 47	82900,00	19 53 46	18753,51 +	49 44
6 9			120,55	
56 5 56	83000,00	19 55 12	18632,96	49 48
6 10			120,41	
56 12 6	83100,00	19 56 38	18512,55	49 52
6 11			120,27	
56 18 17	83200,00	19 58 5	18392,28 +	49 55
6 12			120,12	
56 24 29	83300,00	19 59 31	18272,16 +	49 59
6 13			119,97	
56 30 42	83400,00	20 0 58	18152,19	50 2
6 14			119,83	
56 36 56	83500,00	20 2 24	18032,36	50 6
6 15			119,69	
56 43 11	83600,00	20 3 50	17912,67 —	50 10
6 16			119,55	
56 49 27	83700,00	20 5 17	17793,12	50 13
6 18			119,40	
56 55 45	83800,00	20 6 43	17673,72 —	50 17
6 19			119,26	
57 2 4	83900,00	20 8 10	17554,46 —	50 20
6 20			119,12	
57 8 24	84000,00	20 9 36	17435,34	50 24
6 21			118,98	
57 14 45	84100,00	20 11 2	17316,36 +	50 28
6 22			118,83	
57 21 7	84200,00	20 12 29	17197,53 —	50 31
6 23			118,70	
57 27 30	84300,00	20 13 55	17078,83	50 35
6 24			118,55	
57 33 54	84400,00	20 15 22	16960,28 —	50 38
6 25			118,41	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
6 25			118,41	
57 40 19	84500,00	20 16 48	16841,87 —	50 42
6 26			118,28	
57 46 45	84600,00	20 18 14	16723,59 +	50 46
6 27			118,13	
57 53 12	84700,00	20 19 41	16605,46	50 49
6 28			117,99	
57 59 40	84800,00	20 21 7	16487,47 —	50 53
6 30			117,86	
58 6 10	84900,00	20 22 34	16369,61 +	50 56
6 31			117,71	
58 12 41	85000,00	20 24 0	16251,90 —	51 0
6 32			117,58	
58 19 13	85100,00	20 25 26	16134,32	51 4
6 33			117,44	
58 25 46	85200,00	20 26 53	16016,88 —	51 7
6 35			117,31	
58 32 21	85300,00	20 28 19	15899,57 +	51 11
6 36			117,16	
58 38 57	85400,00	20 29 46	15782,41	51 14
6 37			117,03	
58 45 34	85500,00	20 31 12	15665,38	51 18
6 39			116,89	
58 52 13	85600,00	20 32 38	15548,49 +	51 22
6 40			116,75	
58 58 53	85700,00	20 34 5	15431,74	51 25
6 41			116,62	
59 5 34	85800,00	20 35 31	15315,12	51 29
6 42			116,48	
59 12 16	85900,00	20 36 58	15198,64 —	51 32
6 44			116,35	
59 19 0	86000,00	20 38 24	15082,29	51 36
6 45			116,21	
59 25 45	86100,00	20 39 50	14966,08 —	51 40
6 46			116,06	
59 32 31	86200,00	20 41 17	14850,02 —	51 43
6 47			115,95	
59 39 18	86300,00	20 42 43	14734,07 +	51 47
6 49			115,82	
59 46 7	86400,00	20 44 10	14618,25 +	51 50
6 51			115,68	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
6 51			115,68	
59 52 58	86500,00	20 45 36	14502,57	51 54
6 52			115,53	
59 59 50	86600,00	20 47 2	14387,04 —	51 58
6 53			115,41	
60 6 43	86700,00	20 48 29	14271,63	52 1
6 54			115,27	
60 13 37	86800,00	20 49 55	14156,36	52 5
6 56			115,14	
60 20 33	86900,00	20 51 22	14041,22	52 8
6 57			115,01	
60 27 30	87000,00	20 52 48	13926,21 —	52 12
6 59			114,88	
60 34 29	87100,00	20 54 14	13811,33	52 16
7 1			114,74	
60 41 30	87200,00	20 55 41	13696,59	52 19
7 3			114,72	
60 48 33	87300,00	20 57 7	13581,87	52 23
7 4			114,38	
60 55 37	87400,00	20 58 34	13467,49 +	52 26
7 6			114,35	
61 2 43	87500,00	21 0 0	13353,14	52 30
7 8			114,22	
61 9 51	87600,00	21 1 26	13238,92	52 34
7 9			114,09	
61 17 0	87700,00	21 2 53	13124,83 +	52 37
7 10			113,96	
61 24 10	87800,00	21 4 19	13010,87	52 41
7 12			113,83	
61 31 22	87900,00	21 5 46	12897,04	52 44
7 13			113,70	
61 38 35	88000,00	21 7 12	12783,34 —	52 48
7 15			113,57	
61 45 50	88100,00	21 8 38	12669,77 —	52 52
7 16			113,45	
61 53 6	88200,00	21 10 5	12556,32	52 55
7 18			113,31	
62 0 24	88300,00	21 11 31	12443,01 —	52 59
7 20			113,19	
62 7 44	88400,00	21 12 58	12329,82	53 2
7 22			113,06	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
7 22			113,06	
62 15 6	88500,00	21 14 24	12216,76 +	53 6
7 24			112,93	
62 22 30	88600,00	21 15 50	12103,83 +	53 10
7 26			112,80	
62 29 56	88700,00	21 17 17	11991,03	53 13
7 27			112,68	
62 37 23	88800,00	21 18 43	11878,35 +	53 17
7 29			112,55	
62 44 52	88900,00	21 20 10	11765,80 +	53 20
7 31			112,42	
62 53 23	89000,00	21 21 36	11653,38	53 24
7 33			112,29	
62 59 56	89100,00	21 23 2	11541,09 —	53 28
7 35			112,17	
63 7 31	89200,00	21 24 29	11428,92 —	53 31
7 37			112,05	
63 15 8	89300,00	21 25 55	11316,87	53 35
7 40			111,92	
63 22 48	89400,00	21 27 22	11204,95	53 38
7 42			111,79	
63 30 30	89500,00	21 28 48	11093,16 —	53 42
7 44			111,67	
63 38 14	89600,00	21 30 14	10981,49	53 46
7 46			111,55	
63 46 0	89700,00	21 31 41	10869,94 +	53 49
7 48			111,42	
63 53 48	89800,00	21 33 7	10758,52	53 53
7 50			111,29	
64 1 38	89900,00	21 34 34	10647,23 —	53 56
7 52			111,18	
64 9 30	90000,00	21 36 0	10536,05	54 0
7 54			111,05	
64 17 24	90100,00	21 37 26	10425,00	54 4
7 56			110,92	
64 25 20	90200,00	21 38 53	10314,08 —	54 7
7 58			110,81	
64 33 18	90300,00	21 40 19	10203,27 +	54 11
8 1			110,68	
64 41 19	90400,00	21 41 46	10092,59	54 14
8 3			110,56	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
8 3			110,56	
64 49 22	90500,00	21 43 12	9982,03 +	54 18
8 6			110,43	
64 57 28	90600,00	21 44 38	9871,60	54 22
8 9			110,32	
65 5 37	90700,00	21 46 5	9761,28 +	54 25
8 12			110,19	
65 13 49	90800,00	21 47 31	9651,09	54 29
8 14			110,07	
65 22 3	90900,00	21 48 58	9541,02	54 32
8 17			109,95	
65 30 20	91000,00	21 50 24	9431,07	54 36
8 19			109,83	
65 38 39	91100,00	21 51 50	9321,24	54 40
8 21			109,71	
65 47 0	91200,00	21 53 17	9211,53	54 43
8 24			109,59	
65 55 24	91300,00	21 54 43	9101,94	54 47
8 26			109,47	
66 3 50	91400,00	21 56 10	8992,47	54 50
8 29			109,35	
66 12 19	91500,00	21 57 36	8883,12 +	54 54
8 32			109,23	
66 29 51	91600,00	21 59 2	8773,89 +	54 58
8 35			109,11	
66 29 26	91700,00	22 0 29	8664,78	55 1
8 39			108,99	
66 38 5	91800,00	22 1 55	8555,79	55 5
8 42			108,87	
66 46 47	91900,00	22 3 22	8446,92 —	55 8
8 46			108,76	
66 55 33	92000,00	22 4 48	8338,16	55 12
8 49			108,63	
67 4 22	92100,00	22 6 14	8229,53 —	55 16
8 51			108,52	
67 13 13	92200,00	22 7 41	8121,01 —	55 19
8 54			108,40	
67 22 7	92300,00	22 9 7	8012,61 —	55 23
8 58			108,29	
67 31 5	92400,00	22 10 34	7904,32	55 26
9 1			108,16	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
9 1			108,16	
67 40 6	92500,00	22 12 0	7796,16 —	55 30
9 4			108,05	
67 49 10	92600,00	22 13 26	7688,11 —	55 34
9 8			107,94	
67 58 18	92700,00	22 14 53	7580,17 +	55 37
9 13			107,81	
68 7 31	92800,00	22 16 19	7472,36 —	55 41
9 15			107,70	
68 16 46	92900,00	22 17 46	7364,66 —	55 44
9 20			107,59	
68 26 6	93000,00	22 19 12	7257,07	55 48
9 23			107,47	
68 35 29	93100,00	22 20 38	7149,60	55 52
9 27			107,35	
68 44 56	93200,00	22 22 5	7042,25	55 55
9 31			107,24	
68 54 27	93300,00	22 23 31	6935,01	55 59
9 35			107,12	
69 4 2	93400,00	22 24 58	6827,89 —	56 2
9 39			107,01	
69 13 41	93500,00	22 26 24	6720,88 —	56 6
9 44			106,90	
69 23 25	93600,00	22 27 50	6613,98	56 10
9 48			106,78	
69 33 13	93700,00	22 29 17	6507,20	56 13
9 53			106,67	
69 43 6	93800,00	22 30 43	6400,53 +	56 17
9 57			106,55	
69 53 3	93900,00	22 32 10	6293,98	56 20
10 3			106,44	
70 3 6	94000,00	22 33 36	6187,54	56 24
10 7			106,32	
70 13 13	94100,00	22 35 2	6081,22 —	56 28
10 12			106,22	
70 23 25	94200,00	22 36 29	5975,00	56 31
10 17			106,10	
70 33 42	94300,00	22 37 55	5868,90	56 35
10 22			105,99	
70 44 4	94400,00	22 39 22	5762,91	56 38
10 27			105,87	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
10 27			105,87	
70 54 31	94500,00	22 40 48	5657,04 —	56 42
10 34			105,77	
71 5 5	94600,00	22 42 14	5551,27	56 46
10 39			105,65	
71 15 44	94700,00	22 43 41	5445,62	56 49
10 46			105,54	
71 26 30	94800,00	22 45 7	5340,08	56 53
10 51			105,43	
71 37 21	94900,00	22 46 34	5234,65	56 56
10 57			105,32	
71 48 18	95000,00	22 48 0	5129,33	57 0
11 4			105,21	
71 59 22	95100,00	22 49 26	5024,12	57 4
11 12			105,09	
72 10 34	95200,00	22 50 53	4919,03 —	57 7
11 18			104,99	
72 21 52	95300,00	22 52 19	4814,04	57 11
11 24			104,88	
72 33 16	95400,00	22 53 46	4709,16	57 14
11 31			104,77	
72 44 47	95500,00	22 55 12	4604,39 +	57 18
11 39			104,65	
72 56 26	95600,00	22 56 38	4499,74 —	57 22
11 47			104,55	
73 8 13	95700,00	22 58 5	4395,19	57 25
11 54			104,44	
73 20 7	95800,00	22 59 31	4290,75	57 29
12 3			104,33	
73 32 10	95900,00	23 0 58	4186,42	57 32
12 12			104,22	
73 44 22	96000,00	23 2 24	4082,20	57 36
12 23			104,11	
73 56 45	96100,00	23 3 50	3978,09	57 40
12 30			104,01	
74 9 15	96200,00	23 5 17	3874,08 +	57 43
12 40			103,89	
74 21 55	96300,00	23 6 43	3770,19 —	57 47
12 50			103,79	
74 34 45	96400,00	23 8 10	3666,40	57 50
13 0			103,68	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
13 0			103,68	
74 47 45	96500,00	23 9 36	3562,72	57 54
13 12			103,57	
75 0 57	96600,00	23 11 2	3459,15 —	57 58
13 25			103,47	
75 14 22	96700,00	23 12 29	3355,68	58 1
13 38			103,36	
75 28 0	96800,00	23 13 55	3252,32	58 5
13 50			103,25	
75 41 50	96900,00	23 15 22	3149,07 —	58 8
14 3			103,15	
75 55 53	97000,00	23 16 48	3045,92	58 12
14 16			103,04	
76 10 9	97100,00	23 18 14	2942,88	58 16
14 30			102,93	
76 24 39	97200,00	23 19 41	2839,95	58 19
14 45			102,83	
76 39 24	97300,00	23 21 7	2737,12	58 23
15 1			102,72	
76 54 25	97400,00	23 22 34	2634,40	58 26
15 18			102,62	
77 9 43	97500,00	23 24 0	2531,78	58 30
15 37			102,51	
77 25 20	97600,00	23 25 26	2429,27	58 34
15 55			102,41	
77 41 15	97700,00	23 26 53	2326,86 +	58 37
16 17			102,30	
77 57 32	97800,00	23 28 19	2224,56	58 41
16 40			102,20	
78 14 12	97900,00	23 29 46	2122,36 +	58 44
17 5			102,09	
78 31 17	98000,00	23 31 12	2020,27	58 48
17 31			101,99	
78 48 48	98100,00	23 32 38	1918,28	58 52
17 59			101,88	
79 6 47	98200,00	23 34 5	1816,40 —	58 55
18 27			101,78	
79 25 14	98300,00	23 35 31	1714,62 —	58 59
18 58			101,68	
79 44 12	98400,00	23 36 58	1612,94	59 2
19 34			101,58	

Arcus Circuli cum differentiis.	Sinus seu Numeri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexagenariæ.
19 34			101,58	
80 3 46	98500,00	23 38 24	1511,36 +	59 6
20 12			101,47	
80 23 58	98600,00	23 39 50	1409,89 +	59 10
20 53			101,37	
80 44 51	98700,00	23 41 17	1308,52 +	59 13
21 42			101,26	
81 6 33	98800,00	23 42 43	1207,26	59 17
22 53			101,17	
81 29 26	98900,00	23 44 10	1106,09 +	59 20
24 6			101,06	
81 53 32	99000,00	23 45 36	1005,03 +	59 24
25 6			100,96	
82 18 38	99100,00	23 47 2	904,07 +	59 28
26 28			100,85	
82 45 6	99200,00	23 48 29	803,22 —	59 31
27 54			100,76	
83 13 0	99300,00	23 49 55	702,46	59 35
30 20			100,65	
83 43 20	99400,00	23 51 22	601,81	59 38
32 40			100,56	
84 16 0	99500,00	23 52 48	501,25 +	59 42
36 30			100,45	
84 52 30	99600,00	23 54 14	400,80	59 46
41 9			100,35	
85 33 39	99700,00	23 55 41	300,45	59 49
48 54			100,25	
86 22 33	99800,00	23 57 7	200,20	59 53
1 3 42			100,15	
87 26 15	99900,00	23 58 34	100,05	59 56
2 33 45			100,05	
90 0 0	100000,00	24 0 0	000000,00	60 0

JOANNIS KEPLERI,

IMP. CÆS. FERDINANDI II. MATHEMATICI,

SUPPLEMENTUM CHILIADIS LOGARITHMORUM,

CONTINENS

PRÆCEPTA DE EORUM USU.

AD

ILLUSTRISS. PRINCIPEM ET DOMINUM,

DN. PHILIPPUM, LANDGRAVIUM HASSIÆ, &c.

Prima hujus tractatûs editio impressa fuit Marpurgi, et excusa typis Casparis Chemlini, Anno Domini MDCXXV.

JOANNIS KEPLERI SUPPLEMENTUM CHILIADIS LOGARITHMORUM,

CONTINENS

PRÆCEPTA DE EORUM USU.

LECTORI S.

CUM anno 1621, veniſſem in Germaniam ſuperiorem, paſſimque cum peritis rerum Mathematicarum de Logarithmis Neperianis contuliſſem; deprehendi eos, quibus ætas prudentiam addebat, promptitudinem minuebat, ſuper hoc genere numerorum, loco Canonis Sinuum in uſum recipiendo, cunctari: quod dicerent turpe eſſe Profeſſori Mathematico, ſuper compendio aliquo calculi pueriliter exultare, interimque ſine demonſtratione legitimâ formam calculi in uſum recipere, quæ olim, cùm minimè metueres, in erroris inſidias te pertrahere poſſet. Querebantur Neperi demonſtrationem niti figmento motus cujuſdam Geometrici; cujus lubricitas & fiuxibilitas inepta eſſet, in quâ ſolidus ille ſtilus Rationis Demonſtrationumque firmum poneret veſtigium. Hæc mihi cauſa fuit, ſtatim tunc concipiendi rudimentum aliquod Demonſtrationis legitimæ; quod poſterius, ut primùm Lincium reverſus ſum, excolui diligentius: maximè poſtquàm initio anni 1622, ſupra ea re cum Illuſtri & Generoſo D. D. Petro Henrico à Stralendorff, S. Cæſ. Majeſt. in conſiſtorio Imperiali Aulico, vice Præſide, &c. contuliſſem; qui ut eſt harum artium avidiſſimus, rogando plurium me rerum admonuit, quàm liber aliquis benè craſſus præcipiendo admonere poſſet. Per illam igitur hyemem ſolennem rei demonſtrationem ſum aggreſſus; ſubjectum hujus ſpeculationis, quodnam eſſet genuinum, deſcripſi; quodque id non eſſet verè ſub genere vel linearum, vel motus fluxuſque, aut cujuſquam alterius quantitatis ſenſilis, ut ſic dicam, ſed ſub genere Relationum, quantitatiſque mentalis, evidentibus enunciationibus conſtitui: deinde, cùm,

ut omnes cæteræ, ſic etiam mentalis iſta quantitas (λόγος Græcis dicta, quod Latini *Rationem* minùs uſitatè, crebriùs *Proportionem* transferunt) diviſionem in infinitum recipiat: metas etiam hujus diviſionis congruas, ex eodem, ſcilicet, genere rerum, conſtitui: Lineæ enim punctis, motus articulis temporum dividi ſolet; at Proportionem diſpeſcunt termini inter extremos magnitudine medii. Hiſce ſic conſtitutis feliciter, procedebat demonſtratio: vera proportionum omnium communicantium communiſque menſura, facta eſt & ipſa de genere proportionum: locus eſt factus Arbitrio, in eligendo proportionis Elemento, quodnam deberet haberi pro minimo, proque menſura: demonſtratum etiam eſt plerunque eſſe proportiones inter ſe incommunicabiles; eoque minimum unius Elementum quod placuiſſet, non poſſe eſſe menſuram genuinam proportionis cujuſcunque alterius: ſemper enim peccari vel exceſſu, vel defectu. Ea de cauſa factus jam eſt hic alter locus Arbitrio, ut quinam defectus, quæ minimorum Elementorum incommunicabilium differentia diſſimulari poſſet, inque profundum inſenſibilitatis demergi, conſtitueretur: ut ſic tandem in uſum ipſum numerorum redundaret, abſurda quidem demonſtranti, ſed utiliſſima computanti, *incommunicabilium proportionum communis menſura*, *Logarithmus dicta*.

Simul autem fuit ipſo opere monendus Lector meus, Logarithmos non primum naſci cum Sinubus, ſeu rectis in Circulo, quod Neperiana deſcriptio incautius inſpecta præ ſe ferre videtur, ſed foris extra Geometriam Circuli conſtitui, tanquam intra metas libri Quinti Euclidis: inde verò tranſſumptos, applicari Sinubus; eoque concipiendam eſſe animo (quam ipſe dudum deſcripſeram in Tabula) Matricem quandam Logarithmorum, ex qua omnis per numeros expreſſa quantitas, etiam Sinus ipſi, ſuos Logarithmos peterent, ea ipſa numerorum fide, quâ Sinus ipſi ſunt in Geometria Circuli definiti: ut ſi qui Sinus non legitimo numero ſit expreſſus, idem etiam ex hac Matrice Logarithmum imperfectum hauſurus ſit, nullo Matricis, ſed ſuo proprio vitio. Hoc igitur propoſito & computata eſt à me Chilias Logarithorum, & adornata etiam demonſtratio; quæ ordinem præcipuè numerorum Chiliadis eſt complexa, cauſaſque ejus in lucem profert. De uſu Chiliadis populari cogitatio æqualis quidem tempore, naturâ tamen fuit poſterior. Chiliada cum demonſtratione, ut primùm perfecta fuit, tranſmiſi Lincio tanquam ad PHILIPPUM LANDGRAVIUM HASSIÆ, ſed per ambages itinerum, in quibus libellus hæſit in tertium annum uſque. Spes interdum affulſit mihi, libellum excuſum iri curâ peritorum: ſed ea ſubinde iterum extincta fuit. Itaque refrixit etiam apud me ſtudium libelli exornandi. Tandem in manus Patroni ſui Illuſtriſſimi perlatus, me penitus ignaro, nec quicquam tale opinante, typis mandatus fuit; ut prius ex Catalogo

Nundinarum

Nundinarum Francofurtensium, quàm ex Literis Celſ. ſuæ (quas adhucdum in itinere eſſe nuper admodum reſcivi) quid ageretur perceperim. Non dubito, quin Celſ. ſua in literis me moneat, ſi quid de uſu Chiliadis conceperim, ut id ſubſidio mittam, quo commendatior & utilior libellus exeat. Nam illi jam ſuperius commemorati fines ſcriptionis, reconditiorem ſenſum habent, adque paucos pertinent. Et habet ſanè ipſa Chilias tres columnas ſuperadditas, quæ totæ ſe à ſubtilitate illa demonſtrationum ſubducunt, adque uſum conferunt. Ut igitur, quod in hac editione, temporum et locorum culpâ fuit omiſſum, ſuppleatur, ut ſcilicet Chilias iſta, è penetralibus illis ſpeculationum & demonſtrationum, quarum præcipuè cauſâ compoſita fuit, etiam ad populares uſus educatur, prius ſunt aliqua mihi præfanda de Titulo. Is profitetur Demonſtrationem legitimam ortus Logarithmorum. Et ſpero faſſuros eſſe Geometras, quod in chartis excuſis hunc ſcopum ſim conſecutus. Profitetur verò etiam Demonſtrationem uſus Logarithmorum. Hic ſubſiſte, Lector. Dum enim rei demonſtrationem polliceor, rem ipſam præſuppono jam antea notam, vulgoque tritam. Quænam verò res iſta? quis uſus Logarithmorum? Nimirum is ipſe, qui jam ante decem annos à primo authore Nepero fuit traditus, quique tribus verbis poteſt concipi: *Ubicunque occurrunt in Arithmetica communi, inque Regula Trium duo numeri inter ſe multiplicandi, ibi ſunt eorum Logarithmi in unam ſummam addendi; ubi verò Numerus abſolutus jubetur factum dividere, ibi Logarithmus illius eſt auferendus à Logarithmorum ſummâ: ut Logarithmus illic acervatus, hic reſiduus, monſtret numerum in qualibet operatione quæſitum.* Hic, inquam, eſt uſus Logarithmorum. Hujus uſus demonſtrationem habes in Propoſitionibus XVIII, XIX, XX, ut vides in earum Corollariis expreſſis verbis indicatum. Perpende rem, dices, ſcio, nihil promiſſum eſſe in tituli verbis allegatis, quod non ſit præſtitum in his tribus Propoſitionibus. Quod verò pulli Arithmeticorum implumes, facilitatum avidi, roſtra ad hanc uſus mentionem in immenſum pandunt, velut hauſuri omnes præceptiuncularum particularium bolos, quibus exſatientur; id equidem præſtare in chartis iſtis, quæ demonſtrationi fundamentorum erant deſtinatæ, non potui. At dixeris, ſequentia verba Tituli de novâ Arithmeticâ omninò plura polliceri? Video equidem irritamenta cupiditatum talium, & rideo; mirorque non incidiſſe mihi, cum ſubito tabellarii deceſſu ad properandum compulſus, illud Elogium inventi Neperiani, quo Typographus aliquis illiceretur, titulo libelli mei ſubjungerem; fore ut in majus acciperetur, veluti propria mea gloriatio. Sed utcunque quis hoc meum conſilium accipiat, de intentione ſaltem, me excuſat conſtructio Grammatica. Non de Chiliadis libello, non de mea demonſtratione uſus, dictum vides, ſed de Logarithmis ipſis, invento ſcilicet Neperiano; quod iis nova tradatur

O Arithmetica.

Arithmetica. Nec ego hæc ſequentia præcepta de uſu Chiliadis hujus in id ſubmittenda libello jam excuſo ſtatui, ut hanc poſteriorem tituli partem penitus ad ipſam meam Chiliada traducerem: ſegregetur ea pars tituli a labore meo, adque inventum ipſum Neperianum, & ad Canonem Logarithmorum Quadrantis referatur. Non titulus iſte ſubitò conceptus, ſed utilitas ipſa emptoris me monet, ut quomodo ſingulæ Chiliadis columnæ poſſint ad uſum popularem referri & quatenus, pauculis præceptis docerem. Ea verò referam in certa Capita. In primo paucula præmittam de Tabulâ Chiliadis. In ſecundum Caput referam comparationem Columnæ arcuum, cum Columnâ Rotundorum. Tertio dicam de comparatione Columnæ ſecundæ Sinuum, cum tertia Quadrivicenaria, & quinta ſexageſimorum. Quarto comparabo Quadrivicenariam cum Sexagenariâ. In quinto Capite erit uſus Quadrivicenariæ, & Logarithmicæ columnarum junctarum. In ſexto Sexagenariæ & Logarithmicæ. In ſeptimo comparabitur Columna prima arcuum, cum quartâ Logarithmorum. In octavo jungentur Columna ſecunda Abſolutorum, & Quarta Logarithmorum. In nono denique Columna Logarithorum jungetur duabus aliis columnis.

PRIMUM CAPUT.

DE TABULA CHILIADIS, EJUSQUE COLUMNIS.

PRIMUM ſciat Lector, exemplar meum in formâ folii ſcriptum, diviſum fuiſſe in veſtibulum Chiliadis, & in Chiliada ipſam. In veſtibulum accenſentur lineæ 36, ut Chiliadis ipſius principium habeatur ibi, ubi eſt numerus 100,00. Hac monitione opus eſt. Nam veſtibuli non eſt eadem proportionum ſequela, quæ Chiliadis ipſius, ut apparet ex ipſis Logarithmorum differentiis: quarum ſeries quater ad idem revertitur initium. Quare nec potuit eſſe uſus idem trium columnarum in veſtibulo, qui in Chiliade, eaque de cauſa relictæ ſunt vacantes: denique totum veſtibulum abeſſe poſſet, niſi capite octavo tribuendum id eſſet analogiæ cum arithmeticâ communi, & integrationi numerorum omnium.

Jam columnas Chiliadis ipſius vides eſſe quinque, lineas mille. Prima eſt columna arcuum quadrantis, qui habent pro ſinubus rotundos illos ſecundæ columnæ numeros. Arcus iſti ſunt expreſſi ternis membris, in primo ſunt gradus, in ſecundo ſcrupula prima, in tertio ſcrupula ſecunda, quæ, cum ſuis tertiis & quartis careant, non poſſunt exacta eſſe, præterquam in unico arcu 30° inque fine quadrantis. Quanquam inepta eſt cura de parte unius ſecundi, cùm ipſa applicatio arcuum horum ad ſuas lineas, rotundis numeris expreſſas, eoque effabiles, non ſit ſecuta demonſtrationis alicujus ſubtilitatem; ſed ſolùm popularem inquiſitionem partis proportionalis ex canone ſinuum. Itaque fieri poteſt, præſertim circa finem quadrantis, ut ſcrupula ſecunda paucula paſſim vel deficiant, vel abundent. Inter lineas inſertæ ſunt differentiæ, quantæ arcubus per ſingulas milleſimas totius ſemidiametri accreſcunt; earum minima in principio quadrantis eſt $3' \, 26'' +$, maxima in fine quadrantis eſt $153' \, 45''$, ſeu $2^{\circ} \, 33' \, 45''$, quadragecupla quintupla minimæ.

Sequitur ſecunda columna numerorum: ipſorum rotundorum. Rotundos hic appello, qui continent exactè partes milleſimas ſui maximi in fine poſiti, ſcilicet 100000,00, hoc eſt, Rotundi habentur, qui terminantur in quatuor minimum cyphras rotundas, nullas dictas.

Viciſſim numeri ſcrupuloſi habentur reſpectu chiliadis hujus, qui definunt in cyphras ſignificativas, aut qui minus quàm quatuor nullas habent in fine. Ut 96300,00 eſt rotundus, ſic etiam 90000,00. At 96350,00, & 96357,00, & 96300,68, & 90000,05, ſunt ſcrupuloſi. Atque hi rotundi noſtræ Chiliadis, reſpectu quidem arcuum ad latus ſiniſtrum poſitorum, appellantur Sinus, reſpectu verò columnarum cæterarum ad dextram appellantur Abſoluti, quia in cæteris columnis intelliguntur numeri ſignificatione aliquâ certâ revincti.

O 2 Hic

Hic venit obſervanda cauſa, cur, cùm Chiliada ſcribere propoſuerim, numerus tamen maximus non ſit 1000, ſed potius 100000,00, ejuſque unitas non 1, ſed 100,00. Deinde cur ſemper ultimæ duæ numeri cujuſque figuræ interpoſito puncto ſint ſeparatæ, & velut abſciſſæ à reliquis verſus ſiniſtram. Cauſa igitur utriuſque rei eſt eadem, quia nimirum in vulgato canone ſinuum, maximus ſeu ſemidiameter circuli conſtat figuris totidem, vel enim 100000, vel 10000000, ſolet uſurpari. Et quia radius ille ad quinque cyphras continuatus plerunque ſufficit ad calculum, alterius qui ſeptem cyphris exprimitur, rarior eſt neceſſitas: hinc poſtremi canonis ſinuum excultores cœperunt duas ultimas, tanquam minus neceſſarias, puncto ſeparare. Atque hoc ego mihi ratus ſum obſervandum in rotundis meis, ut qui ſunt etiam loco ſinuum. Si quis iis, ut abſolutis, velit uti, poterit quatuor ultimas cyphras rejicere, quoties id uſus aut præcepta ſequentia exegerint.

Porrò hæc cauſa attinet etiam columnam quartam, quæ præcipua eſt, Logarithmorum ſcilicet cum differentiis, de qua nunc plura dicam.

Primum operæ pretium fecerit computiſta, qui hâc Chiliade eſt uſurus, ſi ſubordinationem differentiarum: ſub ſuos Logarithmos ſubtilibus lineolis adjuvet, ut ſingulæ figuræ illarum ſingulis horum reſpondeant.

Deinde notet calculator, numeros hos non eſſe ἀριθμοὺς abſolutos, ſed eſſe λογαρίθμους, numeros ſcilicet relatos ſingulos ad binos alios, numeros proportionem ſignificantes, quæ eſt inter abſolutum ad latus ſiniſtrum poſitum, & inter maximum Chiliadis ſeu 100000,00.

Tertiò hi Logarithmi omnes ſunt ſcrupuloſi, hoc eſt, non rotundi. Non tamen exhauſta eſt omnis eorum ſcrupuloſitas, continuatione hâc eorum uſque ad duas figuras ultra punctum. Atque hoc indicatur + ibi per ſigna + plus & − minus paſſim appoſita.

Nam quibus nullum tale adeſt ſignum, iis intellige non unam quadrantem unitatis neque deficere, neque abundare: qui verò habent ſignum +, ii habent inſuper plus quàm quadrantem, minus quàm ſemiſſem unitatis, aut ſummum ſemiſſem: qui denique habent ſignum −, eos intellige minuendos aliquâ particulâ unitatis, quæ ſit inter quadrantem & ſemiſſem. Qui igitur vult accuratiſſimè computare; is ſi duos logarithmos eſt additurus cum ſigno +, ſummæ poterit addere unum inſuper ſemiſſem: Sin duos cum ſigno −, de ſummâ detrahet ſemiſſem. Et in ſubtractione, ſi qui eſt cum ſigno − ſubtrahendus erit ab aliquo cum ſigno +, eis reſiduum faciet ſemiſſe auctius. Sin è contrario is qui habet +, ſubtrahi debet ab eo qui habet −. Reſiduum ſemiſſe minuendum erit.

Exempla additionis.	12783,34 −	59783,70 +
	230258,52 −	61618,61 +
	243041,85½	121402,31½
	ſive 86 −	ſive 32 −
Exempla ſubtractionis.	59783,70 +	231263,55 −
	12783,34 −	1005,03 +
	47000,36½	230258,51½
	ſive 37 −	ſive 52 −

Hæc

Hæc ſcrupuloſitas locum habet tantùm in logarithmis, qui ſunt expreſſi in Chiliade, in aliis intercidentibus præſtari non facilè poteſt: nuſpiam verò eſt abſolutè neceſſaria. Immò plerumque contemni poſſunt duæ ultimæ poſt punctum. Denique quantò major logarithmus, tantò minus periculi in neglectu figurarum differentiæ, poſt duas ad ſiniſtram primas omnium.

Reſtant columnæ tertia & quinta. Et in tertia quidem ſunt numeri logiſtici non abſoluti; maximus habet 24° quot ſcilicet ſunt unius diei horæ, etſi poſſunt ſignificare promiſcuè vel horas, vel gradus.

In quinta rurſum ſunt numeri logiſtici, complectentes collectionem ſexagenariam: itaque maximus eſt $60'$. Tertia quidem numeros exhibet ſuos tribus membris, in primo ſunt integri gradus vel horæ, in ſecundo ſcrupula prima unius integri, in tertio ſcrupula ſecunda. Quinta duobus membris eſt contenta, ſcrupulis nimirum & ſecundis. Cùm autem diviſio millenaria non ſit apta numeris 24^{a} & $60'$, fit ut in his duabus columnis, ultima unitas ſcrupulorum ſecundorum non ſit exacta: nam minutiæ ejus, quæ minus ſemiſſe efficiunt, neglectæ ſunt, quæ plus, ea completione unitatis ſunt confuſæ & obliteratæ. Pars enim milleſima de 24° eſt $0^{\circ}\ 1'\ 26''\ 24'''$, pro qua ſcripta ſunt $0^{\circ}\ 1'\ 26''$. Duæ tales partes ſunt $2'\ 52''\ 48'''$, pro hoc ſcripta ſunt $2'\ 53''$. Et in ſexagenaria pars milleſima eſt $0'\ 3''\ 36'''$, pro qua ſcripta ſunt $0'\ 4''$ Duæ milleſimæ ſunt $0'\ 7''\ 12'''$, pro quo ſcripta ſunt $0'\ 7''$. Itaque quaternis lineis ſemper exceſſu vel defectu peccantibus, ſolæ quintæ ſunt exactæ. Propriam verò & accommodatam hæ columnæ ſubdiviſionem ſortientur in tabulis Rudolphi. In præſens Chiliadi dandæ fuerunt partes potiores: et electus eſt à me millenarius, tanquam propria collectio numerorum abſolutorum: quia, ut initio dictum, ſcopus meus fuit, monere lectorem, quod proportiones, earumque menſuræ, logarithmi, accidant primò numeris abſolutis. Nihil tamen impedit has duas columnas, tertiam & quintam etiam per hanc diſpoſitionem minus exactam, & impropriam, millenariam ſcilicet, fieri utiles: ut in ſequentibus dicetur.

CAPUT II.

De Aſſociatione Columnæ Arcuum, & Columnæ Rotundorum.

DIXI cur in columnâ numerorum abſolutorum ſtatuerim progreſſionem arithmeticam continuam ab unitate ad mille, ſeu à 100,00 ad 100000,00, & qua ratione, cuilibet ex hiſce rotundis in columnâ arcuum, aſſociaverim ſuum arcum.

Cùm autem arcus hi omnes ſint ſcrupuloſi, fit creberrimè, ut oblato arcu non ſcrupuloſo, aut ſcrupuloſo quidem, ſed ſic, ut non exactè reperiatur inter arcus Chiliadis, ſed inter binos intercidat, ut his, inquam, arcubus jubeamur aſſignare ſuos ſinus; ut ita Chilias noſtra, quadamtenus etiam ſerviat loco canonis ſinuum, ejuſque vices ſuppleat. Id autem fieri poterit hac ratione.

Per

Per arcum Chiliadis proximè minorem propofito excerpe finum rotundum; eundem arcum aufer à propofito: refidua fcrupula, prima & fecunda colloca in regula detri ad dextram, differentiam verò binorum arcuum, inter quos pofitus intercidit, colloca ad finiftram, in medio colloca numerum progreffionis perpetuum fcilicet 100,00, & operare per regulam detri prodibunt enim figuræ 4, ultimæ fcrupulofæ adjiciendæ ex fcripto finui rotundo, feu loco deletarum ejus 4, cyphrarum finalium fcribendæ.

Viciffim dato finu fcrupulofo, indagabimus etiam ejus arcum; fi cum finu rotundo Chiliadis proximè minore quàm eft propofitus, hoc eft, cùm ejus tribus primis figuris ex feptem, excerpamus arcum, deinde de differentia interjecta inter duos arcus proximos, fumamus partem proportionalem fcrupulofitati, quam 4, ultimæ finus dati figuræ complectuntur.

Hic cùm mifceantur numeri diverforum generum, facilè compendium aliquod invenies ex teipfo, fi bonus es arithmeticus, ingenioque polles. Si verò nefcis compendium, utere noftro, quod huic rei ferviturum in cap. IX num. VII & VIII rejectum eft.

Non quidem erit exactiffima vel una, vel altera operatio, præfertim in fine quadrantis; quia differentia tunc fit fubitò magna. Itaque conduceret, decem ultimas differentias in denas fubdividere. Et interpofitis finubus ufque ad quartam figuram fcrupulofis, fuos illis ex canone quadrantis affignare arcus, quod quilibet privatâ operâ poterit. Exceptis verò decem ultimis differentiis, per reliquum quadrantis perveniemus in notitiam finuum ad 5, faltem figuras continuatorum, non multò incertius, facilius etiam, quàm per canonem finuum, ad fingula fcrupula prima extenforum: quia laborem quærendæ partis, quæ fit proportionalis adhærentibus fecundis, per logarithmos fublevamus, ut capite IX doceberis. Quomodo etiam fecantes arcuum, quodammodo haberi ex hâc Chiliade poffint, infra apparebit capite VIII.

Jam quorfum fint nobis utiles finus arcuum; id non eft hujus loci docere, ubi omnia per logarithmos perficimus. Adeantur libri qui tradunt doctrinam triangulorum, per rectos quadrantis.

CAPUT III.

De Comparatione Columnæ fecundæ Sinuum, cum tertiâ Quadrivicenariâ, vel cum quintâ fexagefimorum.

FIT interdum, fic exigente parte arithmeticæ, quæ logiftica dicitur; ut femidiameter circuli aliter divifa offeratur, quàm in partes denarias, centenarias, millenarias, &c. Verbi caufâ diameter folis vel lunæ deficientis dividitur in partes 12, quas digitos appellamus. Quod fi igitur detur aliqua portio diametri in partibus, qualium eft femidiameter 100000. Ea portio quæfita inter abfolutos, ftatim monftrat è regione fub quadrivicenariâ quadruplum numerum digitorum, quos valet illa portio.

Ut ſi deficiat ſexta pars de ipſâ circumferentiâ lunæ; ad ſciendum quot digiti ſuperſint in lumine, poſito quod umbra terræ ſecet diametrum, ut recta linea ad rectos angulos: ſuperſunt igitur in lumine 300 gradus. Horum dimidium eſt 150°, hoc eſt 90°, & 60°. Partes igitur diametri reſpondentes erunt 100000, & 86603. Illa in quadrivicenariâ oſtendit 24° iſta 20° 47′ 2″ ſumma 44° 47′ 2″. Hinc quarta pars eſt 11° 11′ 45″. Et tot ſunt digiti reſciſſi de diametro.

Viciſſim ſi deficiat digitus, quid is valet in dimenſione, qualium ſemidiameter, hoc eſt, 6 digiti, ſunt 100000? Quia igitur 24, numerus hujus columnæ maximus eſt quadruplum de 6, ſume etiam quadruplum digiti unius, id eſt, 4°. Hoc verò quæſitum in quadrivicenaria, offert inter abſolutos 16675,00 circiter.

Eodem modo fit interdum, ut ſemidiameter valeat authoribus non 100000, ſed 60′, ut ſolet Ptolomæus, et plerique aſtronomorum in indicatione proportionis orbium. Ii authores, ſi exprimant in hac dimenſione ſinus arcuum, aut partes quaſcunque ſemidiametri; quæſitæ eæ partes in ſexagenariâ, exhibent in abſolutis, valorem illius partis in dimenſione hodiernâ ſinuum progreſſionis denariæ.

Exempli cauſa, quæritur 38′ 47″, quantum faciat ſinum, qualium 60′ eſt 100000, reſpondetur 64630 circiter.

Sic etiam viciſſim quæritur 9265, eccentricitas Martis, quot faciat ſcrupula, qualium 100000 facit 60′, reſpondetur 5′ 33″.

CAPUT IV.

De Comparatione Quadrivicenariæ cum Sexagenariâ.

SOLET eſſe pars tabularum aſtronomicarum converſio horarum in ſcrupula diei, & viciſſim.

Hæc converſio habetur per has duas tabulas ſatis exactè, præſertim in quintis lineis, quas rotundas dicere poſſumus.

Verbi cauſâ horæ 6° 28′ 54″, quot valeant ſcrupula diei? In columna tertia occurrunt 6° 28′ 48″, quæ faciunt in quinta 16′ 12″ exactè, quia linea eſt rotunda & quinarii ultima. Reſiduum igitur 6″ unius minuti horarii, quæſita ut horæ, dant 15′, 0″, quæ jam valent 15‴, junctis ſcilicet apicibus reſidui horarii 6″, et horum 15′ horis 6° reſpondentium. Ergò totus numerus 6° 28′ 54″ valet ſcrupula diei 16′ 12″ 15‴.

Viciſſim 47′ 39″ ſcrupula diei quot valent horas?

In quinta 47′ 38″ + valent horas 19° 3′ 22″ — reſiduum ſecundum unicum ſeu 60‴, quæſita ut 60′ oſtendunt 24″ (duobus apicibus, quos illis 60‴ detraxeramus, numero oſtenſo 24° ſuperpoſitis.) Ergò propoſita ſcrupula 47′ 39″ valent horas 19° 3′ 46—Non prodit enim hic exactum, quia linea eſt ſcrupuloſa, quarta ſcilicet ſui quinarii.

Sed

Sed ſi memor eris eorum, quæ capite primo ſunt dicta de his linearum quinariis, quod ſcilicet in primis intelligantur accedere quadrivicenario numero $24'''$, in ſecundo deficere $12'''$, in tertio accedere $12'''$, in quarto deficere $24'''$. Et ſexagenario numero in prima linea deficere $24'''$, in ſecunda accedere $12'''$, in tertia deficere $12'''$, in quarta accedere $24'''$, poteris etiam ex quatuor prioribus quinarii lineis exactè excerpere converſiones iſtas. Nam exactè $47'\ 38''\ 24'''$ valent $19^\circ\ 3'\ 21''\ 36'''$. Ergò reſidua $36'''$, unius ſecundi quæſita, $36''$ oſtendunt horas $14^\circ\ 24'$, valentque $14''\ 24'''$. Itaque totus $47'\ 39''$ valet exactè horas $19^\circ\ 3'\ 36''\ 0'''$.

In tabulis Rudolphi exacta erit iſta converſio in lineis omnibus, quia ſervabitur naturalis, & propria diviſio numerorum 24° & 60° in lineis omnibus.

Alia quoque converſio eſt in uſu crebro, horarum 24 in gradus æquatoris 360, & viciſſim. Et poteſt hac quoque converſio perfici ope harum 2 tabularum in hunc modum.

Quæritur horæ $19\ 25'\ 37''$, quot faciunt gradus, & ſcrupula æquatoris. In columna igitur tertia $19^h\ 26'\ 24''$, dat $48'\ 36''$ exactè. Sed temporaria $47''$, quæſita ut temporaria $47'$ dant $1'\ 58''$, quæ valent $1''\ 58'''$. Ergo propoſitæ $19^h\ 25'\ 37''$ dant $48'\ 34''\ 2'''$, quæ multiplica in 6, & diminue apices, roveniunt $291^\circ\ 24'\ 12''$, gradus æquatoris.

Viciſſim gradus æquatoris ſic convertes in horas: divide eos per 6, & auge apices: quotiens in ſexagenariam immiſſus, oſtendit è regione horas quæſitas.

Quæruntur $259^\circ\ 34'\ 17''$ quot ſint horæ?

Sexta pars eſt — 43 15 43. Quære igitur $43'\ 15''\ 43'''$, & monſtrabunt in tertia $17^h\ 18'\ 14''$—.

CAPUT V.

De Quadrivicenariâ cum Logarithmis compoſita.

I. TABULA diurnorum & horariorum ſolet Ephemeridibus præfigi, prolixa admodùm: ubi docemur, dato diurno excerpere horarium, ad datum horarum numerum. Excerptio ipſa licet non fatiget mentem multiplicatione, fatigat tamen & mentem additione, & oculos manuſque excerptione. At hîc conjunctio columnæ quartæ cum tertia, & tollit neceſſitatem tabulæ diurnorum, & opus peragere docet longè facilius, quippe additione ſimplici (non logiſtica) logarithmorum.

Quære enim tam diurnum, quàm horas in quadrivicenaria, & logarithmos ab iis monſtratos adde in unam ſummam; quæ inter logarithmos quæſita, monſtrabit è regione in columna tertia portionem motus competentem horis.

Utile eſt hoc præceptum, ubi magnus eſt diurnus, non major tamen quàm gr. 4, ut in lunâ & cometarum nonnullis.

Exem-

EXEMPLUM.

Sit diurnus lunæ 14° 23.′ Quæritur quantum
competat horis 19 42
Logarithmus ad 14 23 0″ eſt 51200 circiter.
Logarithmus ad 19 42 0 eſt 19730 circiter.

Summa 70930

Hæc quæſita inter logarithmos, oſtendit in quadrivicenariâ 11° 48′ 0″. Tot gradus competunt horis 19 42′.

ALIUD EXEMPLUM.

Sit diurnus 2° 15′. Horæ 938′.
Logarithmus 2 15 eſt 237000 circiter.
Logarithmus 9 38 eſt 91200 circiter.

Summa 328200 dat 0° 54′ 0″.

Poteſt verò hoc exemplum, ubi diurnus eſt adeò parvus, aliter etiam & faciliori viâ computari, quam referam in caput IX.

II. Huc pertinet etiam ratio computandi adſpectus lunæ cum ſtellis. Ut ſi detur ſeparatio diurna duorum planetarum, & diſtantia eorum, minor quàm illa; quæraturque quot horis ea conficiatur?

Tunc enim à logarithmo diſtantiæ, auferatur logarithmus ſeparationis diurnæ, reſiduum inter logarithmos quæſitum, oſtendit è regione ſub quadrivicenariâ horas competentes.

EXEMPLUM.

Sit mercurii diurnus retrogradus 2° 15′
Lunæ directus - - 15 2
Separatio igitur diurna 17 17
Logarithmus eſt - - 32840 circiter.
Sit diſtantia lunæ & mercurii 12 23
Logarithmus - - 66170 circiter.

Reſiduum 33330 dat 17° 12′.

Tot igitur horis fit aſpectus lunæ, ante vel poſt momentum, quo invenitur hæc diſtantia, prout luna poſt vel ante mercurium fuerit.

Si uterque directus, vel uterque retrogradus eſſet; minor diurnus à majori ſubtraheretur, ad eruendam ſeparationem diurnam, uti docemur in aſtronomia. De aſpectibus tardiorum inter ſe infra, cap. IX.

III. Diurnum indagare (minorem quidem quàm 24^h) ex motu aliquot horarum.

Tunc logarithmum horarum aufer à logarithmo motus competentis, reſtabit logarithmus motus diurni.

EXEMPLUM.

Promotum deprehendatur ſidus aliquod in horis 4 27′.
Gradus 1 53′ quantus erit diurnus?
Gradus 1 53 logar. 254500 circiter
Hor. 4 27 logar. 168400

Reſiduum 86100 dat gr. 10 9′ diurnum.

Infra cap. IX, erit alius modus pro diurnis parvis.

IV. Datis tribus ſtellis in unâ rectâ, datiſque ſingularum latitudinibus, duarum verò ſolarum longitudinibus; indagare & tertiæ longitudinem.

Idem, ſi pro latitudinibus notæ fuerint declinationes, pro longitudine verò ſumatur aſſenſio recta.

Cùm hæ rectæ plerumque ſint breves & infra 24° nihil nobis nocebit, abuſus ἄτεχνος curvilineorum ut rectilineorum triangulorum.

Pone in regula detri, ad ſiniſtram, differentiam latitudinum duarum, in medio differentiam longitudinum earundem; ad dextram differentiam iterum latitudinis ſtellæ tertiæ & priorum unius; quæſitiſque his arcubus in quadrivicenariâ, exſcribantur logarithmi; additiſque ſecundo & tertio, de ſumma dematur primus; reſtabit logarithmus differentiæ longitudinis ſtellæ tertiæ.

Uſus hujus præcepti peropportunus eſt in tractatione obſervationum per lineas rectas trium ſtellarum.

CAPUT VI.

De Sexagenariâ cum Logarithmis compoſitâ.

TABULÆ hexacontádôn uſus eſt in logiſticâ multiplex, præcipuus quidem ad multiplicationes & diviſiones logiſticas, ſcrupuloſitatis infinitæ, ubi fit progreſſio ad tertia, quarta, quinta, ſexta, & ſic deinceps. Tantam ſubtilitatem non profitetur uſus Chiliadis noſtræ. Maneat igitur hæc utilitas propria tabulæ hexacontádôn.

Sed quia hæc tabula præfigi ſolet etiam tabulis aſtronomicis & ephemeridibus,

dibus, in quibus non fit progreſſio ultra ſecunda, vel ad ſummum tertia: hunc ejus uſum tabula noſtra ſexagenaria pleniſſimè præſtat, ejuſque neceſſitatem penitus tollit: quod docebo præceptis ſequentibus.

I. Multiplicationes logiſticas perficere compendioſiſſimè.

Primum capiant numeri, qui ſunt in ſe mutuò multiplicandi, denominationes ſeu apices familiares tabulæ Chiliadis. Nam quia in logiſtica apices antecedentes integra 1° verſus ſiniſtram, ſunt indices ſexagenarum, ſequentes verò integra 1° ſunt indices ſexageſimorum ſcrupulorum, ſeu fractionum unius integri: ſciendum igitur, numeros in columnâ ſexagenariâ propriè intelligi de ſexageſimis ſcrupulis primis & ſecundis, tanquam fractionibus unius integri. Ergò ſi offerantur gradus & minuta in datorum alterutro, pro gradibus ſcribantur minuta, pro minutis ſecunda. Tunc iis in ſexagenaria quæſitis, excerpantur logarithmi, addanturque in unam ſummam; quæ inter logarithmos quæſita exhibebit ſub ſexagenaria factum in primis & ſecundis. Sed ſi prius apices multiplicandorum fuerunt mutati, nunc etiam facti apices erunt mutandi in contrarium, nam quot apices utrique junctim additi ſunt, totidem jam uni ſoli quotienti ſunt adimendi, & viciſſim.

EXEMPLUM.

Sint multiplicandi 25′ 35″ log. 85240
& 49′ 50″ log. 18566

Summa 103806 dat 21′ 14″ factum.

ALIUD.

15° 38′ 40″ ſint multiplicanda in 53′ 49″.
Scribe 15′ 38″ 40‴ log. 134400. Hic apices aucti ſunt unitate.
& 53′ 49″ log. 10900

Summa 145300 dat 14′ 2″.
Sed ſcribe 14° 2′ detracta ab apicibus unitate viciſſim.

ALIUD.

Sexagenæ. Sexageſimæ.
39° 20′ ſint multiplicanda ˋ46 15°
Scribe 39′ 20″ log. 42227. Hic apices aucti unitate
46′ 15″ log. 26028. Hic apices aucti binario.

Summa 68255 dat 30′ 19″ 0‴.

Sed ſcribe ˵30 ˋ19 0° detracto ternario ab apicibus ſexageſimariis.
Scilicet hic factus non eſt 30′ ſcrupula ſexageſima prima, & 19″ ſecunda, ſed ˵30 ſexagenæ ſecundæ & ˋ19 primæ, &c.

ALIUD EXEMPLUM.

Sint multiplicandæ ſexagenæ $`37\ 41^{\circ}$ in ſexageſima $32''\ 29'''$.
Scribe $37'\ 41''$ log. 46513. Hic apices ſunt aucti binario
$32'\ 29''$ log. 61362. Hic apices deminuti unitate

Summa 107875 dat $20'\ 24''$
Sed ſcribe $20^{\circ}\ 24'$.

Nam propter primum detrahendus eſſet ab apicibus binarius, propter ſecundum addenda unitas viciſſim. Compenſatione igitur contrariorum facta, adhuc detrahenda fuit à facti apicibus unitas.

II. Quadrare numerum logiſticum facillimè.

Rurſum eſto memor, quod numeri in ſexagenariâ ſint fractiones, eoque numerus quadratus prodiens, quod æſtimationem apicum attinet, minor ſeu minoris valoris ſit, quàm quadrandus. Propriiſſimè quidem quadratus, de quo quæritur: ſic eſt ad quadrandum, ut hic ad unitatem 1° ſeu $60'$ ſcr. ut ſit quadrare logiſticè nihil aliud, quàm tertiam proportionalem à maximo columnæ ſexagenariæ invenire.

Numeri igitur quadrandi in ſexagenariâ quæſiti logarithmum duplica, duplum in logarithmis quæſitum exhibet ex ſexagenariâ quadratum, quod quærebatur.

EXEMPLUM.

Sit quadrandus $49'\ 53''$. Ejus logarithmus 18450 dupletur ut ſit 36900 hoc in logarithmis quæſitum exhibet $41'\ 29''$. Dico igitur $41'\ 29''\ 0'''\ 0''''$ eſſe quadratum numeri $49'\ 53''$.

Si quid mutandum in apicibus, ut quadrandus inveniri poſſit in Chiliade, tunc valent præcepta eadem, quæ prius in multiplicatione, tantummodò ut memineris, quadrandum eſſe vice duorum in ſe multiplicandorum.

III. Diviſiones logiſticas perficere compendioſiſſimè.

Primò obſerventur eadem de aptatione apicum, quæ prius circa multiplicandos. Deinde obſerva, num dividendus (ſic aptatus ſi opus fuit) major fuerit, quàm diviſor, an minor. Nam ſi major, non propriè pertinet operatio ad Chiliada; ſed ejus loco dividendus eſt, vel exceſſus ejus ſupra diviſorem, vel ejus pars aliquota, minor diviſore.

Si igitur hoc pacto dividendus logiſticus fuerit minor ipſo diviſore, tunc logarithmus diviſoris, auferatur à logarithmo dividendi; reſtabitque logarithmus quotientis. Atque is quotiens vel erit exceſſus itidem, addendus dividendo, ut conſtituatur verus quotiens, vel erit itidem pars æquè multiplex quotientis, vel denique, ſi nulla permutatio facta in datis numeris, erit ipſe quotiens.

At mutatio in apicibus facta adhuc eſt compenſanda. Nam quantum fuit additum vel ſubtractum apicibus diviſoris, tantum etiam addendum vel ſub-

trahendum apicibus quotientis. Quantum verò apicibus dividendi vel additum, vel ſubtractum : tantum contrariâ ratione ſubtrahendum vel addendum apicibus quotientis, ſeorſim utrumque.

EXEMPLUM.

Dividantur 29′ 30″ log. 70995
in 59′ 0″ log. 1681

69314 dat 30′ 0″

Hic nulla eſt facta mutatio apicum, nec in diviſore, nec in dividendo, quippe dividendus erat minor diviſore, caſuſque proprius Chiliadi. Ergò 30′ 0″ erit quotiens juſtus. Quod mirari non debes $29'\frac{1}{2}$ diſtributa in partes 59′, facere portiones magnitudine 30′ majore, quàm erat totus dividendus. Debes enim cogitare, illa 59′ non eſſe integra, ſed fractionem unius integri, proinde quotiens 30′ eſt portio debita non uni 1′ ſcrupulo primo, ſed uni 1 integro.

ALIUD.

Dividantur Gr. 6 0′	Scribe 6′ 0″ log. 230258
In dies 59 3′	59′ 3″ log. 1550
	Reſiduum 228708

Hoc reſiduum oſtendit 6′ 8″. Si ergò dati habuiſſent illos apices, cum quibus excerpſimus logarithmos, tunc quotiens hic fuiſſet. At quia apicibus dividendi ſunt adjectæ duæ unitates, ſic ut ex 0° fierent ſexageſima ‶0 ex ſexagenis verò ‵6 fierent ſexageſima ‶6 : Viciſſim igitur quotienti huic 6′ 6″ adimendi duo apices ſexageſimarii, ut fiat ‵6 6° Quia verò etiam diviſoris apicibus fuit adjecta unitas ſexageſimaria, ut pro 59° ſcriberentur 59′, & pro 3′ ſcriberentur 3″, rurſum igitur idem eſt faciendum quotienti primo mutato, ut pro ‵6 6° ſcribatur itidem 6° 6′.

EXEMPLUM UBI DIVIDENDUS MAJOR.

Dividantur 57′ 23″ major
in 41′ 15″ ut minorem

Subtrahe manet 16′ 8″ reſiduum jam minus diviſore
Ergò divide 16′ 8″ log. 131350
in 41′ 15″ log. 37470

Reſiduum 93880 oſtendit 23′ 29″

Cui adde diviſorem ipſum totum ſemel, quia ſemel tantum erat ſubtractus à dividendo, colligitur 1° 4′ 44″. Quotiens debitus uni integro, cujus diviſor 41′ 15″ erat pars ſeu fractio.

EXEMPLUM DE PARTE ALIQUOTA.

Sint dividendi $3^\circ\ 45'\ 13''$ per $57'\ 8''$. Cùm igitur contineatur diviſor in dividendo craſſâ æſtimatione, minus quàm quater: operabor per dividendi partem quartam, quæ facilè ſumitur, eſtque

$56'\ 18''\ 15'''$ log. 6360
Diviſoris $57'\ 8''$ log. 4900

Reſiduum 1460

Hoc oſtendit quotientem $59'\ 8''$. Hujus igitur ſumendum eſt iterum quadruplum ſcilicet $3^\circ\ 56'\ 32''$.

Nota hoc exemplum potuiſſet etiam tractari aliter pro $3^\circ\ 45'\ 13''$ ſcribendo

$3'\ 45''\ 13'''$ log. 277000
Diviſor $57'\ 8''$ log. 4900

Reſiduum 272100

Hæc enim oſtendit $3'\ 56''\ 30'''$. Quia verò apicibus dividendi fuit appoſita unitas: detrahatur viciſſim apicibus quotientis unitas, fietque quotiens $3^\circ\ 56'\ 30''$ ut prius.

Quomodo compendioſè ſit agendum, ſi dividendus fuerit aliquoties major diviſore, tradetur infra capite IX, modus etiam alius.

IV. Operari per regulam proportionum, detri dictam, in logiſticis.

Si trium datorum unus ſit pura unitas, ſic ut ea poſſit aptari pro 1° ſeu $60'$: ſiquidem hæc unitas fuerit primo loco ad ſiniſtram; tunc operatio abſolvitur per meram additionem logarithmorum, loco primo traditam: ſin autem fuerit hæc pura unitas 1° ſeu $60'$ loco ſecundo vel tertio; tunc operatio perficitur per ſimplicem ſubtractionem logarithmorum loco ſecundo propoſitam. At ſi nuſpiam fuerit pura unitas, tunc ſiniſtimi logarithmus, aufertur â ſumma duorum logarithmorum reſiduorum, ſi poteſt: vel quod idem eſt, ſiniſtimi logarithmus aufertur ab uno duorum reſiduorum, ſi poteſt; reſiduum addetur logarithmo tertii: utroque modo conficitur logarithmus quotientis.

EXEMPLUM.

$29'\ 45''$ - dat - $15'\ 43''$ - quid - $58'\ 47''$?	Vel 133970
log. 70150 133970 2050	Aufer 70150
Adde 133970	
	Reſid. 63820
Summa 136020	Adde 2050
Aufer 70150	
	65870
Reſiduum 65870	ut prius.

Quotiens oſtenditur $31'\ 3''$.

Si

Si ſiniſtimi logarithmus à ſummâ reliquorum ſubtrahi non poteſt, ſi nimirum ſiniſtimus datus, minor fuerit utroque reliquorum dato ſeorſim: id indicio eſt, quotientem excreſcere ſupra 1° ſeu 60′. Quare operare per ſecundi vel tertii partem aliquotam, quotientiſque prodeuntis ſume æquè multiplicem: vel operare per exceſſum alterutrius datorum, ſupra datum ſiniſtimum; quotientique emergenti adjunge ipſum ſiniſtimum: uti in diviſione doctus es.

V. A numero logiſtico propoſito, radicem extrahere quadratam facillimè.

Ne turberis, quod radix ſit major ſuo quadrato: fit enim hoc propterea, qui numeri ſexagenariæ ſunt fractiones unius integri: ut ſuprà dictum. Radix enim logiſtica nihil eſt aliud, quàm medium proportionale inter integrum 1° & numerum logiſticum integro minorem, vel etiam majorem.

Ergò numeri propoſiti logiſtici logarithmum bipartire, ſemiſſis enim quæſitus inter logarithmos in ſexagenariâ oſtendet radicem quæſitam.

EXEMPLUM.

Quadratus eſto 50′ 27″ logar. 17360
ſemiſſ. 8680.

Hic ſemiſſis oſtendit radicem 55′ 0″ eritque ut 1° 0′ 0″ - ad 55′ 0″ - ſic hoc ad 50′ 27″ ferè.

Quod ſi quadratus habuerit alios apices, quàm in ſexagenariâ, quæratur ejus partis quadratæ, quæ minor fuerit integro, puta quartæ, vel nonæ, vel ſedecimæ, &c. radix, eaque inventa viciſſim duplicetur, triplicetur, vel quadruplicetur, &c.

Sit quadratum 1° 39′ 20″. Hic cùm excurrat ſupra integrum, ſic ut inveniri non poſſit in ſexagenariâ, tantò ejus quadrantem 24′ 50″. Hic cùm jam inveniatur in ſexagenariâ, ejus ergò log. 88190, ſemiſſis 44095 oſtendit 38′ 36″ radicem quadrantis; ejus ergò duplum 1° 17′ 12″ eſt radix quæſita; ſeu magis propriè eſt medium proportionale inter 1° 0′ 0″ & 1° 39′ 28″.

Mirabitur hoc imperitus, quomodo 1° 17′ ſit radix de 1° 39′, reputans illic eſſe 77′ hic 99′, & verò radicem de 99′ eſſe paulò minorem quàm 10′. At memineris integrum in ſexagenariâ non eſſe unum ſcrupulum, ſed unum gradum: proinde hujus unitatis linearis quadratum, itidem eſt unitas ſuperficiaria valens 1° ſeu 60′. Et ſic unitatis linearis, cum fractione appendice, per 17′ expreſſâ, quadratum rectè fit unitas ſuperficiaria cum fractione appendice per 39′ expreſſâ. Si verò cogitationem ab hâc unitate gradus transferas ad unitatem ſcrupulariam; tunc unitas linearis ſcrupularia, quadratum habet, unitatem ſuperficiariam, quæ valet ſcrupulum: Et ſic prioris unitatis graduariæ 60′ ſcrupula in formam redacta quadratam, latus habebunt paulò brevius 8, unitatibus lineæ ſcrupulariæ. Unde elucet conſenſui rei utriuſque.

VI. Inter

VI. Inter duos numeros logiſticos medium proportionale conſtituere.

Si datorum unus eſt 1° 0′ 0″ jam modò doctus es idem facere per extractionem radicis. Hæc eſt enim medium proportionale. Si verò non eſt integrum ſexagenariæ, ſcilicet 1° interdatos, adde datorum logarithmos, ſummæ ſemiſſem quære inter logarithmos, & excerpes ex ſexagenariâ quæſitum medium proportionale.

EXEMPLUM.

Sint dati 25′ 35″ logarith. 85240
& 49′ 50″ logarith. 18566

Summa 103806
Semiſſis 51903 dat 35′ 43″
Erit igitur ut 25′ 35″ ad 35′ 43″, ſic hoc ad 49′ 50″.

VII. In ſpecie hìc docemur partem proportionalem venari, in tabulis æquationum & alibi.

Fit autem ſecundum præcepta præmiſſa multò compendioſiſſimè, quoties totum, quod debetur uni gradui, ſeu horæ, ſeu 60 minutis, non ſuperat 60′ minuta. Adduntur enim logarithmi 1. Differentiæ uni gradui, vel horæ reſpondentis, & 2 ſcrupulorum, integris gradibus vel horis adhærentium, ſumma inter logarithmos quæſita exhibet, ad latus ſub ſexagenariâ partem proportionalem quæſitam.

Sit anomalia 136° 47′ 14″, & excerpatur cum integris 136° æquatio — 4° 15′ 23″. Sitque differentia æquationum duarum vicinarum decreſcentium 37′ 29″. Quæritur pars proportionalis ſcrupulis 47′ 14″.

Logarith. 47′ 14″ eſt 23920 circiter
Logarith. 37′ 29″ eſt 47050 circiter

Summa 70970 dat 29′ 30″ partem proportionalem decrementi, ablata igitur hæc à 4° 15′ 23′ relinquit 3° 41′ 55′ æquationem correctam.

EXEMPLUM ALIUD.

Sit horarius 31′ 24″, ſint minuta unius horæ 41′ 48″, quæritur quantum iis debeatur de horario? Adde logarithmos 64770, & 36150. Summa 100920, oſtendit partem proportionalem 21′ 52″.

VIII. Dato

VIII. Dato horario lunæ à ſole, datiſque ſcrupulis incidentiæ, moræ dimidiæ, vel durationis dimidiæ: eruere tempus incidentiæ, moram dimidiam, vel durationem dimidiam in eclipſibus.

VEL,

Dato horario, datoque arcu per currendo, indagare numerum horarum & minutorum, intra quos arcus percurritur.

Primum aufer horarium lunæ à ſole quoties potes à datis ſcrupulis, totieſque ſcribe unam horam.

Deinde à logarithmo reſidui, quod minus erit horario, aufer logarithmum horarii, reſtabit logarithmus minutorum horis integris adjiciendorum.

Sint ſcrupula 53′ 16″
Horar. lunæ à ſole 31′ 24″ aufer ſemel & ſcribe hor. 1

Reſiduum 21′ 52″ logar. 100920
Horarii 31′ 24″ logar. 64770

Reſiduum 36150 dat 41′ 48″

Ergò tempus eſt 1^h 41′ 48″.

IX. Dato numero horarum & minutorum, cui reſpondeat datus minor numerus graduum & ſcrupulorum, inquirere horarium: oportet autem horarum numerum infra 60′ eſſe.

Quære datos (diminutis ſignis) in ſexagenariâ, & logarithmum horarum, aufer à logarithmo graduum; reſiduum eſt logarithmus horarii ex eadem ſexagenaria excerpendi cum ipſis ſignis.

Ut ſi horis 50 26′ luna promoveatur 28° 3′
Diminue ſigna, ut ſtent illic 50′ 26″ hic 28′ 3″
Jam igitur à logarithmo 28′ 3″ ſc. 76036
Aufer logarithmum 50′ 26″ ſc. 17469

Reſiduum 58567 oſtendit horarium 33′ 24″, retentis ſignis.

APPENDIX.

Si plures eſſent gradus quàm horæ, aufer numerum horarum à numero graduum quoties potes, totieſque ſcribe unum gradum in quotiente. Deinde cum eo, quod de gradibus fuerit reſiduum, operare, ut jam dictum, prodi-

buntque ſcrupula & ſecunda, gradibus integris in quotiente ſcriptis, addenda.

Sed hæc cap. IX tradantur per alium modum ſimpliciorem, & magis proprium, ut mutatione apicum non ſit opus.

COMPENDIUM.

Conducit etiam ad brevitatem, ſi major numerus fuerit infra 30, uti tunc operemur per utriuſque duplum.

Ut ſi horis 25 13′ luna promoveretur gr. 14 $1\frac{1}{2}$, hoc perinde eſſet ac ſi horis 50 26′ reſponderent gr. 28 3′

X. Si triangulum rectangulum, ſeu planum, ſeu ſphæricum, omnia latera minora habuerit, quàm 1° ſeu 60′: tunc datis duobus lateribus circa rectum, per logarithmos invenire latus tertium recto ſubtenſum, ſeu baſin.

Facilius quidem fit, in ſphæricis quidem per antilogarithmos canonis, ſeu ut cap. VII dicatur, per logarithmos complementi: ſi tamen per opportunitatem etiam ſexagenariâ placeat uti: ſic poterit operatio inſtitui.

Logarithmos laterum excerptos ſingulos ſeorſim duplica, cum his duplis, ut logarithmis, excerpe laterum quadrata ex ſexagenariâ, eaque in unam ſummam adde, ſumma quæſita in ſexagenaria, exhibebit è regione logarithmum, cujus ſemiſſis, ut logarithmus quæſitus, ex eadem ſexagenariâ exhibebit baſin quæſitam.

Ut ſi ſint latera 28′ 17″ & 50′ 31″, eorum logarithmi 75200 & 17200, dupli horum 150400 & 34400, oſtendunt quadrata laterum 13′ 20″ & 42′ 32″. Summa utriuſque eſt 55′ 52″; hujus logarithmus 7140, et hujus ſemiſſis 3570, oſtendit baſin 57′ 54″.

XI. Viciſſim datâ baſi, in ſic comparatis, & latere alterutro, invenire latus reliquum.

Rurſum duplicia logarithmos baſis & lateris dati; per hæc dupla excerpe quadrata ex ſexagenariâ, minuſque à majori ſubtrahe; reſidui logarithmus eſt duplus logarithmi lateris quæſiti: ut ſi baſis 57′ 54″, latus 28′ 17″, logarithmi 3570,57200, duplicati ſunt 7140,150400, quadrata 55′ 52″ & 13′ 20″, reſiduum 42′ 32″, logarithmus 34400, dimidium 17200, quod oſtendit 50′ 31″ latus quæſitum.

CAPUT

CAPUT VII.

De Copulatione Columnæ Arcuum cum Columnâ Logarithmorum.

Per ſolos Logarithmos Arcuum & Angulorum, omnia Triangula Sphærica ſolvere, omnia ſcilicet quæſita ex tribus datis eruere.

PRIMA autem ſunt rectangula, eorumque caſus ſedecim, in quibus inter tria data intelligitur ſemper ipſe rectus angulus.

Hic autem vox complementi crebrò uſurpanda, cùm ſic ſolitariè ſumitur, ſemper dat ſubintelligi ad quadrantem: Baſis verò vox pro latere maximo ſumitur, quod ſcilicet angulo recto opponitur. Et loquimur de talibus rectangulis, quorum latera ſunt minora quadrante ſingula.

I. Ex baſi & angulo latus oppoſitum.

Addantur invicem logarithmi datorum, conficitur logarithmus quæſiti lateris.

II. Ex lateribus baſin.

Addantur invicem logarithmi complementorum datorum laterum, conficietur logarithmus complementi quæſitæ baſis. Vide cap. VI numero X modum alium.

III. Ex angulo & latere adjacente angulum reliquum.

Addantur invicem logarithmi anguli, & complementi lateris, conficietur logarithmus complementi quæſiti anguli.

IV. Ex baſi & latere angulum oppoſitum.

Auferatur logarithmus baſis à logarithmo lateris, reſtabit logarithmus anguli oppoſiti.

V. Ex angulo & latere oppoſito baſin.

Auferatur logarithmus anguli à logarithmo lateris oppoſiti, reſtabit logarithmus baſis.

VI. Ex angulo & latere ſubtenſo angulum reliquum.

Auferatur logarithmus complementi lateris dati à logarithmo complementi anguli dato oppoſiti, reſtat logarithmus ipſius anguli reſidui quæſiti.

VII. Ex latere & basi latus reliquum.

Auferatur logarithmus complementi lateris à logarithmo complementi basis, restabit logarithmus complementi lateris reliqui. Vide suprà cap. VI num. XI modum alium.

VIII. Ex angulis latus.

Auferatur logarithmus anguli, qui lateri quæsito adjacet, à logarithmo complementi anguli oppositi, restat logarithmus complementi lateris quæsiti.

IX. Ex lateribus angulum.

Pro his compositis casibus, aptior est canon ipse logarithmorum quadrantis, quàm Chilias nostra: quia in illo non opus est excerptione arcus, quippe cum ejus complementum statim cum logarithmo suo è regione in conspectum veniat. Ut jam non dicam quòd mesologarithmi canonis, quibus caret Chilias nostra, præstent hos compositos etiam simplices.

Addantur in unam summam logarithmi complementorum datorum, laterum summa excerpat arcum: Ejus arcus complementi logarithmus, auferatur à logarithmo lateris quæsito angulo oppositi, restat logarithmus anguli quæsiti.

X. Ex basi & angulo residuus angulus.

Addantur in unam summam logarithmi datorum, summa excerpat arcum: ejus complementi logarithmus auferatur à logarithmo complementi anguli dati, restabit logarithmus anguli quæsiti.

XI. Ex angulo & latere adjacente, latus oppositum reliquum.

Addantur in unam summam logarithmi, anguli & complementi lateris, summa excerpat arcum: ejus complementi logarithmus auferatur à logarithmo complementi anguli dati, restabit logarithmus complementi lateris quæsiti.

XII. Ex basi & angulo latus adjacens.

Addantur logarithmi datorum, summa excerpat arcum; ejus complementi logarithmus auferatur à logarithmo complementi basis, restabit logarithmus complementi lateris quæsiti, angulo adjacentis.

XIII. Ex basi & latere angulum adjacentem.

Auferatur logarithmus complementi lateris à logarithmo complementi basis, residuum excerpat arcum, ab hujus complementi arcu, auferatur logarithmus basis, restabit logarithmus anguli adjacentis.

XIV. Ex

XIV. Ex angulo & latere oppoſito latus reliquum.

Auferatur logarithmus anguli à logarithmo lateris, reſiduum excerpat arcum, à cujus complementi logarithmo, auferatur logarithmus complementi lateris · reſtabit logarithmus complementi lateris reliqui.

XV. Ex angulo & latere adjacente baſin.

Addantur invicem logarithmi anguli & complementi lateris; ſumma excerpat arcum; hujus complementi logarithmus auferatur à logarithmo lateris dati, reſtabit logarithmus baſis.

XVI. Ex angulis baſin.

Auferatur logarithmus anguli unius à logarithmo complementi anguli alterius; reſiduum excerpat arcum, ab hujus complementi logarithmo, aufer logarithmum ipſius anguli alterius, ſeu poſterius hîc adhibiti, reſtat logarithmus baſis.

EXEMPLUM CASUUM OMNIUM.

Sit Baſis 72° 0′. Angulus 50° 0′.

				Logarithmi.	Complementa.	Logarithmi.
Baſis	72°	0′	F	5030 A	18° 0′ L	117440 O
Angulus	50	0	G	26650 B	40 0 M	44190 P
Latus oppoſitum	46	45	H	31680 C	43 15 N	37805 Q
Latus majus	63	12	I	11370 D	26 48 R	79635 T
Angulus major	69	48	K	6340 E	20 12 S	106350 V

Hoc exemplum ſic fuit conſtructum.

F, G, ſunt ex arbitrio, quæ dant L, M. Ergò habentur A, B, O, P, jam per I ex A, B, habetur C hinc H hinc N hinc Q. Tunc per VII. ex O, Q habetur T, hinc R hinc I hinc D, etiam per IV. ex A, D habetur E hinc K hinc S hinc V. Reliqua præcepta omnia erunt loco probationis, verbi causâ ex prim. Ex baſi F & angulo K latus I illi oppoſitum, invenitur enim D qui dat I.

Ex primo.	Ex ſecundo.	Ex Tertio.	
5030 A	37805 Q	26650 B	6340 E
6340 E	79635 T	79635 T	37805 Q
11370 D	117440 O	106285 V	44145 P
		pro 106350	pro 44190

Ex

Ex quarto.

5030	A
31680	C
26650	B

Ex quinto.

26650	B	6340	E
31680	C	11370	D
5030	A	5030	A

Ex Sexto.

44190	P	106350	V
37805	Q	79635	T
6385	E	26715	B
pro 6340		pro 26650	

Ex feptimo.

117440	O
79635	T
37805	Q

Ex octavo.

26650	B	6340	E
106350	V	44190	P
79700	T	37850	Q
pro 79635		pro 37805	

Ex nono.

73805	Q			
79635	T			
117440	O	18° 0′	117440	O
31680	C		11370	D
5030	A	72 0	5030	A
26650	B		3640	E

Ex decimo.

5030	A	5030	A
26650	B	6340	E
31680	C	11370	D
73805	Q	79635	T
44190	P	106350	V
6385	E	26715	B
pro 6340		pro 26650	

Ex undecimo.

26650	B	6340	E
79635	T	37805	Q
106285	V	44145	P
6340	E	26650	B
44190	P	106350	V
37850	Q	79700	T

Ex duodecimo.

5030	A	5030	A
26650	B	6340	E
31680	C	11370	D
37805	Q	79635	T
117440	O	117440	O
79635	T	37805	Q

Ex decimo tertio.

37805	Q	79635	T
117440	O	117440	O
79635	T	37805	Q
11370	D	31680	C
5030	A	5030	A
6340	E	26650	B

Ex decimo quarto.

26650	B	6340	E
31680	C	11370	D
5030	A	5030	A
117440	O	117440	O
37805	Q	79635	T
79635	T	37805	Q

Ex decimo quinto.

26650	B	6340	E
79635	T	37805	Q
106285	V	44145	P
6340	E	26650	B
11370	D	31680	C
5030	A	5030	A

Ex decimo fexto.

26650	B	6340	E
106350	V	44190	P
79700	T	37850	Q
11370	D	31680	C
6340	E	26650	B
5030	A	5030	A

Primi

Primi igitur octo casus perficiuntur per operationem simplicem; reliqui octo per duplicem, in quibus scilicet semper aliquid aliud quæritur ante id, quod proponitur, per unum ex octo prioribus.

Resolvuntur itaque 9 in 2 & 4. Sic 10 in 1 & 6. Sic 11 in 3 & 8. Sic 12 in 1 & 7. Sic 13 in 2 & 14. Sic 14 in 5 & 7. Sic 15 in 3 & 5. Sic 16 in 8 & 5.

Excerpsi autem studio logarithmos rudes, ut si arcus propositus caderet medius inter duos expressos in Chiliade, medium etiam aliquid inter duos illorum logarithmos eligerem, rotundo fine; ut monerem, etsi neque arcus, neque eorum logarithmi in Chiliade rotundi sint, nihil tamen opus esse, ut calculator sese ubique maceret, minutias consectando, quod equidem grave esset in hâc chiliade: facilius aliquantò in canone quadrantis; in quo arcus rotundi sunt, & singula minuta suos habent logarithmos.

Aliud etiam compendium non contemnendum (ut ad marginem monui) habet canon præ meâ chiliade, quod is arcuum complementa cum suis logarithmis exhibet è regione. Itaque quod ego in chiliade cogor circumloqui longius, logarithmum scilicet complementi arcus, lateris, basis, vel anguli: id in canone brevius exprimimus, antilogarithmum arcus, lateris, basis & anguli dicentes: ut innuamus excerpendum esse antilogarithmum è regione logarithmi.

Tertium compendium canonis est in eo, quod is habet etiam mesologarithmos, pro quos, casus octo posteriores fiunt simplices, qui sunt hic duplices: verum hoc canonis compendium apud inexercitatos conjunctum est cum dispendio, quod distrahitur animus additionibus & subtractionibus mesologarithmorum cossicis, hoc est, multitudine cautionum, quibus arithmetica cossica constat.

Sed quod attinet logarithmos, paulò exactiores ex ipsâ etiam chiliade excerpendos, ut etiam in hoc satisfiat curiositati quorundam: tradam cap. sequenti VIII modum elaborandi logarithmum cuique sinui respondentem: capite verò IX modum alium subsidiarium, capiendi pro logarithmis partem proportionalem vulgarem, idque sine labore, mediantibus aliis logarithmis.

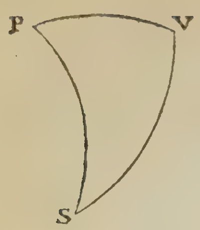
Hactenus igitur de triangulis sphæricis rectangulis egimus. Nunc ad obliquangula transeamus.

DE TRIANGULIS OBLIQUANGULIS SPHÆRICIS; IN QUIBUS CASUS SUNT 12.

I. Si dentur duo latera, & angulus uni oppositus: quæraturque angulus alteri datorum laterum oppositus: ad logarithmum anguli, adde logarithmum lateris dati, quæsito oppositi, à summa aufer logarithmum lateris angulo dato oppositi, restabit logarithmus anguli quæsiti, recto seu majoris, seu minoris.

EXEMPLUM CASUS PRIMI, LOGARITHMIS RUDIBUS SEU ROTUNDIS.

In triangulo PVS data sunto latera duo PV 38° 30′, VS 40° 0′, angulus VPS 31° 34′, oppositus ipsi VS. Quæritur angulus VSP lateri VP oppositus.



VPS 31° 34′ logarith. 64720
VP 38 30 logarith. 47480

Summa 112200
VS 40° 0′ logarith. 44190 ſubt.

Reſiduum 68010 ut.
Logarithmus dat VSP 30° 26′ circiter.

Hic ſitne ipſe quæſitus, an ejus reſiduum ad ſemicirculum, facilè diſcerni poterit ex habitudine laterum, præſertim oppoſiti PV. Non eſt enim mei inſtituti minimas cautiones hâc vice conſectari.

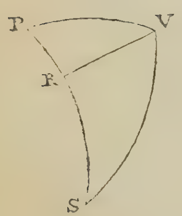
II. Si iiſdem datis, quæratur latus tertium : Tunc adde logarithmos anguli dati, & lateris attigui dati, prodit logarithmus perpendiculi dictus. Ejus perpendiculi logarithmus complementi ablatus à datorum laterum logarithmis complementi, relinquit logarithmos duorum arcuum, quorum vel differentia, vel complementorum ſumma, eſt latus quæſitum tertium.

III. Si iiſdem datis, quæratur angulus inter latera data comprehenſus.

Tunc quæratur latus tertium, ut præcepto ſecundo horum. Ejus logarithmo adde logarithmum anguli dati, à ſummâ aufer logarithmum lateris, cui datus angulus opponitur : reſtat logarithmus anguli comprehenſi, ejuſque complementi ad duos rectos.

EXEMPLUM II. ET III. CASUUM.

Iiſdem datis, quæratur PS & PVS.



			Logarith.				
VS	40°	0′	44190	Compl. 50°	0′	Log.	26660
VPS	31	34	64720				
VP	38	30	47480	Compl. 51	30	Log.	24510
VR	19	0	112200	Compl. 71	0	Log.	5600
SR	35	54		Compl. 54	6	Log.	21060
PR	34	8		Compl. 55	52	Log.	18910
SP	70	2	Log. 6190				
			70910				
SVP	49	57	log. 26720				

Reſiduum hoc eſt, ſeu complementum anguli ad duos rectos, quòd facilè patet ex magnâ proportione PS ad PV, VS. Quando verò dubitari poteſt, tunc quærenda ſunt ejus elementa PVR, RVS.

IV. Si

IV. Si datis duobus lateribus & angulo comprehenſo, quæratur latus tertium.

Tunc ad logarithmum lateris minoris, adde logarithmum comprehenſi, prodit logarithmus perpendiculi in latus majus, quod perpendiculum conſtituit duo illius lateris elementa: Hujus enim perpendiculi logarithmus complementi ablatus à logarithmo complementi lateris minoris, dat logarithmum arcus, cujus complementum eſt alterum ex illis elementis, eoque vel addi, vel auferri debet à toto latere majori, ſummæ vel reſidui, ut elementi alterius, logarithmum complementi, adde logarithmo complementi perpendiculi, fietque logarithmus complementi lateris tertii quæſiti. Prior caſus eſt, quando angulus comprehenſus & datus obtuſus eſt; poſterior, quando acutus.

V. Si datis duobus lateribus & angulo comprehenſo, quæratur angulorum reſiduorum unus.

Tunc quære latus tertium, ut in quarto præcepto horum. Ejus logarithmum aufer à ſumma logarithmorum anguli initio dati, & lateris angulo quæſito oppoſiti, reſtabit logarithmus anguli quæſiti.

EXEMPLUM IV. ET V. CASUUM.

Dentur VP, PS & VPS, quæratur VS & PVS.

				Logar.	Compl.		Logar.
	PV	30°	30′	47480	51°	30′	24510
	VPS	31	34	64720			
Perpendiculum	VR	19	0	112200	71	0	5600
Elementum unum	PR	34	8		55	52	18910
	PS	70	2	6190			
Elementum alterum	RS	35	54		54	6	21070
Quæſitum latus	VS	40	0	44190	50	0	26670
Summa log.	VPS & PS			70910			
Quæſit. angulus	PVS	49	57	26720.			

VI. Si datis tribus lateribus quæritur angulus.

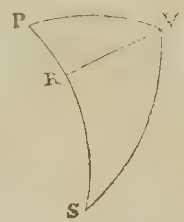
Tunc laterum circa quæſitum angulum exquire differentiam; illam à latere tertio aufer, eidemque adde, & tam reſidui quàm ſummæ conſtitue ſemiſſes. Hâc præparatione præmiſsâ, jam excerpe quatuor logarithmos, duos ſemiſſium, & duos laterum circa quæſitum angulum, ſummamque horum aufer à ſumma

illorum:

illorum: Differentiæ, quæ futura eſt, ſemiſſis, ut logarithmus, oſtendet ſemiſſem anguli quæſiti.

EXEMPLUM SEXTI CASUS.

Dentur VP, PS, SV quæratur VPS.



	VP 38° 30′		Log. 47480	
	PS 70 2		Log. 6190	
Diff.	31 32		53670	ſumma
	VS 40 0	Semiſſes		
Reſiduum	8 28	4 14	Log. 260400	
Summa	71 32	35 46	Log. 53700	
			314100	ſumma
			260430	ſummar. diff.
Quæſitus	VPS 31 34	15 47	Log. 130215	ſemiſſis, quæratur.

Quæratur PVS.

			Logarithmi.	
	VP 38° 30′		47480	
	VS 40 0		44190	
	1 30		91670	
	PS 70 2	Semiſſes		
	68 32	34 16	57440	
	71 32	35 46	53700	
			111140	
			19470	ſummar. diff.
Quæſitus	PVS 130 14	65 7	9735	dimid.

Quæratur

Quæratur PSV.

				Logarithmi.
SV	40°	0′		44190
PS	70	2		6190
	30	2		50380
	38	30	Semiſſes	
	8	28	4 14	260400
	68	32	34 16	57440
				317840
				267460 ſummar. diff.
Quæſitus VSP	30	26	15 13	133730 dimidium.

VII. Si datis duobus angulis, & latere alteri datorum ſubtenſo, quæratur latus reliquo ſubtenſum: Ad logarithmum lateris dati, adde logarithmum anguli adjacentis dati, à ſummâ aufer logarithmum anguli, lateri dato oppoſiti; reſtabit logarithmus lateris quæſiti, ejuſque complementi ad ſemicirculum: ubi pro ratione quantitatis datorum, vel ipſe logarithmi arcus, vel ejus complementum ad ſemicirculum ſumendum eſt pro latere.

EXEMPLUM VII. CASUS.

Dentur PVS. VPS, VS, quæratur PS.

			Logarith.
VS	40°	0′	44190
PVS	130	3	26720
			70910
VPS	31°	34′	64720
Quæſitum VS	70	2	6190

ALIUD.

Latus	40°	0′	44194
Ang.	30	28	67917
			112111
Ang.	80°	0′	1531
Lat. quæſ.	19°	19′	110580

VIII. Si iiſdem datis quæratur angulus tertius.

Tunc adde logarithmos lateris dati, & anguli alterius adjacentis: prodit logarithmus perpendiculi ex angulo quæſito demiſſi; cujus logarithmus complementi ablatus à datorum angulorum logarithmis complementi, relinquit logarithmos duorum elementorum anguli quæſiti, à perpendiculo conſtitutorum: quorum ideò vel ſumma conſtituit quæſitum, ſi perpendiculum cadit intra triangulum, vel differentia, ſi extra.

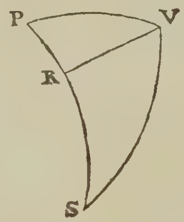
IX. Si iisdem datis, quæratur latus interjectum datis angulis.

Tunc quæratur angulus tertius, ut præcepto VIII. horum. Ejus logarithmo adde logarithmum lateris dati; à summa aufer logarithmum anguli, qui lateri illi dato opponitur, restabit logarithmus lateris interjecti.

EXEMPLUM VIII. ET IX. CASUUM.

Dentur PVS, VPS, VS, quæratur VSP & VP.

		°	′	Logarith.	°	′	
	VS	40°	0′	44190	Compl.		Logar.
	PVS	130	3	26720	40°	3′	44080
Perpendic. ex angulo	S	29	29	70910	60	31	13870
	VPS	31	34		58	26	16000
Elementum unum					47	40	30210
Elementum alterum					78	14	2130
	VSP quæsitus				30	34	per subtractionem.



		°	′	Logarith.
	VSP	30°	34′	67630
				111820
	VPS			64720
Quæsitus	VP	38	37	47100

NOTA.

Arcus 130° 3′ & 49 57 } Logarithmus est idem.
At 40 3 est minoris quadrante complementum; majoris quadrante excessus.

ALIUD EXEMPLUM.

	°	′	Logarith.	Compl. °	′	Logarith.
Latus	40°	0′	44194			
Ang. adjac.	30	28	67917	59	32	14859
Perpendic. ex angulo	19	1	112111	70	59	5613
Ang. alter. lateri oppos.	80	0	1531	10	0	175072
Elementum unum				56	$44\frac{1}{2}$	99247
Elementum alterum				10	35	169470
Ang. quæsitus				76	19	per addit.
Angulus tertius	76°	19′	2879			
			47073			
Latus quæsitum angulo tertio quæsito oppositum.	39	22	45542			

X. Si

X. Si datis duobus angulis & latere interjecto, quæratur angulus tertius, lateri dato oppositus. Tunc ad logarithmum anguli minoris, adde logarithmum lateris interjecti, prodit logarithmus perpendiculi ex majori angulo; cujus perpendiculi logarithmus complementi & asservari debet, et auferri ab anguli illius ejusdem logarithmo complementi, relinquetur logarithmus unius elementi illius anguli, unde perpendiculum est demissum. Hoc igitur ablatum ab angulo ipso toto, vel ille ab hoc, relinquit ejus elementum alterum.

Hujus iterum logarithmus addatur logarithmo asservato, ita fiet logarithmus complementi anguli tertii quæsiti, vel ejus excessus supra quadrantem.

Si datorum angulorum alter est major quadrante, latus datum minus esse debet quadrante.

Si latus datum est quadrante majus, tunc elementum anguli, quod illi respondet, est etiam quadrante majus, & sic triangulum obtusangulum, ubi perpendicularis cadit extra.

XI. Si datis duobus angulis, & latere interjecto, quæratur laterum residuorum unum: Tunc quære angulum tertium, ut in præcepto x horum, ejus logarithmum aufer à summa logarithmorum lateris initio dati, & anguli lateri quæsito oppositi, restabit logarithmus lateris quæsiti.

EXEMPLUM X. ET XI. CASUUM.

Dentur PVS, VPS, VP, quæratur VSP & VS.

				Logar.	Compl.			
Ang. tot.	PVS	130°	3′	26720	40°	3′	Log.	
	VPS	31	34	64720	58	26	16000	
	VP	38	30	47480				
Perpendic. ex	V	19	1	112200	70	59	5610	asservandus.
Unius elementi ang.		64	23		64	23	10390	
Alterum elementum ang.		65	40	9300				
				5610 asservatus				
		59	29	14910				
Quæsitus	VSP	30	31	67780				
Quæsitum	VS	39	54	44420				

XII. Si datis tribus sphærici angulis, quæratur laterum aliquod.

Tunc primum pro angulo maximo scribe & usurpa ejus complementum ad semicirculum, deinde angulorum latus quæsitum attingentium exquira differentiam: illam ab angulo tertio aufer, eidemque adde, & tam residui quàm summæ constitue semisses.

Horum

Horum logarithmos adde in unam ſummam, à quâ aufer ſummam logarithmorum duorum angulorum latus attingentium: reſidui ſemiſſis, ut logarithmus, dat ſemiſſem lateris quæſiti ex columnâ arcuum; excipe caſum unum, ſi nimirum in acutangulo triangulo, latus quæſitum ſubtenſum fuerit angulorum maximo: tunc operatione per ejus complementum ad ſemicirculum peractâ, ut prius, prodit ſemiſſis, non ipſius lateris quæſiti, ſed ejus itidem complementi ad ſemicirculum.

At in obtuſangulo, etiam ſi latus quæſitum ſubtenſum fuerit obtuſo, & ſic maximo, valet nihilominus regula generalis, proditque ſemiſſis lateris quæſiti ipſius.

EXEMPLUM XII. CASUS.

Dentur anguli PVS, VSP, SPV, quæratur PS.

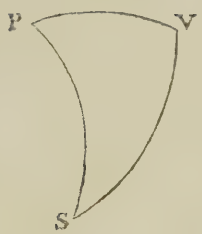
						Logarithmi.	
	VSP	30°	28′			67910	
	VPS	31	34			64720	
Differentia		1	6			132630	ſumma
Complementum ad ſemicirc.	PVS	49	57				
				ſemiſſ.			
Differentia		48	51	24	25½	88309	
Summa		51	3	25	31½	84187	
						172496	ſumma.
						39866	diff. ſum.
Quæſitum	PS	70	0	35	0	19933	ſemiſſ.

Quæratur PV.

	PVS	130°	3′			Logarithmi.
Compl. ad ſemicirc.		49	57			26725
	VPS	31	34			64724
		18	23			91449
	VSP	30	28	ſemiſſes.		
		12	5	6	2½	225142
		48	51	24	25½	88309
						313451
						222002
Quæſitum	VP	38	29	19	14½	111001

Quæratur

Quæratur VS.



PVS	130° 3′			Logarithmi.
Compl. ad ſemicirc.	49 57			26725
VSP	30 28			67917
	19 29			94642
VPS	31 34	ſemiſſes		
	12 5	6	$2\frac{1}{2}$	225142
	51 3	25	$31\frac{1}{2}$	84187
				309329
				214687
Quæſitum VS	39 57	19	$58\frac{1}{2}$	107343

ALIUD EXEMPLUM.

Sint anguli circa latus quæſitum.

	30° 28′			67917
	76 19			2879
Differentia	45 51			70796 ſumma
Ang. oppoſ.	80 0 maximus trium			
Compl. ad ſemicirc.	100 0			
	54 9	27°	$4'\frac{1}{2}$	78712
	145 51	72	$55\frac{1}{2}$	4508
				83220 ſumma
				12424 diff. ſum.
	140 2	70	1	6212 ſemiſſis.
Quæſitus	39 58.			

Viciſſim

Viciſſim ſint anguli circa quæſitum hi :

					Logarithmi.	
Maximus trium	80°	0′			1531	
Compl. ad ſemicirc.	100	0				
Alter	76	19			2879	
Differentia	23	41			4410	ſumma
Tertius	30	28	ſemiſſes			
Differentia	6	47	3	23′½	282750	
Summa	54	9	27	4½	78712	
					361462	ſumma
					357052	diff. ſum.
Quæſitum latus	19	19	9	39½	178526	ſemiſſ.

Atque hæc de obliquangulis, & in univerſum de ſphæricæ ſuperficiei triangulis curvilineis, quæ formantur arcubus circulorum maximorum.

De rectilineis verò triangulis, ſeu quæ formantur in ſuperficiei planâ, capite nono agetur.

CAPUT VIII.

De Copulatione Columnæ Sinuum, ſeu Numerorum Abſolutorum, & Columnæ Logarithmorum.

PRÆCEPTUM I.

Logarithmum excerpere accuratum Numeri ſcrupuloſi propoſiti.

ET ſi rariſſimè eſt opus tantâ ſcrupuloſitate, quoties tamen eſt opus, fit per prop. XXVII corollarium III in hunc modum.

Propoſiti numeri ſcrupuloſi exceſſum ſuper rotundum chiliadis proximè minorem, continua, appoſitis verſus dextram ſeptem cyphris; ſic continuatum divide per hunc ipſum proximè minorem, elaboratum tamen prius ſeu continuatum per unam inſuper vel duas figuras ſcrupuloſas, quæ contineant dimidium duarum primarum figurarum exceſſus, vel aliquid proximè minus dimidio:

dimidio: Quotientem aufer à logarithmo ad dictum rotundum proximè minorem, restabit logarithmus propositi scrupulosi.

EXEMPLUM.

Propositus esto scrupulosus	23456,78	
Chilias exhibet ad rotundum proximé minorem	23400,00	Log. 145243,42
Igitur excessus scrupulosi est	56,78	242,36 quotiens auferatur
		145001,66 hic est logarith. quæsitus accuratius
Ei apponantur 7 cyphræ	5678000000	
Hunc divide per inventum rotundum chiliadis proximi minorem	2340000	
Sed elaboratum per duas figuras significativas, quæ sint dimidium de 56 primis duabus figuris in excessu scilicet per 28. Ut	.2342800	

Divisio compendiosa

56780	
23428	
46856	2
9924	
9371	4
553	
469	2
84	
70	3
14	
14	6

ALIUD EXEMPLUM.

Sing. gr. 81 scrupulosus	987	68,83$\frac{2}{5}$ log.
Minor chiliadis	987	00,00,1308,52
Excessus appositis cyphris		68834000000
Divisor		98734 00

592404	6
959360	
888606	9
707540	
691138	7
164020	
197468	2 Quotiens.

Logarith. 1308,52
Quotiens 69,72

1238,80 quæsitus

Hîc prodeunt tantum quatuor figuræ.

ALIUD EXEMPLUM.

Sit ſinus 9°	15643,45		
Rotundi ergò	15600,00		Log. 185789,93
Exceſſus	43,45		278,15 quotiens ſubt.
Diviſor	15621		
	31242		185511,78 log. quæſitus.
		2	
	12208		
	109347		
		7	
	127330		
	124960		
		8	
	23620		
	15621		
		1	
	80000	5	

CAUTIO.

Raro uſu venit, ut opus nobis ſit logarithmo tam ſcrupuloſo & accurato: plerunque ſufficit logarithmus rotundus, cui non ſint plures à ſiniſtris figuræ ſignificantes, quàm in quot figuris logarithmi bini chiliadis, proximi invicem, nihil inter ſe differunt, unâ plus.

Ut quia logarithmus ad 23400,00 eſt	1452	43,40
Sed logarithmus ad 23500,00 eſt	1448	16,97
Sufficit igitur logarithmus rotundus	1450	00,00
Quia 1450 eſt medium inter 1452 &	1448	

CAUTIO ALIA.

Si vero omninò eſſet opus logarithmo accuratione, tunc ſcito, quod in logarithmis valdè magnis, ad numeros parvos chiliadis, diviſor debeat eſſe viciniſſimus medio geometrico, inter numerum datum ſcrupuloſum, & inter proximè eo minorem chiliadis: paulò tamen major medio geometrico: ut docet allegata propoſitio.

EXEMPLI CAUSA.

Si quæratur logarithmus numeri 150,00. Hic numerus ſcrupuloſus eſt, quia cadit inter duos chiliadis proximos, ſcilicet inter 100,00 & 200,00. Exceſſus eſt 50,00. Si hunc divideres per 12500, quod eſt inter 15000 datum ſcrupuloſum,

ſcrupuloſum, & inter datum proximè minorem 10000 medium arithmeticum, prodiret quotiens 40000,00 auferendus à 690775,54, reſtaretque 650775,54. Sin autem dividas per 12000, geometricum medium inter 15000 & 10000 quotiens erit 41666,67, qui ablatus à logarithmo proximè minoris, ſcilicet à 690775,54, relinquit 649108,87, quaſi logarithmum ad propoſitum abſolutum ſeu ſinum 125,00.

Quod verò logarithmus iſte ſit jam minor juſto, ſic patet per uſum præcepti ſequentis.

Pro 150,00 ſume centuplum 15000,00, ejuſque logarithmum 189712,00 adde logarithmo centuplicationis 460517,03. Sic colligitur logarithmus numeri 150,00, ſcilicet 650229,03.

Quare quotiens 4166667 quam ſubtrahebamus, erat major juſto, ac proinde medium proportionale inter ſcrupuloſum, & chiliadis proximè minorem, ſcilicet 120,00, erat diviſor minor juſto.

ALIA EXEMPLARIS OBSERVATIO,

Pro ſcrupuloſorum magnorum logarithmis accuratis indagandis.

Sit numerus ſcrupuloſu		
Seu ſinus 81°	98768,834	log. accuratus eſt 1238,80
Sagitta	1231,166	
Secantis complementi (9°) exc.	1246,513	
Summa	2477,679	
Horum med. arithmeticum	1238,840	
Differentia medii arith. & ſagitt.	7,673	
Adde ad ſagittam fit	1238,839	logarithmus ferè: qui differt
Vel differentia exceſſ. & ſag.	15,347	à logarithmo accurato ſolis 4.
Ejus dimidium	7,673	unitatibus loco ſecundo poſt

punctum, ſcilicet quia logarithmus eſt adhuc parvus.

Sic per complementum ſcrupuloſi parvum.

Sin 84° 16′	99499,762		
Chiliadis numerus major	99500,000	dat log.	501,25
Complementum ſcrupuloſi parvum	238000000		24 quotientem adde
Diviſor	9950000	$\frac{2}{4}$ quotiens	501,49 log.
Hîc ſagitta	500,238		
& exceſſ. ſecantis in canone	502,753		
Faciunt ſummam	1002,991		
Dimidium	501,496	paulò majus eſt logarithmo accurato.	

Vide ſuper hac re etiam prop. XXII corollaria & præcepta, fol. 21.

Ut compendioſè excerpas logarithmos numerorum quorundam ſcrupuloſorum, ſcilicet ſine diviſione.

CASUS PRIMUS.

Si propoſitus numerus ſcrupuloſus fuerit minor quàm 10000,00, minor ſcilicet quàm decima pars maximi in chiliade: ſic ut logarithmus ejus ſit valdè magnus, & differentiæ logarithmorum circa illa loca non magnæ tantum, ſed inſuper etiam notabilibus incrementis creſcentes. Tunc logarithmo propoſiti decupli (qui facilius elaboratur) adde logarithmum decuplicationis, vel logarithmo centupli, logarithmum centuplationis, &c; ita habetur logarithmus propoſiti numeri.

EXEMPLUM.

Propoſitus eſto ſcrupuloſus 2345,68. Hic cùm ſit minor quàm 10000.00, ergò quære logarithmum ejus decupli, ſc. 23456,80. Is autem invenitur ſuperiori methodo 145001,00. Huic igitur adde logarithmum decuplationis
230258,51

ſumma 375259,51 eſt log. quæſitus numeri 2345,68.

ALIUD EXEMPLUM.

Propoſitus eſto numerus ſcrupuloſus 175,00 ut ejus logarithmus exactus habeatur: quære centupli 17500,00 logarithmum, qui eſt 174296,93
Et pro appoſitis 00 adde centuplat. 460517,03

Emergit numeri 175,00 log. 634813,96

ALIUD EXEMPLUM.

Sinus unius minuti ſolet exprimi hoc numero 29,09 quæritur hujus numeri logarithmus.

Scribe millecuplum	29090,00	
Hoc proximè minor invenitur	29000,00	log. 123787,43
Exceſſus	90,00.	

Hic continuatus ſeptem cyphris, & diviſus per 29045,00 medium arithmeticum inter datum & excerptum: ſeu ille continuatus 5 cyphris, & diviſus per 29045, dat quotientem 309,87, qui ablatus ab exſcripto logarithmo, relinquit logarithmum millecupli 123477,56

Huic igitur adde millecuplatorem	690775,54
Provenit logarithmus quæſitus	814253,10

Quod ſi numerus 29,09 exactè exprimeret ſinum unius minuti, tunc hic etiam exactus eſſet logarithmus unius minuti: at quia in 29,09 ultima unitas non eſt omninò plena, ideò etiam logarithmus unius minuti eſt paulò major: ut quidem Urſinus eum exprimit ſic 814257.

CASUS SECUNDUS.

Idem ferè proceſſus eſt, ſi datus ſcrupuloſus fuerit major quidem quàm 10000,00 parvus tamen etiamnum & minor quàm maximus chiliadis. Tunc enim logarithmo dupli, vel tripli, vel quadrupli (quod quidem non excedat maximum chiliadis) additur logarithmus totuplans.

Ut ſi detur numerus ſcrupuloſus 23456,78.

Hic etſi major eſt quàm 10000,00, eſt adhuc tam parvus, ut ejus quadruplus ſcilicet 93827,12 ſit adhuc minor maximo chiliadis. Quæratur igitur hujus logarithmus:

Adde	93800,00 eſt log.	6400,53	
Appendix ſcrupuloſa	27,12 aufert	28,82	quotiens auferend.
		138629,44	log. 4 druplans add.
Hæc dividitur per	938135		
	18,76270 | 2	145001,15	logarithmus quæſitus.
	8357300		
	7505080 | 8		
	852220		
	835730 | 8.		
	16490 | 2		

Hic

Hic caſus generaliter proponi poteſt, ut circumſpicias, in quâ proportione (terminorum quidem rotundorum) ſit numerus propoſitus ad aliquem proximè minorem maximo chiliadis. Tunc conſtituitur proportionis illius logarithmus, & per eum augetur logarithmus illius proximè minoris.

Sit ſinus quicunque, puta 15643,45; hic continetur in 100000,00 ſexies, ſextuplum enim eſt 93860,70. Huc poteſt accedere triens 5214,48, & conficitur 99075,18. Eſt igitur 15643,45 ad 99075,18 ut 19 ad 3, logarithmus novem decupli eſt 299443,90. Hinc aufer triplicantem 109861,23

Logarithmus igitur proportionis eſt 189582,67.

Quare logarithmus illius multiplicis proximè minoris maximo chiliadis per diviſionem elaboratus eſt iſte:

Exc.	75180000000	
Div.	9903700000	
	693259	7
	385410	
	495185	5
	902250	
	891333	9
	10917	1 quotiens.

	99075,18	1005,03
	99000,00	75,91 quotiens
Exc.	75,18	929,12
		184582,67
Adde log. prop.		185511,79

Log. quæſitus numeri abſoluti 15643,45

CASUS TERTIUS.

Si ſcrupuloſus, minor quàm 50000,00 habuerit non plures quàm 4 figuras ſignificativas, earumque ultima fuerit quinarius.

Tunc dupli vel quadrupli logarithmo in chiliade invento, adde logarithmum duplicantem vel quadruplicantem.

EXEMPLA.

Ut ſi datus ſit numerus ſcrupuloſus 49950,00 minor ſcilicet quàm 50000,00, ejus dupli 99900,00 logarithmo 100,05, adde logarithmum duplicationis 69314,72, fit 69414,77 logarithmus quæſitus.

ALIUD.

Suprà caſu primo quærendus fuit logarithmus ad numerum 17500,00, propter ſcilicet ejus centeſimæ 175,00 logarithmum indagandum. Hic ſi non poſitus eſſet in ipsâ chiliade, poſſet ſic inveſtigari.

Duplum

Duplum eſt 35000,00, quadruplum 70000,00. Hujus verò logarithmus eſt 35667,49.

Huic adde logarithmum quadruplatorem	138629,44
Prodit qui ſupra	174296,93

CASUS QUARTUS.

Si datus ſcrupuloſus, duplicatus continuè, tandem ſortiatur non plures figuras ſignificativas, quàm tres primas à ſiniſtris, ſed tunc exceſſerit maximum chiliadis, tum logarithmo partes 10 vel 100 de hoc ſic multiplicato, adde logarithmum duplicationis vel quadruplicationis, &c, à ſummâ aufer logarithmum decuplationis vel centuplationis, &c.

Ut ſi proponatur numerus ſcrupuloſus	51250,00
Hujus duplus eſt	102500,00
Et hujus duplus eſt	205000,00 habens

ſignificativas figuras non ultra tres.

Hic verò jam eſt major quàm 100000,00. Ergò ejus partis decimæ 20500,00

Logarithmo	158474,53
Adde logarithmum quadruplicantem hoc loco	138629,44
Et aufer logarithmum decuplantem hoc loco	230258,51
Reſtat numeri 51250,00 logarith.	66845,46

Potuit etiam ſumi duplus ipſius 205, &c, ſcilicet 410, &c, & rurſum hujus duplus 820, &c, hujus log. 19845,09

Adde ſedecuplatorem	277258,88
	297103,97
Et aufer decuplatorem	230258,51
Reſtat idem quod prius	66845,46

CASUS QUINTUS.

Si datus ſcrupuloſus habuerit partem aliquotam, quæ ſolas tres figuras à ſiniſtris habeat ſignificativas: Tunc logarithmo illius partis aufer logarithmum totuplantem.

Ut

Ut ſi proponatur numerus 19980,00 ſcrupuloſus.
Hujus ſemiſſis eſt 9990,00. Hujus numeri logarithmus per præceptum primi caſus invenitur ſic :

Decupli 99900,00. Logarith. eſt	100,05
Huic adde logarithmum decuplationis	230258,51
Ita fit log. 9990,00 partis dimidiæ	230358,56
Hinc aufer logarithmum duplationis	69314,72
Reſtat log. numeri 19980,00 quæſitus	161043,84

PRÆCEPTUM II.

Dato numero abſoluto, qui excedit maximum chiliadis (ut ſunt ſecantes arcuum) invenire ejus logarithmum.

Excedentis dati conſtitue partem aliquotam, quæ non excedat, puta decimam, centeſimam, milleſimam, &c ; vel etiam dimidiam, tertiam, quartam, per Caſ. II præced.

Ejus partis in chiliade quæſitæ logarithmus, auferatur à logarithmo totuplationis (ut decuplationis, centuplationis, millecuplationis, vel etiam duplationis, triplationis, quadruplationis) reſtabitque logarithmus excedentis privativus.

Ut ſi quæratur logarithmus numeri 102400,00. Hic numerus habet loca octo, cum chiliadis maximus 100000,00, habens loca itidem octo, ſit tamen minor. Quare ſumatur numeri dati pars decima 10240,00 utpote quæ jam ſit minor maximo chiliadis. Hic numerus cùm jam ſit in quarto loco ſcrupuloſus, logarithmum exactum non invenit in chiliade, quare per præcept. præcedens elaborandus eſt ejus logarithmus ſic :

Numerus

Numerus 10240,00
In chil. prox. mino. 10200,00 habens log. 228278,25
Quotiens 391,39 subt.

227886,86. Hic est logarithmus partis decimæ: qui ablatus à logarithmo 230258,51 decuplationis, relinquit logarithmum 2371,65 privativum, signo —, numeri scilicet dati excedentis 102400,00.

Excess. nostri prolong.	40,000000000	
Divisor est	1022 000	
Medium scil.	3066	3
Arithmeticum inter datum, & proximè minorem chiliadis	9340	
	9198	9
	1420	
	1022	1
	3980	
	3066	3
	9140	
	9108	9

In hoc exemplo, quia pars decima numeri propositi fuit scrupulosa, præstabat nos uti præcepti prioris casu v.

Ecc. numeri 102400,00 propositi.
Semissis 51200,00 rotundus, habet log. 66943,07 subt.
Logarithmus duplicationis est 69314,72

Restat quod prius 2371,65

In genere, si fuerint tres numeri continuè proportionales, eorumque medius idem, qui & maximus chiliadis; tunc excedens maximum hunc chiliadis logarithmum habet eundem cum minimo trium, sed privativum, signo — præponendo, ne videatur nihil aliud significare, quàm defectum in figurâ ultimâ, ut cap. 1.

Ut si sit 80000,00 ad 100000,00, ut hic ad 125000,00.

Quia igitur numeri 80000,00, logarithmus est 22314,36 positivus, erit ergò & numeri 125000,00, logarithmus 22314,36 sed privativus.

Aliter & facilitate inopinabili.

A dato numero excedente rejice tres ultimas figuras ad dextram. Sic curtatum quære inter logarithmorum differentias interlineares: & logarithmum ipsum ei differentiæ proximum, planeque respondentem exscribe; erit enim hic ipse logarithmus numeri propositi excedentis, sed privativus.

Ut si quæratur, quis sit logarithmus intervalli Martis & solis 152500, seu prolongati 152500,00, ut sit intervallum solis & terræ mediocre 100000,00,

 deletis

deletis ergò tribus ultimis, reſtat 152,50. Huic verò numero proximus invenitur inter logarithmorum incrementa interlinearia iſte 152,55, habens ante ſe logarithmum 42312,00, poſt ſe 42159,45, ut ſic ipſi differentiæ 152,55 intermediæ, reſpondeat etiam intermedius 42230,00 circiter. Hic igitur eſt numeri 152500,00 logarithmus privativus.

ADMONITIO DE SECANTIBUS.

Atque hinc apparet, etiam ſecantes ipſos arcuum complementorum quodammodo haberi per differentias logarithmorum, ad illos arcus, prolongatas tribus locis, quod ſuprà cap. II. promiſi me indicaturum.

ADMONITIO ALIA.

Hic poteſt vicem chiliadis implere, canonis Neperiani columna media, meſologarithmorum, juncto canone tangentum in hunc modum.

Datum numerum excedentem quære inter tangentes, et nota ejus arcum in gradibus & ſcrupulis: cum hoc excerpe ex canone meſologarithmorum, meſologarithmum privativum, habebis ſic excedentis logarithmum privativum.

PRÆCEPTUM III.

Dato logarithmo, qui non exactè reperiatur in chiliade, aſſignare numerum abſolutum juſtum cum ſcrupuloſitate ſuâ, ſicubi eâ ſit opus.

Dato logarithmo proximè majorem exſcribe ex chiliade, cum numero abſoluto rotundo reſpondente, factâque ſubtractione dati ab exſcripto, reſiduum duc in abſolutum exſcriptum ut multiplicantem, factus, abſectis ſeptem ultimis ad dextram locis, collocetur loco quatuor cyphrarum exſcripti abſoluti: Ita prodit quam proximè juſtus abſolutus, dato logarithmo competens. Sed quia ei adhuc deeſt aliquid, corrigetur ſic, ſi facti curtati dimidium colloces loco ultimarum cyphrarum multiplicantis, & multiplicationem repetas.

EXEMPLUM.

Datus eſto log.	145001,10 ſubtr.	
Hoc prox. major		
Chiliadis log.	145243,42	ei reſp. abſol. 23400,00
Reſiduum	242,32	
Duc in abſ. exſcript.	234,00,00	
	4846|4	
	726|96	
	96|9280000	
Factus abſectis 7 ultimis	5670|	Hic eſſet addendus ad exſcrip-

tum

tum absolutum, fieretque 23456,70. Sed quia pars addenda nondum est perfecta; ideò dimidium ejus 2835 appono ad multiplicantem, ut fiat 2342835, repetoque multiplicationem sic:

```
         242,32
         234,2835
         ————————
         4846|4
          726|96
           96|928
            4|8464
            1|93856
             | 72696
             | 121160
         ————————
Factus correctior 5677
```

Compendiosè sic, quia ultimi 35, parum aut nihil efficiunt.

```
 242,32
 234,28
 ——————
 4846|4
  727|0
   96|9
    4|8
    1|9
     |1
 ——————
 5677|
```

Facilè autem videbit exercitatus calculator, non totam sibi multiplicationem esse repetendam, sed multiplicatione per anteriores figuras multiplicantis peractâ, & summâ constitutâ, jam id quod per figuras anteriores semissis de emergente facto, fit, tantummodò subjiciendum esse decenter facto priori, in hunc modum.

```
2423,4
234
——————
4846|4
 727|0
  96|9
——————
5670|3
——
28,4|8
   1|9
——————
5677|
```

Hoc igitur correctiori facto apposito	
ad primo exscriptum	23400,00
Fit absolutus justus	23456,77

CAUTIO.

In logarithmis valdè magnis, non est accuratissima hæc ratio: id tamen, quod peccatur, utcunque magnum, in logarithmis magnis, non efficit tamen sensibile quippiam in iis, quorum causâ sunt logarithmi.

ALIA ADMONITIO.

Plerunque non eſt nobis opus ſcrupuloſitate numeri quæſiti: ſic ut ſufficiat exſcribere aliquem abſolutum medium inter duorum circumſtantium logarithmorum abſolutos. Et tunc etiam ſufficit ſcribere tam numeros, quàm logarithmos uſque ad punctum; quod quidem punctum huic ipſi compendio ſervit.

PRÆCEPTUM IV.

Dato numeri logarithmo privativo, invenire numerum.

Privativum datum aufer à multiplicatore aliquo majore, ut à decuplatore vel centuplatore, &c, vel etiam à duplicante, triplicante, &c; cùm reſiduo quære numerum in chiliade, qui erit quæſiti pars decima vel centeſima, &c, vel etiam dimidia, tertia, &c.

Ut ſi detur logarithmus—2371,65 privativus: ut ſcias quis ei reſpondeat numerus: aufer eum à logarithmo decuplationis 230258,51, quippe qui jam eſt illo major, reſtabit 227886,86. Hic indicat in chiliade numerum 10240,00, qui erit pars decima quæſiti: quare quæſitus ipſe erit 10240,00 ferè.

Vel aufer datum privativum à 69314,72 duplicationis logarithmo, quia etiam hic jam eſt major dato privativo, reſtabit 66943,07, qui oſtendit in chiliade 51200,00, qui eſt pars dimidia quæſiti: quare ipſe quæſitus eſt 102400,00.

Aliter & facilitate inopinabili.

Privativum datum quære inter logarithmos chiliadis, & incrementum duorum proximorum, illi reſpondens, exſcriptum, auge tribus figuris ad dextram, ſic formatus erit numerus logarithmi dati quæſitus.

Ut ſi datus ſit idem, qui prius 2371,65. Hic inter logarithmos quæſitus, cadit medius inter 2429,27 & 2326,86 & differentia horum, itidem media, eſt 102,41. Hanc auge tribus locis, fiet 102410,00, numerus reſpondens, ferè.

ADMONITIO.

Atque hîc iterum vicem chiliadis ſupplet ex canone logarithmorum quadrantis, columna media meſologarithmorum. Quæſitus enim in eâ logarithmus excerpat arcum, gradus ſcilicet in calce (quippe cum ſit privativus) minuta in margine dextro. Hic arcus tranſlatus in canonem ſinuum & tangentum, oſtendit inter tangentes numerum quæſitum.

PRÆ-

PRÆCEPTUM V.

De uſu Differentiarum interlinearium.

Et ſi incrementa logarithmorum chiliadis reſpiciunt (intra eam) ſolos numeros abſolutos, ad latus poſitos, eorumque logarithmos, inter quos ſunt inſerta: ſciendum tamen eſt, illa eadem, eodem ordine à primo chiliadis ipſius (diſſimulato jam ejus veſtibulo) ſervire cuicunque alii tabulæ conſtruendæ, quæ alios numeros (progreſſione tamen arithmeticâ æquabili ſurgentes), alios etiam eorum logarithmos habuerit, dummodò pauciores mille fuerint.

Verbi causâ, ſi pro numero maximo 100000,00 placeat ſtatuere maximum 60′ 0″, & hujus partes facere 720, ſic ut una pars ſit 0′ 5″, pro eo quod in chiliade eſt 100,00 logarithmus in chiliade ad numerum 720 eſt 32850,41. Hunc aufer à logarithmo ad 100,00, ſcilicet 690775,57, reſiduus erit logarithmus ad 0′ 5″ novæ tabulæ, ſcilicet 657925,13. Quod ſi jam ſubtraxeris ordine omnia incrementa (ſeu potius decrementa) logarithmica chiliadis ipſius (excluſo jam ejus veſtibulo) orſus à primo 69314,72 uſque ad ultimum ex 720, ſcilicet 138,98, conſtituti erunt logarithmi ad omnes partium novarum collectiones, puta ad 0′ 10″ & 0′ 15″, &c. Idem fiet, ſi unum & eundem logarithmum, qui eſt ad numerum 720 ſeu 72000,00 ſubtraxeris ab omnibus & ſingulis 720 logarithmis, ordine invicem ſuccedentibus in chiliade.

ALIUD EXEMPLUM.

Si quis in ſubſidium rei monetariæ vellet conſtruere tabulam, cujus maximus numerus ſit una marca, hoc eſt 16 unciæ, hoc eſt 256 oboli; logarithmus ad abſolutum 25600,00 chiliadis ſcilicet 136257,79

Ablatus à logarithmo primo chiliadis ſcilicet	690775,54
Relinquit logarithmum unius oboli ſcilicet	554517,75

Ab hoc logarithmo ablata prima 256 incrementa logarithmica chiliadis, quorum primum ſit 69314,72, ultimum verò 391,39, conſtituent omnium numerorum monetariorum logarithmos.

Uſus verò talis tabulæ facilè intelligatur ex comparatione harum præceptionum.

Ego quidem illum hâc vice prætereo, cùm nulla pars chiliadis à re monetaria nomen ſortiatur.

Alius uſus indicatus eſt præcepto 11.

PRÆCEPTUM VI.

Duos numeros abſolutos in ſe mutuò multiplicare per logarithmos, & factum invenire.

Multa quidem ſequentium præceptorum, ſeipſis inutilia videbuntur, cùm facilius & exactius operationes aliquæ peragantur viâ communi: at propter cognationem cum aliis utilioribus, negligenda non fuerunt, lucis causâ.

Quando igitur numeri duo abſoluti ſunt inter ſe multiplicandi, ut ſciatur factus: tunc hoc eſt perinde, ac ſi in regula trium, poneretur primo loco ad ſiniſtram unitas, loco ſecundo & tertio multiplicans, & multiplicandus; ut factus occupet locum quartum, veluti peractâ regulâ trium, quotientem.

Igitur adde logarithmos duorum numerorum, excerptos ex chiliade: ſumma quæſita rurſum inter logarithmos, monſtrat abſolutum in illorum columnâ numerum, qui ſemper eſt quæſiti pars multiplex, proportionis decuplæ continuæ. Nam ſi uterque numerorum abſolutorum fuit minor maximo chiliadis; tunc numero per ſummam logarithmorum monſtrato ſunt apponendæ ſeptem cyphræ ad dextram, ut ita factus hic rectè comparari poſſit cum facientibus.

EXEMPLUM.

Sint invicem multiplicandi	51200,00	log.	66943,07
&	76800,00	log.	26396,55
Summa			93339,62

Hæc ſumma dat abſolutum 39330,00 circiter.
Igitur factus ex multiplicantibus eſt 393300000000,00 circiter.

Sin autem facientes excedunt maximum chiliadis, tunc per eos excerpi non poteſt, ſed per eorum partes decimas vel centeſimas, &c. & tunc etiam facto, ſic conformato, ut vult præceptum ſuperius, apponendæ ſunt inſuper omnes cyphræ, quæ facientium utrique junctim fuerunt adimendæ, ut excerptio fieri poſſet.

Ut ſi numerorum ſuperiorum alter fuiſſet uno loco longior, ſcilicet 768000,00, etiam factus prodiret uno loco longior, ſcilicet 3933000000000,00. Sin autem etiam alter fuiſſet duobus locis longior, ſcilicet 5120000,00, factus prodiret in univerſum tribus locis longior, ſcilicet 393300000000000,00.

CAUTIO.

Si logarithmus ex additione excreſceret ſupra maximum chiliadis logarithmum, ipſius etiam veſtibuli: aufer ab eo decuplatorem vel centuplatorem, &c. ſic ut remaneat minor maximo chiliadis: per hoc reſiduum excerptus numerus abſolutus, & per præceptum conformatus, debet jam viciſſim curtari unâ, duabus, &c, figuris.

PRÆCEPTUM VII.

Numerum abſolutum quadrare, ſeu ejus quadratum præter propter invenire.

Duplica logarithmum numeri quadrandi, duplum hoc inter logarithmos quæſitum, exhibet ex columnâ ſuâ numerum abſolutum, cui ſunt apponendæ ad dextram ſeptem cyphræ; ſic habebitur quadratum numeri propoſiti, ſaltem in primis ejus figuris ad ſiniſtram. Nam loca reliqua ad dextram cyphris impleta, nullius ſunt momenti, etiam ſi vel ſcrupuloſiſſimè exprimerentur.

EXEMPLUM.

Quadrandus ſit 3100,00 log.	347376,81
Ejus duplum	694753,62
Hoc duplum indicat abſolutum	96,10
Ergò quadratum quæſitum eſt	9610000,0000

VEL,

Sit quadrandus 31000,00 log.	117118,30
Duplum	234236,60
Hoc duplum indicat abſolutum	9610,00
Ergò quadratum quæſitum eſt	961000000,0000.

PRÆCEPTUM VIII.

Numeri abſoluti cubum invenire.

Triplica logarithmum numeri; triplum hoc inter logarithmos quæſitum, exhibebit primas cubi quæſiti figuras, quibus apponendæ ſunt aliæ 14.

EXEMPLUM.

Numeri abſoluti 90000,00, logarithmus 10536,05 triplicatus facit 31608,15, qui ut logarithmus dat ex columnâ numerorum abſolutorum 72900,00.

Ergò cubus eſt 729000000000000,000000.

PRÆCEPTUM IX.

Si duorum numerorum abſolutorum major dividendus eſt per minorem; quotientem eruere per logarithmos.

Quando dividendus eſt major per minorem: tunc hoc eſt perinde ac ſi in regulâ trium, primo loco ad ſiniſtram collocaretur minor, ſecundo major, tertio unitas, quartoque loco ad dextram quotiens. Nam ut minor eſt ad majorem, ſic unitas ad quotientem.

Aufer igitur logarithmum minoris, à log. unitatis abſolutæ in veſtibulo chiliadis, ſcilicet ab 1611809,59, reſiduo adde logarithmum majoris, ſumma inter logarithmos quæſita oſtendit inter abſolutos quotientem.

EXEMPLUM.

EXEMPLUM.

Sit dividendus 99200,00 pro 3200,00. Hujus ergò logarithmum 344201,94 aufer, & viciſſim illius logarithmum 803,22 adde, ad 1611809,59, conficietur logarithmus 1268412,87, qui oſtendit abſolutum 0,31. Ergò 0,31 eſt quotiens.

VEL,

Aufer logarithmum minoris à proximè majori logarithmo unitatis cujuſcunque in chiliade, reſiduo adde logarithmum majoris, provenit logarithmus numeri, à quo ſunt abſcindenda totidem loca, quot cyphras habuit unitas illa, quæ logarithmum dedit.

Ut ſi in exemplo ſuperiori logarithmum minoris 344201,94 abſtuliſſes à 460517,03 logarithmo numeri 1000,00, reſiduus fuiſſet 116315,09

Cui additus majoris log.	803,22
Confeciſſet logarithmum	117118,31

Hic verò indicat 31000,00. Quia verò numerus ſeu unitas illa 1000,00 habet 5 cyphras, decurta igitur hunc viciſſim 5 ultimis locis, manetque 31.

OBSERVATIONES.

Si per diviſoris decuplum diviſeris; prodibit pars decima quotientis quæſiti: & ſic conſequenter.

Si unus vel ambo exceſſerint maximum chiliadis: decurtatis ambobus æquali numero locorum, ſic ut fiant minores maximo chiliadis, operatio peragatur, prodibitque quotiens juſtus.

EXEMPLA DIMISSIS FIGURIS POST PUNCTUM.

Dividendus ſit 100000 per 4362.

Hic poſſum uti diviſoris decuplo 43620, cujus log. cùm ſit 82965, auferam eum â logarithmo proximè majori unitatis alicujus ex chiliade, ſcilicet à numeris 10000 logarithmo 230259, reſtat 147294, cui jam nihil additur, quia logarithmus numeri 100000 eſt 0. Hic ergò logarithmus oſtendit abſolutum 22950. Sed quia unitas uſurpata habet quatuor cyphras, debent ergò viciſſim rejici ab hoc abſoluto, quatuor quidem loca, ut formetur pars decima quotientis, tria verò, ut ipſe quotiens. Erit igitur ille 23 vel $22\frac{95}{100}$.

Quod ſi dividendus fuiſſet 1000000, diviſor 4362: tunc pro 1000000, chiliadis maximum (uſque ad punctum) ſuperante, ſcripſiſſem 100000, ut qui non ſuperat, & pro 4362, $436\frac{1}{5}$ utrumque ſcilicet curtaſſem æqualiter.

Tunc uſus priore proceſſu, pro $436\frac{1}{5}$ uſus eſſem ejus centuplo 43620. Et prodiſſet $2\frac{259}{1000}$ centeſima quotientis, ipſe igitur quotiens $225\frac{9}{10}$.

ALIUS DIVIDENDI MODUS.

Dividendum majorem decurta, uti curtatus fiat minor divisore. Tunc à logarithmo curtati aufer logarithmum divisoris, residuum erit logarithmus numeri, qui decurtandus est tot locis, ut decurtatio utraque deleat loca 7 (vel, si omittimus figuras post punctum, loca 5.)

Ut si sit dividendus 99200 per 3200.

Rejectis à dividendo duabus figuris fit 992 jam minor divisore. Jam igitur per observationes superiores sume utriusque decuplum, ut fiant proximi maximo chiliadis.

Ergò	32000 logar.	113943
	9920 logar.	231062
Residuum		117119

Est logarithmus numeri 31000. Quia igitur à dividendo rejectæ sunt figuræ duæ, rejiciantur jam à quotiente tres, ut rejectarum sint 5 formabitur quotiens 31.

PRÆCEPTUM X.

Ex numero absoluto, ut quadrato, radicem extrahere quadrati.

A numero absoluto proposito reseca loca bina & bina, tantisper, quoad loca residua ad sinistram representent numerum non majorem maximo chiliadis: numeri sic decurtati logarithmum adde logarithmo unitatis puræ in vestibulo chiliadis: summæ semissis quæsitus inter logarithmos, ostendet è regione inter absolutos, primas figuras radicis quæsitæ, cui numero pro binis locis prius resectis, restituendæ sunt singulæ cyphræ. Ita formatur radix quadrata quæsita.

EXEMPLUM.

Quadratum esto 961000000, quæritur ejus radix quadrata. Numerus igitur iste habet loca 9, cùm maximus chiliadis habeat tantùm 8. Abscinde duo ultima (numero pari) ut restent septem.

Ergò 96100,00 habet log.	3978,00
Sed puræ unitatis in vestib. log.	1611809,59
Summa	1615787,59
Semissis	807893,80

Hic semissis indicat absolutum 31,00, huic verò pro duobus locis prius rejectis jam restitue unam cyphram, fietque vera radix 310,00.

ALIUS MODUS SINE LOGARITHMO UNITATIS.

A numero propoſito rejice loca ad dextram numero impari, ut fiat minor maximo chiliadis :

Hujus numeri decurtati logarithmum ipſum bipartire, cùm ſemiſſe inter logarithmos quæſito, excerpe ex columnâ abſolutorum numerum competentem, qui offert primas ad ſiniſtram figuras radicis quæſitæ : ſed niſi ſeptem omninò loca dempta fuerint initiò, nondum erit hic numerus radix ipſa. Pro binis enim minus quàm ſeptem illic ademptis, ſingulæ cyphræ hîc ſunt adimendæ, pro binis pluſquàm ſeptem ademptis, ſingulæ reſtituendæ.

Et nota, quod etiam ſi numerus ipſe ſtatim initio fuerit minor maximo chiliadis ; tamen unus illi locus ſit demendus, ut radici demantur tria loca, vel unus adjiciendus, ut radici demantur quatuor, vel ſi tam eſt parvus, tres adjiciendi, ut radici demantur quinque.

EXEMPLUM.

Ut ſi quadratus proponatur 961000000. Hic cùm habent loca 9, ſi auferres unum, adhuc haberet nimium, octo ſcilicet, & in primo novenarium, cùm maximus chiliadis habeat octo quidem & ipſe, ſed in eorum primo unitatem. Aufer ergò loca tria (numero ſcilicet impari) reſtant ſex, numerus ſcilicet 961000. Hujus logarithmus eſt 234236 uſque ad punctum, ſemiſſis hujus 117118 oſtendit 31000,00.

Quia igitur quadrato, loca ſunt dempta 4 minus quàm 7 demenda jam ſunt huic numero duo ; formaturque radix 310,00.

Si quadratus fuiſſet 9610000000000, conſtans ſcilicet locis 13, ut igitur minor reſtet numerus, quàm maximus chiliadis : oportet reſcindere loca 7, numero ſcilicet impari : & tunc peractâ operatione de invento abſoluto reſcinderetur nihil, quia loca à quadrato reſciſſa, fuerunt numero ſeptenario, quot cyphras habet maximus chiliadis.

Si verò quadratus habuiſſet loca 15, reſcindenda fuiſſent loca 9, numero ſcilicet impari, ut ſemper in hoc modo : ut ſcilicet reſtaret numerus minor maximo chiliadis. Cùm autem 9 excedat 7 binario, jam igitur ad numerum per operationem inventum, viciſſim fuiſſet apponenda una cyphra.

Denique ſi quadratus fuiſſet 9610000, adhuc rejiciendus fuiſſet ab illo locus unus, ſemper enim in hoc modo aliquid impari numero mutandum eſt. Unum verò à 7 ablatum relinquit 6, tres igitur cyphræ tunc fuiſſent rejiciendæ, ut radix tunc fiat 31,00.

Et ſi quadratus fuiſſet 96100, rejici poterit unus locus, poterit & addi unus, ut prolongatus nihilominus ſit minor maximo chiliadis. Sed tunc qui apponit unum, is demit uno minus quàm nihil. Differentia verò inter 7, & inter unum minus quàm nihil, eſt 8, quatuor ergo demeret loca de invento 31000,00, ut ſic formetur ipſius 96100 radix 310.

Ita ſi quadratus fuiſſet 9610, potuiſſent addi loca tria, fuiſſetque 96100,00

minor maximo chiliadis, habens logarithmum 3978. Semiſſis verò 1989 oſtendiſſet abſolutum 98029,00.

Tria verò loca prius appoſita pro VII delendis, faciunt decem: quinque igitur hîc dele, ut reſtet radix 98 plus, numeri quadrati 9610 ferè.

PRÆCEPTUM XI.

Medium proportionale inter duos abſolutos datos invenire.

Quia hoc fit in arithmetica vulgari, multiplicatis in ſe mutuò duobus datis, factique radice quæſitâ; fit igitur per logarithmos ſimiliter, conjunctione in unum, ſexti & decimi præceptorum; in hunc tamen modum commodiſſimè.

Abſolutus uterque acquirat loca 7, niſi fortè unus eorum fuerit ipſe maximus chiliadis, huic relinqui poſſunt loca octo. Tunc ſic aptatorum logarithmi adduntur, cum ſummæ ſemiſſe, ut logarithmo, excerpitur numerus abſolutus; cui ſunt reſtituenda loca prius abſciſſa, vel demenda quæ erant prius adjecta facientibus utriſque.

EXEMPLUM.

Sint duo numeri 987654321 & 59643, quæritur eorum medius proportionalis. A primo rejice loca duo ad dextram, ad ſecundum appone duas cyphras:

Ergò	9876543	log.	1242,25
	5964300	log.	51679,34
		Summa	52921,59
		Semiſſis	26460,80.

Hic dat abſolutum 76750,65. Cùm igitur uni datorum dempta ſint loca duo, alteri totidem appoſita: viciſſim huic invento duo ſunt apponenda, duo viciſſim demenda, hoc eſt, compenſatione factâ nihil mutandum. Ita hic ipſe eſt medium proportionale quæſitum.

Hic operæ precium eſt videre, quantus error fuiſſet commiſſus, ſi logarithmi fuiſſent excerpti per proximè minores vel majores abſolutos ſine elaboratione ſcrupuloſâ.

Numero 98765,43
Vicinior chiliadis & major 98800,00 dat log. 1207,26 minorem juſto.
Et numero 59643,00
Vicinior chiliadis & minor 59600,00 dat log. 51751,46 majorem juſto.

Summa 52958,72 propè vera
Semiſſis 26479,36

Jam logarithmus chiliadis hoc proximè major, ut qui propior, dat abſolutum 76700,00. Ergò ſemiſſe ipſi reſpondebit plus aliquid. Itaque ſi

 medium

medium eligam inter 76700 & 76800, puta 76750,00, id erit proximum vero. Sanè suprà vides inventum 76750,65.

PRÆCEPTUM XII.

Propositi numeri radicem invenire cubicam, ejusque radicis quadratum.

Quia unitas est ad radicem cubicam, ut hæc ad quadratum ejus, & hoc ad cubum ejus: ergò à numero absoluto, qui ut cubus proponitur, rejice ad dextram loca terna & terna tantisper, quoad usque loca residua repræsentent non majorem maximo chiliadis: Numeri sic decurtati logarithmum, aufer à logarithmo unitatis puræ in chiliadis vestibulo, residui tertiam partem, si adjeceris logarithmo numeri, constitues logarithmum utilem indagando cubicæ radicis quadrato, si duas tertias adjeceris, fiet logarithmus pro ipsa radice cubicâ. Excerpe igitur absolutos, cùm utraque summâ constitutâ, inter logarithmos quæsitâ, & pro ternis locis prius demptis, appone radicis quidem numero cyphras singulas, ejus verò quadrati numero binas: ita formabitur, & radix cubica & ejus quadratum.

EXEMPLUM.

Propositus esto cubicus numerus 729 cum 18 cyphris, loca scilicet habens 21. Rejice loca 15 quinquies scilicet terna: quia si 12 rejiceres, restarent 9, plura scilicet quàm habet maximus chiliadis.

Jam igitur 72900,00 dat log.	261866,63
Sed unitatis primæ in vestib. log. est	1611809,59
Differentia	1349942,96
Seu quod idem est 72900,00	
Dat log.	31608,15
Sed unitatis secundæ in vestibus log.	1381551,08
Differentia	1349942,93
Hujus pars tertia	449980,98
Summa pro quadrato	711847,61
Summa pro rad. cub.	1161828,59

Ergò summa prior dat 81,00, cui appone propter loca 15 prius rejecta, cyphras 10 fiet 8100000000000. Quadratum radicis cubicæ.

Sic posterior summa dat 90, cui appone pro 15 locis prius rejectis jam cyphras 5, ita fit radix cubica 90000,00.

PRÆ-

PRÆCEPTUM XIII.

Invenire logarithmum, indicem proportionis, inter duos numeros datos.

Magnum habet uſum hoc præceptum ad triangula plana ſolvenda, de quibus agemus capite ſequenti.

Sunt autem caſus varii. Aut enim uterque datorum terminorum invenitur expreſſus in chiliade; aut alter ſolum, aut neuter. Et ſi uterque in chiliade; tunc ii aut proximi ſunt invicem, ut 237, 238 ſeu 23700,00, 23800,00, & tunc differentia logarithmorum, numeris ad latus reſpondentium, eſt quæſitus proportionis index; ut in hoc exemplo 421,05 indicat proportionem inter 237, 238. Aut non ſunt invicem proximi, ſed diſtant interjectis aliis: tunc logarithmus majoris auferatur à logarithmo minoris, reſtabit logarithmus proportionis inter datos terminos.

EXEMPLA.

Sint termini proportionis 1 & 60.
Numeri 1000,00 logar. 460517,03
Numeri 60000,00 logar. 51082,56 auf.

Reſtat log. proportionis inter datos 409434,47 qui ideò & ſexagecuplator dici poteſt.

Sic numeri 1000,00 logar. 460517,03
Et numeri 24000,00 logar. 372970,14

Reſtat log. viginti quadruplæ proport. 87546,89

Sin vel neuter ex terminis, vel alteruter ſolùm invenitur expreſſus in chiliade: tunc vel cadit talis inter duos chiliadis, proximos invicem, eſtque ſcrupuloſus, vel excedit maximum chiliadis.

In priori caſu elaboretur prius logarithmus numeri ſcrupuloſi; & tunc per eum operatio eſt eadem, quæ prius.

In poſteriori caſu ſumantur numeri utriuſque æquè multiplices partes, minores maximo chiliadis, & cum eorum logarithmis agatur ut ſuprà.

Ut ſi quæratur proportio inter 102400,00
& 100000,00
Decurta eos æqualiter, ut ſint eorum partes decimæ ſcilicet
10240,00 log. 227886,86
10000,00 log. 230258,51

Erit proportionis log. 2371,65

NOTA.

Si numerorum alter fuerit ipſe maximus chiliadis, tunc ipſe logarithmus alterius numeri, eſt index proportionis inter utrumque ſeu poſitivus fuerit logarithmus, ſeu privativus. Logarithmus enim definitus eſt poſt prop. xx. nihil aliud quàm hæc ipſa proportio.

PRÆ-

PRÆCEPTUM XIV.

Proportionem cum terminis ſuis datam ſecare in partes, quæ ſint ad invicem in alia proportione propoſitâ.

Data ſit proportio inter terminos 37 & 53.

Hæc proportio ſit ſecanda in partes duas; quarum una ſit ad alteram, ut 5 ad 2.

Igitur conſtituatur quantitas proportionis ſecandæ per præcedens præceptum.

	Logarithmi		Logar.
Termini	5300—293746	vel	53000—63488
	3700—329684		37000—99425
Proportio	35938		35 937

Jam ſecundæ proportionis, ſecundum quam ſecanda eſt prima, termini ſunt 5 & 2, ſumma 7. Divide igitur proportionis primæ quantitatem in 7, erit pars ſeptima 5134, & duæ ſeptimæ 10268. Ergò quinque ſeptimæ 25670.

Quantitate partium inventâ, jam duplici viâ poteſt ſecari in has partes prima proportio: Aut enim ut pars major ſtet à plagâ termini majoris.

Igitur termini majoris logarithmo	63488
Adde partem majorem	25670
Summa	89158

Hæc ut logarithmus oſtendit abſolutum 41000 ferè. Ergò ut ſe habet 5 ad 2, ſic ſe habere fecimus proportionem inter 53, 41 ad proportionem inter 41, 37.

Aut ut pars minor ſectæ, ſtet à plagâ termini majoris.

Eidem ergò termini majoris logarithmo	63488
Adde partem minorem	10268
Summa	73756

Hæc ut logarithmus oſtendit 47830 circiter.

Ergò ut ſe habet 2 ad 5, ſic fecimus ſe habere proportionem inter 53, $47\frac{83}{100}$ ad proportionem inter $47\frac{83}{100}$, 37.

Idem efficiemus etiam per logarithmum termini minoris, ſubtrahentes ab eo.

ALIUD

ALIUD EXEMPLUM ET NOBILE QUIDEM.

Datur proportio inter dies 687 periodi Martis & $365\frac{1}{4}$ periodi ſolis (terræ Copernico), ſit hæc proportio ſecanda in partes duas, ſic ut major ad minorem ſe habeat, ut 2 ad 1, & ut major pars proportionis diviſæ ſtet à plaga termini minoris, ut ita proportio tota ſit ejus partis majoris ſeſquialtera.

Ergò termini majoris 68700	log.	37542
Termini minoris 36525	log.	100740
Quantitas ergò proportionis		63198

Proportionis verò diviſoriæ termini 2, 1 faciunt 3, quâ ſummâ, diviſa quantitas proportionis, facit ejus trientem 21066, & duos trientes 42132
Hos aufer à logarithmo partis minoris 100740

Reſtat 58608

Hoc ut logarithmus oſtendit numerum abſolutum 55650. Ergò ut 1 ad 2, ſic fecimus proportionem inter 68700, 58608 ad proportionem inter 58608, 36525.

Quod ſi ex termino minori 36525 fiat 100000, diſtantia mediocris ſolis & lunæ, tunc ex hoc invento termino majori 58608, fiat 1524,00 ferè: diſtantia ſolis & Martis mediocris.

Et ſit manifeſtum, quod proportio periodicorum temporum ſit ſeſquialtera proportionis intervallorum mediocrium: ut indicavi in Epitomen Aſtronomiæ Cop. libro IV.

PRÆCEPTUM XV.

Regulam proportionem ſeu detri abſolvere per logarithmos, ſeu ſine multiplicatione & diviſione.

Poſitis in regula tribus datis numeris, in chiliade comprehenſis, ut docet vulgaris arithmetica, aufer logarithmum ſiniſtimi à logarithmo medii, vel hunc ab illo: differentiam logarithmo dextimi adde in primo caſu, ſubtrahe in ſecundo, ſi poteſt: ſumma illic, vel hic reſiduum, eſt logarithmus quarti quæſiti ſeu quotientis poſitivus.

Rurſum autem ſi logarithmus tertii fuerit minor reſiduo, ſubtrahe viciſſim illum ab hoc, remanebitque privativus logarithmus, numeri ſcilicet abſoluti, ſeu quotientis, ſuperantis maximum chiliadis.

EXEMPLUM PRIMI CASUS.

77100 dat 53200 quid 92500

Log. 63111

Log. 26007 ſubt.

Reſiduum 37105

Logar. 7796 adde

Logarithmus 44901 quotientis 63800 plus.

EXEMPLUM SECUNDI CASUS.

876 dat 941 quid 765

Log. 6081 ſubtrahatur ut minor

Log. 13239

Reſiduum 7158 ſubtrahe quia ſecundus ſubtractus

Logarithmus 26788

Logarithmus 19630 quotientis 822 minus.

EXEMPLUM CASUS TERTII.

889 dat 996 quid 960

Log 401 ſubtrahatur ut minor

Log. 11766

Reſiduum 13365 ſubtrahendum erat, quia ſecundus ſubtractus fuit

Logarith. 4082 ſubtrahe hunc

quia minor eſt reſiduo.

Logarith. 7283 privativus, quia tertius ſubtractus fuit. Hic igitur privativus per IV præceptum, dat quotientem 1075 majorem maximo chiliadis 1000, ut hîc quidem operamur per curtatos.

NOTA.

Licet autem commoditatis cauſa, omnes tres in regulâ poſitos, vel prolongare, vel curtare, ut fiant proximè minores maximo chiliadis: & tunc quotiens, qui pro ſic accommodatos elicitur, idem debet pati, quod paſſus eſt ſiniſtimus: eos verò locos, qui ſecundo & tertio fuerunt adempti vel appoſiti ſolus, quotiens omnes contrariâ ratione vel recipere debet, vel amittere.

CAPUT

CAPUT IX.

De Copulatione Columnæ Logarithmorum cum duabus aliis junctis.

PRÆCEPTUM GENERALE.

QUOTIESCUNQUE in regulam proportionum offeruntur diversarum columnarum numeri, ficut duo quidem ex iis fint ex unius columnæ genere, unus verò refiduus cum quotiente quæfito competat in columnam alteram: tunc operatio per logarithmos perfici poteft; tam in vulgaribus feu abfolutis numeris, quàm in logifticis.

Hujus mixturæ dabo aliquot exempla, tanquam præcepta particularia.

I. PRÆCEPTUM.

Dividere horas vel gradus per fexagenaria fcrupula.

Quia enim eft, ut divifor fexagenarius, ad dividendum, fic integrum feu 60′ ad quotientem: Divifor igitur & integrum, funt ex genere eodem fexagenario, dividendus verò & quotiens poffunt effe ex quadrivicenariâ. Poffunt igitur illa exempla capitis VI, præcepti III; ubi dividendus aliquot integra continet, poffunt, inquam, plerunque per quadrivicenariam perfici.

Ut fi dividendi, ut fuprà, 3° 45′ 13″ in 57′ 8″, fic age:

	3°	45′	13″	log. 185470 circ.	ex quadrivic.
Scr.	57	8		log. 4900	ex fexagenar.

Refiduum 180570 dat ex quadrivicenaria quotientem 3° 56′ 30″ ut & cap. VI.

II. PRÆCEPTUM.

Partem proportionalem fcrupulis fexagenariis, capere de differentiâ, excedente integrum.

Addantur logarithmi fcrupulorum in fexagenaria, & differentiæ excedentis in quadrivicenaria, quæfitorum fumma ex quadrivicenaria oftendit partem proportionalem.

X Ut

Ut ſi uni gradui anomaliæ mediæ reſpondeant 1° 24′ 17″ coæquatæ, quid competit minutis 49,23.

1° 24′ 17″	log. 284000	circiter	
Se. 49 23	log. 19480		
Summa	303480	dat 1° 9′ 15″	tantùm competit ſcrupulis 49,23.

III. PRÆCEPTUM.

Dato diurno ſolis, vel planetarum v unius, colligere aliquot horarum motum.

Si diurnus minor eſt quàm 60′, ut in Saturno, Jove, Marte, & in ſole ; quære eum in ſexagenaria, numerum horarum in quadrivicenaria, & adde eorum logarithmos ; ſumma in logarithmis quæſita, eruet ex ſexagenaria partem horis competentem.

In ſole, Venere, Mercurio, ſi diurnus eſt inter 1 & 2°, quære proportionalem partem de exceſſu ſupra 1°, eamque adde ad id, quod horis datis in ſexagenaria reſpondet per caput IV. In Mercurio verò, ſi ejus diurnus ſuperat 2° partem proportionalem de exceſſu, adde duplo illius, quod horis datis reſpondet in ſexagenaria.

Aliter, quia diurnus plerunque conſtat integris ſcrupulis, ſine appendice ſecundorum : quærere illa poteris non in ſexagenariâ, ſed inter abſolutos, prolongata priùs, ut cap. VIII docui : & tunc quæ ex logarithmis colligitur ſumma, quæſita in logarithmis, rurſum inter abſolutos oſtendet partem proportionalem decurtandam iterum totidem locis.

EXEMPLA.

		VEL	
Sit diurnus 17′, horæ 19 48′ 0″			19237
Horar. 19 48′ log.	19237	Ex abſolutis 17000	
Ser. 17′ ex ſexag. log.	126200	Log.	177200
Summa	145437	Summa	196437
Pars proportionalis ex ſexagenariâ	14′ 1″	Ex abſol. colum.	14020

Jam ſi Veneris diurnus fuiſſet 1° 17′, tunc è regione 19^h 48′ invenio ſcrupula 49′ 30″, quæ addenda ſunt ad 14′ 1″ partem proportionalem de exceſſu, eſſetque arcus quæſitus 1° 3′ 1″. Et ſi Mercurii diurnus fuiſſet 2° 17′, duplicanda fuiſſent illa 49′ 30″ addendaque 14′ 1″, ita collegiſſemus 1° 53′ 1″.

IV. PRÆCEPTUM.

Horarium elicere ex motu aliquot horarum datarum, minore quàm 60′.

Aufertur logarithmus horarum datarum, in quadrivicenariâ quæſitarum, à logarithmo ex ſexagenariâ ſcrupulorum, horis competentium; reſiduum, ut logarithmus, ex ſexagenaria exhibebit horarium. Exempla ſunt facilia & obvia.

V. PRÆCEPTUM.

Datâ ſeparatione diurnâ minore quàm 60′ & diſtantiâ planetæ, ab alterius aſpectu, minore ea quàm eſt diurnus, prodere horarum intervallum reſpondens.

In ſexagenariâ, vel etiam inter abſolutos, quæſitis diurna ſeparatione & diſtantiâ, logarithmus illius aufertur à logarithmo hujus, reſiduum, ut logarithmus, exhibet horas ex quadrivicenariâ, reſpondentes diſtantiæ ſeu intervallo.

VI. PRÆCEPTUM.

Hoc modo etiam ingreſſus ſolis in ſigna computatur: Per diurnum ſcilicet ſolis, & per diſtantiam ejus à principio ſigni, in meridie vicino, in ſexagenariâ quæſitis, & logarithmo illius ablato à logarithmo hujus. Reſiduum enim, ut logarithmus, oſtendit in quadrivicenariâ, horas ante vel poſt meridiem.

Plurima alia per hanc combinationem columnarum perfici poſſunt: ſed ſex iſta exemplorum loco ſufficiant. Jam enim ad rariora & ſcrupuloſiora nonnulla transeundum, ſuprà dilata, & in hunc locum rejecta.

VII. PRÆCEPTUM.

Cuilibet arcui ſinum ſuum ſcrupuloſum aſſignare per logarithmos.

Subſidium, ut vides, paratur hîc capiti 11, ut in ejus tractatione promiſeram. Quanta igitur ſit operationis certitudo, vide ibi: nunc modum doceo.

Ab arcu dato aufer proximè minorem chiliadis, & exſcribe ejus ſinum rotundum, ut jam corrigatur, ſuamque acquirat ſcrupuloſitatem. Tunc in ſexagenariâ, quæſitis & exceſſu arcus dati, & differentiâ binorem arcuum chiliadis, inter quos arcus propoſitus intercidit, auferatur logarithmus hujus à logar. illius: Reſiduum immiſſum in logarithmos, oſtendit inter abſolutos numerum,

 qui

qui præcisis tribus ultimis locis ad dextram, fit augmentum scrupulosum exscripti sinus rotundi.

EXEMPLUM.

Propositus esto arcus	37°	49′	53″		
Proximè minor chil.	37	48	26	dat rotund.	61300,00
Residui	0	1′	27″	log. 372900 circ.	
Differentiæ inter duos chil.		4	21	log. 262400 circ. aufer.	
				Residuum 110500	

Hoc quæsitum, ut logarithmus, dat inter absolutos numerum 33190,00 circiter. Ergò præcisis 3 locis, acquirimus 33,19, & fit sinus scrupulosus arcui proposito conveniens 61333,19.

CAUTIO.

In fine quadrantis, ubi differentia in chiliade superat 60′, uti liceret columnâ quadrivicenariâ, nisi tum simul etiam proportionalitas, quàm hoc præceptum præsupponit, insigniter turbaretur.

VIII. PRÆCEPTUM.

Vicissim cuilibet sinui scrupuloso, suum assignare arcum.

Primùm cum sinu rotundo, proximè minori, exscribe arcum & differentiam subjectam. Tunc ad 4 ultimas figuras sinus scrupulosi (ad excessum scilicet sinus dati suprà rotundum chiliadis proximè minorem) appone tres cyphras, & quæsito numero sic formato inter absolutos, differentiâ verò in sexagenariâ, adde eorum logarithmos; summa ut logarithmus ostendet in sexagenaria, scrupula prima & secunda, addenda arcui exscripto.

EXEMPLUM.

Ut si detur sinus scrupulosus 61333,10, proximè minor rotundus 61300,00 dat arcum 37° 48′ 26″. Et differentiam 4′ 21″ cujus logarithmus 262400 circiter. Quatuor autem ultimæ sinus scrupulosi figuræ hic sunt 3310, prolongatus hic numerus, ut sit 33100,00 dat logarith. 110500, qui additus ad priorem, facit summam 372900 circ.

Hæc ostendit partem proportionalem 1′ 27″. Ergò arcus erit 37° 49′ 53.″

IX. PRÆ-

IX. PRÆCEPTUM.

Cuilibet arcui ſuum aſſignare logarithmum ſine magno labore.

Rurſum hic ſubſidium paratur doctrinæ triangulorum ſphæricorum, cap. VII traditæ, pro curioſis, qui dedignantur operari cum logarithmis rudibus. Præceptio erit verboſa quidem; at non valdè neceſſaria, ne metuas calculator.

Cùm enim in hâc chiliade, tam arcus, quàm logarithmi exhibeantur ſcrupuloſi; & differentiæ, inter arcus quidem interjectæ, creſcant continuò inter logarithmos verò, decreſcant; ut ita neutrobique maneat unus & idem numerus conſtans: laborem equidem haud levem ſumpſerit, qui, quotieſcunque opus fuerit logarithmo exacto, toties multiplicatione ſcrupulorum & ſecundorum exceſſus arcus, in differentiam logarithmorum, ut in numerum apicibus carentem (diverſi ſcilicet generis) & rurſum, facti diviſione in differentiam, ex ſcrupulis & ſecundis conſtantem, venari velit partem proportionalem.

Hic igitur juvent logarithmi ſeipſos, ut contribules. Cùm arcu chiliadis, qui proximè minor propoſito arcu vel angulo fuerit, exſcribe logarithmum exactum; eumque arcum ſubtrahe à propoſito: & quæſitis in ſexagenariâ, tam exceſſu propoſiti, quàm differentia inter duos arcus chiliadis, inter quos cadit propoſitus, logarithmum hujus ruditer ſaltem excerptum, aufer à logarithmo rudi illius: reſiduum junge logarithmo rudi differentiæ, vicinorum duorum logarithmorum prolongatæ ſecundum præcepta præcedentis capitis, & quæſitæ inter abſolutos: Summa ut logarithmus oſtendit, rurſum inter abſolutos, numerum qui decurtatus prodit partem proportionalem, auferendam à logarithmo exacto arcus proximè minoris, primum exſcripto, ut juſtus conſtituatur logarithmus arcus propoſiti.

EXEMPLUM.

Cupio logarithmum arcus	71°	40′	10″	
In chiliade invenio proximè minorem	71	37	21	log. 5234,65
Exceſſus dati		2′	49″	log. 305800
Differentia duorum in chiliade		10	57	log. 170000 rudes.
			Reſiduum	135800

Differentia duorum logarithmorum 105,32 prolongata ut fiat 10532,00, inter abſolutos quæſita dat logarithmum 225000 rudem.

Adde reſid. 135800

Summa 360800

Hæc ut logarithmus oſtendit inter abſolutos 2725,00. Deletis verò iterum duobus ultimis, fit pars proportionalis 27,25, quibus ablatis ab exſcripto 5234,65

5234,65, reſtat 5207,40 logarithmus, quantum proportionalitas locum habet, exactus. Nam capite VIII veriores, & plerunque exactiſſimos docuimus aſſignare ſinubus, qui per eos ſuis arcubus tranſcribi poſſunt.

X. PRÆCEPTUM.

Viciſſim logarithmo ſcrupuloſo arcum ſuum aſſignare.

Primùm cum logarithmo chiliadis, qui proximè major fuerit propoſito, exſcribe arcum ſcrupuloſum, cum differentia ſubjecta. Tunc ſubtrahe logarithmum propoſitum ab exſcripto proximè majore: Exceſſum illius, & ſimul differentiam duorum in chiliade, prolonga æquali locorum numero, ut ſint tamen minores maximo chiliadis. Sic igitur formatis iis, & inter abſolutos quæſitis, logarithmum hujus ab illius logarithmo aufer, reſiduo adde logarithmum differentiæ arcuum in ſexagenariâ quæſita; ſic accumulatur logarithmus partis proportionalis ex ſexagenariâ itidem excerpendæ, et addendæ arcui exſcripto.

EXEMPLUM.

Sit logarithmus	5207,40		
Prox. major	5234,65	dat 71° 37′ 21″ & diff. 10′ 57″	
Exceſſus	27,15		
Diff. logarithmor.	105,32		
Hi num. prolongati	2715,00	dant rudes	360800
	10532,00		225000
		Differentia	135800
		Differentiæ arcuum log.	170000
		Summa	305800 oſtendit partem proportionalem 2′ 49″
		Ergò arcus	71° 40′ 10″

Habent igitur curioſi quod agant, ſi rudibus logarithmis, minimo cum decimo, uti dedignantur. Veruntamen qui tales futuri ſunt, iis conſultum eſt, ut privatâ quilibet operâ, & in veſtibulo chiliadis columnam arcuum hîc vacantem impleat, & in fine quadrantis, decem ultimas arcuum differentias, in denas ſubdividat, ſinuum quidem differentias ſtatuens æquales, eis verò ſinubus ex canone ſinuum exacto ſuos adſcribens arcus, & logarithmos ſinubus ſuos ex doctrina capitis prioris accommodans.

Nam pars proportionalis, uſitato modo quæſita, locum non habet in fine quadrantis: ut capite ſecundo monui.

XI. PRÆ-

XI. PRÆCEPTUM.

Per folos logarithmos, abfolutorum, & arcuum feu angulorum, omnia triangula plana folvere: omnia fcilicet quæfita ex tribus datis eruere.

Ad hoc caput propriè pertinet hæc doctrina, quia jungi debet columna abfolutorum (non ut ii funt finus arcuum) & columna ipforum arcuum. Sunt autem cafus fex.

I. Si datis angulis & uno latere, quæritur latus.

Tunc prolongato vel decurtato latere, ut capite VIII doctus es, adde logarithmos lateris dati & anguli quæfito lateri oppofiti; à fummâ aufer logarithmum anguli, dato lateri oppofiti, reftat logarithmus lateris quæfiti, fimiliter prolongati.

Si fubtractio fieri non poteft, viciffim fummam fubtrahe, reftabitque privativus logarithmus, lateris excedentis maximum chiliadis.

EXEMPLUM.

Dentur anguli 50° 0′ & 60° 0′, datur igitur etiam refiduus, ut complementum illorum duorum ad femicirculum fcilicet 70° 0′, detur & latus oppofitum medio angulo 573, quæritur latus oppofitum maximo.

Lateris 57300,00 (prolongati)	log.	55687
Anguli 70° 0′	log.	6220 circ.
	Summa	61907
Anguli 60° 0′	Log.	14380 circ.
	Refiduum	47527
	dat	62175,00
	Latus ergò eft	622 ferè.

Quod fi rectus fuerit angulorum unus, & detur ei fubtenfum latus, fufficiet fola additio.

Sin autem quæratur latus recto fubtenfum, fufficiet fola fubtractio log. anguli à log. lateris oppofiti. Reftabit enim log. lateris recto fubtenfi.

II. Si datis duobus lateribus, & angulo uni eorum oppofito, quæruntur anguli reliqui.

Tunc lateribus æqualiter prolongatis vel decurtatis, ut fint proximè minora maximo

maximo chiliadis, adde logarithmos anguli & lateris unius adjacentis; à ſummâ aufer logarithmum lateris dato angulo oppoſiti, reſtat logarithmus anguli lateri alteri oppoſiti, qui angulus interdum tàm poteſt eſſe minor quadrante, quàm major: & ſic complementum ad ſemicirculum ejus, quem logarithmus excerpit: quantiſper non plura dantur. Tertius verò angulus eſt duorum junctorum complementum ad ſemicirculum.

EXEMPLUM.

Ut ſi dentur latera 622,507. Et huic oppoſitus 50° 0′ quærantur anguli reliqui.

Anguli	50° 0′	log.	26650
Adjac. lateris	62200,00	log.	47527
Summa			74177
Oppoſiti lateris	50700,00	log.	67957

Reſiduum 6220 dat ang. 70° 0′ alteri lateri oppoſitum.

Si ergò duo ſunt 50° 0′ & 70° 0′: tertius erit eorum ſummæ complementum ad duos rectos ſcilicet 60° 0′

Si verò angulus prodiens ſumeretur non 70° 0′ ſed complementum ad ſemicirculum 110 0

Tertius fiet 20 0

Si datus angulus fuerit rectus; ſufficiet ſola ſubtractio logarithmi lateris recto oppoſiti, à log. lateris reliqui dati: reſtabit enim logarithmus anguli ei oppoſiti.

Si datus dato angulo oppoſitum logarithmum habuerit æqualem ſummæ priorum: rectangulum erit triangulum.

III. Si datis duobus lateribus, & angulo uni eorum oppoſito, quæratur latus tertium.

Tunc primùm, ut in ſecundo caſu, quæratur angulus reliquo lateri dato oppoſitus; additis logarithmis, anguli dati, & lateris ei adjacentis; à ſummâ verò dempto logarithmo lateris dato angulo ſubtenſi: cum reſiduo, ut logarithmo, exſcribatur angulus & jam ſumma duorum angulorum à duobus rectis ablatâ, conſtituitur angulus tertius, cujus logarithmus, ut in primo caſu addendus eſt logarithmo unius ex lateribus, à ſummâ auferendus logarithmus anguli oppoſiti; reſtabitque logarithmus lateris tertii.

Ut

Ut in exemplo priori, ſumpto angulo ſecundo 70° 0′ & invento angulo tertio 60° 0′. Ejus logarithmo 14380

tertio 60° 0′. Ejus logarithmo	14380		14380
Addatur lat. 622 log.	47527	vel lateris 507 log.	67957
A ſummâ	61907	vel	82337
Aufer anguli 70° log.	6220	vel ang. 50° log.	26650
Reſtat	55687		55687

Logarithmus lateris tertii quæſiti 57300,00, ſed decurtati, ut priora, 573.

At ſi angulus in priori præcepto prodiens ſumatur non 70° 0′ ſed 110° 0′, & tertius ideò ſit 20° 0′, hujus igitur logarithmus 107290, additis logarithmis datorum laterum, & ablatis logarithmis angulorum oppoſitorum, dat logarithmum 148600 lateris tertii, angulo ſcilicet 20° 0′ oppoſiti, ſcilicet 22630,00, decurtati igitur $226\frac{3}{10}$.

Si angulus datus fuerit rectus, aufertur logarithmus lateris recto oppoſiti, à logarithmo lateris reliqui, reſtat logarithmus anguli: hujus complementi logarithmus, viciſſim additus logarithmo primo, lateris recto oppoſiti, dat logarithmum lateris reliqui.

IV. Si datis duobus lateribus, et angulo comprehenſo, quærantur anguli reliqui.

Hic via directa ſolvendi hunc caſum utitur meſologarithmis, qui non aliter elici poſſunt ex chiliade, quàm ſi logarithmos arcus ejuſque complementi ab invicem ſubtrahamus: ut ſic datâ eorum differentiâ, ipſi logarithmi partium quadrantis non directè poſſint accipi, ſed poſitione ſit utendum.

Dimiſſo igitur hoc modo; ne tamen manca hîc ſit chilias noſtra: utamur poſitionibus, potius in aliâ viâ facili & indirectâ: et ſi & hæc aptior ſit logarithmus canonicis quadrantis.

Eſt autem talis:

Datis lateribus quæratur eorum proportio: ſit enim facillimè per capitis VIII præceptum XIII.

Hæc verò proportio, cùm ſit differentia inter logarithmos angulorum reſiduorum, quorum angulorum ſumma eſt in anguli dati complemento ad ſemicirculum: Ponatur igitur angulus minor eſſe notus, auferaturque à dati complemento ad ſemicirculum, reſidui logarithmo adde proportionem laterum inventam; ſumma ut logarithmus, ſi dat arcum eundem, quem poſueras, felix fuit poſitio; ſin diſcrepat, muta poſitionem primam, ſumens vel aliquid intermedium, ſi reſiduum ex complemento ad ſemicirculum fuerit quadrante minus: vel aliquid longius à primâ poſitione recedens, quàm quod prodierat; ſi reſiduum ex complemento ad ſemicirculum fuerit majus quadrante; præſertim ſi valdè magnum.

A tali nova poſitione incipiat nova operatio: Id fiat tantiſper, donec prodeat id, quod ultimò fuit poſitum. Ita habebis utrumque angulum ex quæſitis.

Conducit autem primas poſitiones ſic moderari, ut remaneat de complemento ad ſemicirculum aliquid quod exactè reperiatur in chiliade.

EXEMPLA.

Dentur latera 507 & $226\frac{3}{10}$, comprehenſus eſto 110° ut ſit ejus complementum ad ſemicirculum 70°. Quæruntur duo anguli reſidui; quos junctos ſcio facere 70°.

Primùm quæro proportionem laterum.

50700			67957
22630			148600
		Proportio	80643
Compl. ang. ad ſemic.	70° 0′		
Poſitio arbitraria	10° 0′ 10″		
Reſiduum	59 59 50	log.	14387
Prodit	22 45 circ.		95030
Poſitio ſecunda	17° 3′ 37″		
Reſiduum	52 56 23		22565
Prodit	30 52 circ.		103208
Poſitio tertia	19° 0′ 47″		
Reſiduum	50 59 13		25231
Prodit	20 17 circ.		105874

Hoc modo cum medio arithmetico inter poſitum & prodeuntem ſemper poſſemus propius ad conſenſum venire. At quia ſtatim in ſecunda poſitione apparuit, veritatem eſſe ultra medium arithmeticum: ſumemus etiam in quarta poſitione

pofitione aliquid, quod propius fit prodeunti 20° 17′ quam pofito 19° 1′, tentabimus fcilicet 20° 0′

Pofitio quarta	20° 0′	proportio	80643
Refiduum	50 0	logarith.	26650
Prodit	20 0		107293

Hæc ergò tandem felix fuit pofitio, & anguli quæfiti funt 20° 0′ & 50° 0.

EXEMPLUM ALIUD.

Dentur latera 507, 573 & comprehenfus 70° ut refiduum ad femicirculum fit 110° 0′

Quæro proportionem laterum.			
50700			67957
57300			55687
Compl. ang. ad femic.	110°	proportio	12270
Pofitio arbitraria	29° 56′ 14″		
Refiduum	80 3 46	log.	1511
Prodit	60 36	circiter	13781
Pofitio fecunda	45° 58′ 22″		
Refiduum	64 1 38		10647
Prodit	52 40	circ.	22917
Pofitio tertia	49° 4′ 13″		
Refiduum	60 55 34		13467
Prodit	50 38		25737
Apparet ponendum	50° 0′		
Refiduum	60 0		14380
Prodit	50 0		26650

Ergò anguli quæfiti funt 50° 0′, & 60° 0′.

EXEMPLUM TERTIUM.

Dentur latera, ut prius 507, 573, ſed comprehenſus ſit 20° complementum ad ſemicirculum 160°, manente igitur laterum proportione.

Compl. ad ſemic.	160° 0′		12270
Sit poſitio arbitrar.	70°	logarith.	
Reſiduum	90	logarith.	0
Prodit	62° 8′ circiter		12270
Poſitio ſecunda	54° 47′ 45″		
Reſiduum	105 12 15		
	74 47 45	log.	3563
Prodit	58 35		15833
Poſitio tertia	60° 0′		
Reſiduum	100° 0′		
	80 0	logarith.	1530 circ.
Prodit	60° 35′ circ.		13800
Poſitio quarta	61 0		
Reſiduum	99 0		
	81 0		1240
Prodit	60 52		13510
Ergò poſitio felix erit	60° 47′ circiter.		

Hic cum in primâ poſitione 70°. Reſiduum fieret 90° prodiit aliquid minus, ipsâ poſitione, ſcilicet 62° 8′. Reſiduum igitur apparuit futurum majus quadrante. In hoc ergò caſu poſitionem ſecundam longius à primâ diſtare feci, quàm id quod per primam prodiit.

Sic in poſitione ſecundâ 54° 47′ prodiit 58° 35′.

Ergò

Ergò positio tertia 60° 0′ longius distare jussa est, à positione præcedenti 54° 47′ quàm id quod prodierat 58° 35′, minus tamen quàm quod primò prodiit, scilicet 62° 8′. Et in positione tertia 60° 0′ prodiit major 60, 35, feci ergò quartam adhuc majorem, scilicet 61° 0′. Per hanc autem quartam 61° 0′ prodiit jam minus 60° 52′.

Ergò quinta positio adhuc minor est facta, major tamen quàm 60° 35′, quod prius prodierat, scilicet 60° 47′. Est igitur angulus minor 60° 47′, major 99° 13′ ferè.

Si datus fuerit rectus: reliqui duo ex canone Neperiano facilè habentur, quæsitâ proportione laterum inter mesologarithmos, inventa enim ea ostendit utrumque angulum.

V. Si datis duobus lateribus & angulo comprehenso quæratur latus tertium.

Nulla est alia machina, quàm ut prius quærantur anguli, per præcedens. Tunc iis inventis, cadit quæstio in casum primum.

Si tamen angulus datus sit rectus, logarithmos laterum duplica, duplicatorum logarithmorum numeros absolutos adde, summæ logarithmum dimidia; semissis iste, ut logarithmus, dabit inter absolutos quæsitum latus.

VI. Si datis tribus lateribus quæratur angulus.

Tunc ante omnia ex majori angulo resecandum est triangulum æquicrurum, cujus crura sint linea sectionis & latus minus; basis verò segmentum lateris majoris. Id verò obtinetur in hunc modum.

Adde logarithmos & differentiæ, & summæ laterum minorum, à logarithmorum summâ aufer logarithmum majoris lateris, residuum ut logarithmus ostendet inter absolutos, partem de latere majori subtrahendam, ut restet basis æquicruri supradicti.

Constitutâ hac basi, maximus angulorum habetur per sua elementa, si exscripseris logarithmos tam semissis de basi, quàm residui de latere majori, post ablatum basis semissem; & ab illo quidem minoris, ab isto verò medii lateris logarithmos abstuleris, relinquuntur enim logarithmi duarum partium anguli maximi. Duo verò minores anguli sunt horum majoris elementorum complementa.

LEXEM-

EXEMPLUM.

Sint latera	507, 573, 622, quæruntur anguli.			
Latus minus	507			
Medium	573			
Differentia	66	ſcribo 66000	log.	41552
Summa	1080	ſcribo 10800	log.	222562
				264114
Lat. maxim.	622	ſcribo 62200	log.	47482
Prodit	11460	Reſiduum		216632

Ad primum tres cyphræ acceſſerunt, ad ſecundum una, quæ ſunt à prodeunte removendæ: viciſſim ad tertium duæ appoſitæ, ſunt etiam ad quotientem apponendæ: duabus igitur appoſitis, & quatuor remotis, formatur pars lateris majoris $114\frac{6}{10}$.

Pars lateris majoris	114,60			
Latus majus	622,00			
Baſis æquicruri	507,40			
Semiſſis	253,70	log.	137160	
Reſiduum ablato baſis ſemiſſe	368,30	log.		99886
Latus minus	507,00	log.	67924	
Latus medium	573,00	log.		55687

Reſidui manent duo logarithmi	69236, 44199
Duorum elementorum anguli	30° 1′ 40° 0′
Ergo totus angulus iſte eſt	70° 1′

Duo verò minores ſunt complementa elementorum, ſcilicet 59° 59′ & 50° 0′.

Hæc igitur de uſu chiliadis noſtræ hâc vice ſufficiant.

F I N I S.

LOGARITHMO-TECHNIA:

SIVE

METHODUS CONSTRUENDI

LOGARITHMOS

NOVA, ACCURATA, ET FACILIS;

SCRIPTO

ANTEHAC COMMUNICATA, ANNO SC. 1667, NONIS AUGUSTI.

CUI NUNC ACCEDIT

VERA QUADRATURA HYPERBOLÆ, ET INVENTIO SUMMÆ LOGARITHMORUM.

AUCTORE NICOLAO MERCATORE, HOLSATO,
E SOCIETATE REGIA.

HUIC ETIAM JUNGITUR

MICHAELIS ANGELI RICCII EXERCITATIO GEOMETRICA DE MAXIMIS ET MINIMIS:
Hîc ob Argumenti Præstantiam & Exemplarium Raritatem recusa.

Prima hujus Tractatûs editio impressa fuit Londini, Anno M,DC,LXVIII.

LOGARITHMO-TECHNIA.

LOGARITHMUS compoſito vocabulo dicitur à ratione & numero, quaſi rationum numerus, id quod planè cum re conſentit. Eſt enim logarithmus nihil aliud, quàm numerus ratiuncularum contentarum in ratione, quam abſolutus quiſque ad unitatem obtinet. In qua definitione rationes accipimus, tanquam magnitudines partibus conſtantes homogeneis toti; ſtrictiori aliquantò notione, quàm vulgò ſolet. Quamvis enim ratum ſit, rationem omnem ex comparatione quantitatum homogenearum oriri: certè nec quævis comparatio producit rationem; nec quarumvis quantitatum, homogenearum licèt, habitudo eſt ratio quanta, ſeu partibus conſtans. Nam æqualitatis, quæ dicitur, ratio, eſt illa quidem quantitatum homogenearum, atque æqualium, habitudo mutua, unde nec rationis appellatione privandam autumem, cui definitio Euclidea competit non minus ac aliis rationibus, quas inæqualitatis vocant: ſed nihil obſtat, quò minùs generalem iſtam rationis notionem porro dividamus ita, ut quantitatem habere putentur ſolæ rationes, ex inæqualium habitudine ortæ; at æqualitatis ratio in indiviſibili conſiſtat, habeatque ſe in rationibus, quemadmodum punctum in magnitudinibus, aut nullitas in numeris, quæ ſingula quantitate ac partibus carent. Componas enim ſexcentas rationes æqualitatis, non augetur nec minuitur ratio, ſed eadem manet æqualitas: ſecus atque in rationibus inæqualitatis, quæ additæ vel detractæ invicem, faciunt rationem majorem vel minorem. Quantum autem eſt, quod additis vel demtis partibus homogeneis augetur minuiturve. Sed nec quævis comparatio quantitatum homogenearum rationem producit. Veluti cum numerum dividimus per numerum; comparantur utique quantitates homogeneæ, ſpectando quoties altera contineatur in altera; ſed quod inde oritur latus, nec ratio eſt ipſorum numerorum, nec ſanè quantitatem exprimit rationis, quæ utriſque intercedit. Alioquin diviſo numero quovis per æqualem, quæ inde oritur unitas, exprimeret quantitatem rationis æqualium, quam tamen quantitate carere ſuprà adſtruximus. Quinimò, datis pluribus rationibus, v. gr. 4 ad 2, & 9 ad 3, ſi diviſo utriuſque antecedente per ſuum conſequentem, exhiberent orti 2 & 3 veram quantitatem iſtarum rationum; oporteret, ut ex his ortis compoſitus numerus, nimirum 5, exhiberet quoque veram quantitatem rationis compoſitæ. Atqui ratio compoſita eſt 36 ad 6, cujus quantitatem jam exprimeret ortus 6, diverſus ſanè ab iſto 5. Obtinuit tamen uſus, ut rationes denominentur à latere orto; ſic ratio 4 ad 2 dicitur dupla, & 9 ad 3 tripla: verùm hæc nomina arbitrio hominum impoſita, retineri quidem poſſunt, veritati

autem derogare nullo modo debent. Quanquam nec utilitate caret iſte modus denominandi rationes; ſiquidem arguit, rationes eſſe majores, quarum denominator eſt major, & contrà: eodem modo ſinus majores congruunt majoribus arcubus, quorum tamen veram quantitatem exprimere nemini videntur. Cæterùm, ut linea eſt dupla lineæ, quam bis continet; ita, propriè loquendo, dupla foret ratio alterius rationis, quam bis continet; ſed pro eo duplicatam dicere maluerunt ſcriptores, quorum arbitrio ſynonyma alioquin vocabula dupli & duplicati, res planè diverſas ſignificare intelliguntur. Verùm id quod eſt multò maximum, nimirum omnium quantitatum menſuram eſſe quantitatem homogeneam, & in diviſione genuina fieri applicationem menſuræ homogeneæ ad quantitatem menſurandam, ortum vero ex tali applicatione, nihil aliud eſſe, quàm numerum arithmeticum, exprimentem quoties menſura continetur in menſurato; hoc ſcilicet eſt, quod omnem dubitationem excludit. Ita falluntur profectò, qui, applicatâ lineâ rectâ illatabili ad aream datam, putant inveniri latitudinem rectanguli: quaſi non potiùs ſecundum veras diviſionis leges applicaretur rectangulum æquè longum diviſori, & æquè latum unitati aſſumtæ, ad aream extenſam quoque ad longitudinem diviſoris; & quaſi non ortus ex iſta applicatione, numerus eſſet arithmeticus, exprimens quoties rectangulum menſurans contineatur in menſurato, vel (cùm per I, VI eadem ſit ratio) quoties latitudo rectanguli applicati contineatur in latitudine rectanguli menſurati; quò ſcilicet diviſio verè opponatur multiplicationi, quæ reſolvat hujus productum in ſua elementa. Quemadmodum enim omnis multiplicator eſt numerus arithmeticus (ut habet Stevinus in Arithmetica Practica), ita omnis ortus à diviſione eſt ſimiliter numerus arithmeticus. Quæ quidem omnia facillimè præſenti negotio aptantur. Nam multiplicare rationem nihil eſt aliud, quàm replicare aliquam rationem toties, quot ſunt unitates in numero aliquo arithmetico, qui dicitur factor. Et dividere rationem, eſt applicare rationem aliquam ad aliam rationem, ut inveniatur numerus arithmeticus, exprimens quoties menſurans ratio contineatur in menſurata. Id ſi hìc fieret, nihil dubium, quin vera patefieret rationis quantitas. At enimverò, cùm applicatur terminus alicujus rationis ad alterum; num putamus rationem applicari ad rationem? quo pacto igitur ortus ex tali applicatione poteſt exprimere quantitatem datæ rationis? Verum eſt quidem, quòd ortus ex applicatione termini ad terminum, rationem habet ad unitatem eandem, quam dividuus ad diviſorem; ſed hoc modo eadem prodit ratio, quæ ante diviſionem fuerat, aliis tantum terminis expreſſa; nec proinde quantitas rationis datæ invenitur in menſura aliqua priùs nota, quemadmodum in aliis magnitudinibus diviſis aſſolet, & inſtituti noſtri ratio poſtulat: Siquidem tum demum quantitatem rationum exactè determinâſſe videbimur, cùm eas omnes in una aliqua ac eâdem communi menſurâ æſtimare noverimus; id quod logarithmorum ope præſtari, definitione modò traditâ innuere volui. Ex qua porro intelligitur, cùm ſinguli logarithmi numerent particulas rationum inde ab unitate ad ſingulos ordine abſolutos procedendo coacervatarum; fieri non poſſe, quin æqualibus logarithmorum differentiis (id eſt, æqualibus particularum incrementis) congruant quoque æquales rationes abſolutis intercedentes (cùm integra ex æquali numero particularum æqualium conflata, inter ſe ſint æqualia); adeóque logarithmos eſſe in proportione arithmetica, cùm eorum

abſoluti ſunt in geometrica; idcircò poſſe operationem regulæ proportionum in compendium redigi, ſubſtitutâ additione & ſubtractione loco multiplicationis & diviſionis: denique rationis cujuſque bipartitionem, tripartitionem, &c, quæ alioquin requireret extractionem radicis quadratæ, cubicæ, &c, conſiſtere in bipartitione, tripartitione, &c, differentiæ logarithmorum datis terminis congruentium (hoc eſt, ratiuncularum in ratione data comprehenſarum). Qui uſus cùm ſit eximius, patet poſtremò, quo pacto tam utiles numeri artificiales, ſeu logarithmi concinnari poſſint; nimirum inveſtigando, quot ratiunculæ, aſſumtæ magnitudinis, contineantur in ratione cujuſque abſoluti ad unitatem. Sic enim unitatis logarithmus evadit 0, cùm unitatis ad unitatem ratio ſit æqualitatis, quam quantitate carere ſuprà aſſerui. Ita nimirum fiet, ut cùm inter multiplicandum vel dividendum unitas nihil mutet; hujus logarithmus 0 (dum additio & ſubtractio ſubſtituuntur multiplicationi & diviſioni) nihil quoque additione vel detractione ſui mutet. Numerus autem ratiuncularum in ratione decupla contentarum commodiſſimè aſſumitur 1,0000000 (hoc eſt, una decupla ratio in numerum partium decimalium rotundum diſtributa). Ita enim fiet, dum inter unitatem & 10 intercedit una ratio decupla, & porrò inter 10 & 100 altera, inter 100 & 1000 tertia, & deinceps; ut in centupla quidem ratione contineantur ratiunculæ 2,0000000 (hoc eſt, duæ decuplæ rationes in numerum partium rotundum diſtributæ), in millecupla verò ratione contineantur 3,0000000 (hoc eſt, tres rationes decuplæ in numerum partium rotundum diſtributæ); & deinceps. Unde primùm hoc commodi conſequimur, ut abſolutis, iiſdem characteribus expreſſis, iidem competant logarithmi; veluti ſi abſoluto 2 competat logarithmus 0,3010299; etiam abſolutis 20, 200, 2000 competent logarithmi 1,3010299; 2,3010299; 3,3010299. Etenim ſi rationes 2 ad 1, 20 ad 1, 200 ad 1, & deinceps, intelligantur partitæ, illa quidem in rationes 2 ad 1 & 1 ad 1; iſta in 20 ad 10, & 10 ad 1; hæc in 200 ad 100, & 100 ad 1; apparet quòd exceſſus 2 ad 1, quo illa ſuperat rationem 1 ad 1 æqualis ſit exceſſui 20 ad 10, quo iſta ſuperat rationem 10 ad 1, idemque æqualis exceſſui 200 ad 100, quo hæc ſuperat rationem 100 ad 1. Ergo ſi ratio 2 ad 1 præter æqualitatis rationem (quam innuit characteriſtica 0) contineat ratiunculas 3010299, qualium ipſa decupla continet 1,0000000; certè ratio 20 ad 1, præter unam decuplam (quam innuit characteriſtica 1) continebit eundem numerum ratiuncularum excurrentium 3010299; & ratio 200 ad 1, præter duas decuplas (quas innuit characteriſtica 2) continebit ſimiliter eundem numerum ratiuncularum excurrentium 3010299. Unde porro & hoc conſequimur, ut ex inſpecta characteriſtica, uniuſcujuſque abſoluti valorem æſtimare queamus. Nam cùm in decupla contineantur 1,0000000 ratiunculæ numero rotundo, & in centupla 2,0000000 ratiunculæ numero itidem rotundo; oportet, ut quæcunque ſunt inter decuplam & centuplam contineant plus quàm unam decuplam, minùs autem quam duas; quamobrem characteriſtica omnium abſolutorum, qui ſunt inter 10 & 100, ipſiuſque adeò denarii (hoc eſt, omnium numerorum, qui ſcribuntur duobus characteribus) erit 1; & ſic deinceps.

His ita ordinatis, proximum eſt, ut oſtendamus, quomodo inveniatur menſura rationis, quam quiſque abſolutus obtinet ad unitatem, in partibus, qualium decupla continet 1,0000000 (hoc eſt, quomodo cujuſque abſoluti logarithmus

 inveſti-

investigandus sit.) Verbi gratiâ: Scire velim, ratio 100⌊5 ad 1 quot contineat ratiunculas, qualium decupla continet 1,0000000. Dispesco igitur rationem datam 100⌊5 ad 1 in suas partes, nimirum 100⌊5 ad 100, 100 ad 10, & 10 ad 1, quarum posteriores duæ constituunt duas decuplas (unde patet characteristicam fore 2); itaque restat, ut investigemus, quota pars sit reliqua ista ratio 100⌊5 ad 100 ipsius decuplæ. Quod si igitur termini 100⌊5 & 100 ducantur uterque in sese, producti exhibebunt rationem duplicatam rationis 100⌊5 ad 100, cujus (duplicatæ sc. rationis) termini rursus in se ducti procreabunt duplicatam duplicatæ, id est, quadruplicatam rationis 100⌊5 ad 100: atque ita continuatâ multiplicatione terminorum, donec is, qui gignitur ex ductu continuo termini 100⌊5 in seipsum, evadat decuplus ejus, quem ductus continuus termini 100 in seipsum producit; denominator potestatis postremò genitæ ostendet, quot integris vicibus ratio 100⌊5 ad 100 contineatur in decupla. Et cùm alter terminorum sit 100, cujus potestates omnes constant unitate & certo numero cyphrarum; omnis labor reliquus occupabitur circa elevandum alterum terminum 100⌊5 ad eam potestatem, quæ prioris termini (nimirum 100^{rii}) æquè altam potestatem excedat decuplo; cujus operationis compendium exemplo, quàm verbis docere præstat.

100⌊5000 (1)
5001 (1)
———
1005000
5025
———
1010025 (2)
5200101 (2)
———
1010025
10100
20
5
———
1020150 (4)
0510201 (4)
———
1020150
20403
102
51
———
1040706 (8)
6070401 (8)
———
1083068 (16)
8603801 (16)
———
1173035 (32)
5303711 (32)
———
1376011 (64)
1106731 (64)
———
1893406 (128)

1893406 (128)
6043981 (128)
———
3584985 (256)
5894853 (256)
———
12852116 (512)

Hæc potestas plusquam decuplo excedit potestatem æquè altam 100^{rii}; ergo resumo 256^{tam}, eamque duco, non in sese, ut modò, sed in proximè præcedentem, nimirum 128^{vam}, hoc modo:

3584985 (256)
6043981 (128)
———
6787831 (384)
1106731 (64)
———
9340130 (448)
5303711 (32)
———
10956299 (480)

Hæc potestas denuò excedit æquè altam 100^{rii} plusquam decuplo; ergo eandem 448^{vam} duco, non in 32^{dam}, ut modò, sed in proximè præcedentem, hoc modo:

9340130 (448)
8603801 (16)
———
10115994 (464)

Ubi rursus nimium colligitur; ergo eandem adhuc 448^{vam} duco, non in 16^{tam}, ut modò, sed in proximè præcedentem, nimirum 8^{vam}, hoc modo:

9340130 (448)
6070401 (8)
———
9720329 (456)
0510201 (4)
———
9916193 (460)
5200101 (2)
———
10015603 (462)

Quæ potestas rursus excedit limitem; quare eandem 460^{mam} duco, non in 2^{dam}, sed in 1^{mam}, hoc modo:

9916193 (460)
5001 (1)
———
9965774 (461)

Cùm

Cùm igitur 462^{da} poteſtas termini 100[5 excedat æquè altam 100^{rii} plus quam decuplo; at 461^{ma} ejuſdem termini 100[5 excedat æquè altam 100^{rii} minùs quàm decuplo: aio, rationem 100[5 ad 100 contineri in decupla plus quàm 461 vicibus, minùs autem quàm 462^{bus}.

Cæterùm

$$\text{Cùm poteſtas} \left\{ \begin{array}{c} 460 \\ 461 \\ 462 \end{array} \right\} \text{ſit} \left\{ \begin{array}{r} 9916193 \\ 9965774 \\ 10015603 \end{array} \right\} \begin{array}{l} \text{\& differentiæ} \\ 49581 \\ 49829 \end{array} \left\{ \begin{array}{l} \text{propemodum} \\ \text{æquales;} \end{array} \right.$$

Itaque partem proportionalem, quâ poteſtas juſta, nimirum 10000000, excedit proximè minorem 9965774, per regulam auream facilè ac tutò reperire datur, ſumendo nimirum,

juſtæ	10000000
& proximè minoris	9965774
differentiam	34226, & dicendo;

Ut differentia inter proximè minorem & majorem
(nimirum 49829.)
Ad differentiam inter proximè minorem & juſtam
(putà 34226;)

Ita 10000, ad 6868; quæ ſunt partes decimales unius vicis, adeò ut ratio 100[5 ad 100 contineatur in decupla 461686[8 vicibus. Porrò, ſi decupla (ſive ratio 100[5 ad 100 ſumta 461[6868 vicibus) continet ratiunculas 1,0000000; quot ejuſmodi ratiunculas continebit ratio 100[5 ad 100 ſemel ſumta? Prodeunt 21659[7 ratiunculæ, quæ ſunt exacta menſura rationis 100[5 ad 100, quibus ſi addas rationes 100 ad 10, & 10 ad 1, hoc eſt bis decuplam, conſtantem ratiunculis 2,0000000; fit integra menſura rationis 100[5 ad 1 (ſive logarithmus abſoluti 100[5), hic ſcilicet 2,0021659[7.

Pari modo invenietur logarithmus abſoluti 99[5, vel ratio abſoluti 99[5 ad 1, ſi ex ratione 100 ad 1 (quæ æquipollet bis decuplæ), auferas rationem 100 ad 99[5, hoc eſt, ex ratiunculis 2,0000000 auferas numerum ſimilium ratiuncularum in ratione 100 ad 99[5 contentarum. Quæratur igitur primùm, quoties ratio 100 ad 99[5 contineatur in decupla. Ubi rurſus alter terminorum cùm ſit 100, operationis haut indiget; alter verò 99[5 continuo ductu in ſeipſum elevandus eſt ad eam poteſtatem, quæ decuplo minor ſit poteſtate 100^{rii} æquè altâ. En operationem:

995000 (1)
599 (1)

8955000
895500
49750

9900250 (2)
520099 (2)

8910225
891023
198
49

9801495 (4)
5941089 (4)

8821345
784120
980
392
88
5

9606930 (8)
396069 (8)

9229310 (16)
139229 (16)

8518016 (32)

8518016 (32)
6108158 (32)

7255660 (64)
665527 (64)

5264459 (128)
9544625 (128)

2771452 (256)
2541772 (256)

554290 (512)

Hæc poteſtas pluſquam decuplo minor eſt poteſtate 100^{rii} æquè altâ; ergo reſume

2771452 (256)
9544625 (128)

1459018 (384)
665527 (64)

1058613 (448)
6108158 (32)

901728 (480)

Quæ poteſtas rurſus pluſquam decuplo minor eſt poteſtate 100^{rii} æquè altâ; ergo reſume

1058613 (448)
139229 (16)

977026 (464)

Quæ etiam pluſquam decuplo minor eſt poteſtate 100^{rii} æquè altâ; ergo reſume

1058613 (448)
396069 (8)

1017002 (456)
5941089 (4)

996814 (460)

Hæc quoque pluſquam decuplo minor eſt poteſtate 100^{rii} æquè altâ; ergo reſume

1017002 (456)
520099 (2)

1006857 (458)
599 (1)

1001823 (459)

Cùm igitur 460^{ma} poteſtas termini 99⌊5 deficiat ab æquè altâ 100^{rii} pluſquam decuplo; at 459^{na} ejuſdem termini 99⌊5 deficiat ab æquè altâ 100^{rii} minùs quàm decuplo; aio, rationem 100 ad 99⌊5 contineri in decupla pluſquam 459 vicibus, minùs autem quàm 460.

Tum, ut differentia poteſtatum 459^{nae} & 460^{mae} (nimirum 5009) ad differentiam 459^{nae} & juſtæ (putà 1823): Ita 10000, ad 3639. Quare ratio 100 ad 99⌊5 continetur in decupla 459⌊3639 vicibus.

Porrò, ſi decupla (ſive ratio 100 ad 99⌊5 ſumta 459⌊3639 vicibus) continet ratiunculas 1,0000000; quot ejuſmodi ratiunculas continebit ratio 100 ad 99⌊5 ſemel ſumta. Prodeunt 21769⌊3 ratiunculæ, quæ ſunt exacta menſura rationis 100 ad 99⌊5, quâ ſcilicet ratio 99⌊5 ad 1 deficit à ratione 100 ad 1, hoc eſt, à bis decuplâ, quæ cùm conſtet ratiunculis 2,0000000, demtis hinc

hinc 21769⌊3, reſtat menſura rationis 99⌊5 ad 1, hæc ſcilicet 1,9978230⌊7, qui proinde eſt logarithmus abſoluti 99⌊5.

Atque hoc modo æſtimatis rationum quantitatibus in communi quadam menſura, non ſolum natura & uſus logarithmorum clarius eluceſcit; ſed et conſtructio eorundem multò facilior evadit. Id quod magis perſpicuum fiet, cùm oſtendero alterum etiam longè promptiorem modum rationes æſtimandi. Sed amolienda eſt priùs difficultas, quæ, haut ſcio an cuiquam detecta, plures utique in errorem induxit. Cum enim ratio duobus terminis intercedens vulgò haut aliter conſideretur, quàm accipiendo alterutrum terminum ut antecedentem, & alterum ut conſequentem; unde cùm ratio eſt quanta (hoc eſt, cùm termini ſunt inæquales) vel major terminus eſt antecedens, & dicitur ratio majoris inæqualitatis, vel minor eſt antecedens, & dicitur ratio minoris inæqualitatis: Aio ego, eandem rationem iiſdem terminis conceptam poſſe ac debere (ſaltem in Muſicis, atque in hac noſtra logarithmo-technia) alio etiamnum modo conſiderari ita, ut neuter terminorum exiſtimetur tanquam antecedens, vel conſequens, ſed uterque capiatur ſimul pariter tempore atque ordine. Sic v. gr. in Muſicis intervallum diapente, ſive ratio $\frac{3}{2}$ vel $\frac{2}{3}$, poteſt quidem accipi ita, ut numerus undationum ab acutiori phthongo in aëre excitatarum, nempe ternarius, ſit antecedens, & binarius, exhibens numerum undationum pari temporis ſpatio â graviori phthongo effectarum, ſit conſequens, dum intelligatur acutior phthongus tempore (vel ſaltem cogitatione) præcedere graviorem; & vice verſâ: ſed nihil vetat, quò minus etiam ambo iſti phthongi ſimul atque eodem tempore conſonent, adeóque neuter altero ſit vel tempore vel naturâ prior. Cæterum nihilo majus ob hoc vel minus evadit intervallum diapente (ratione ſeſquialterâ conſtans) ſive acutior phthongus præcedat graviorem, ſive contra, ſeu denique ambo ſimul conſonent. Ita, licèt utilis ſit demonſtrationibus geometricis conſideratio vulgaris, quâ minor terminus antecedens ad majorem conſequentem dicitur minorem rationem habere, quàm idem ille major tanquam antecedens ad eundem minorem tanquam conſequentem obtineat: Negari tamen haut poteſt, eundem numerum ratiuncularum contineri in ſeſquialtera ratione, ſive ternarius ſit antecedens, ſive binarius, ſive neuter; adeóque conſiderationem antecedentis & conſequentis in æſtimanda mole vel menſura rationum nullum inſtar habere. Non ſecùs ac quinarius negatus (-5) mole haud differt à quinario affirmato ($+5$), cùm uterque conſtet quinque unitatibus; diſſimulatâ nimirum affectione, propter quam negatus vel affirmatus cenſetur, & ſolâ mole vel quantitate ſimpliciter æſtimatâ: cùm tamen accipiendo quinarium negatum, prout ſigno negationis affectus eſt, verum ſit, eum minorem eſſe, non modò quovis numero affirmato, ſed & omni negato, qui à nihilo minùs differat, quàm ipſe, quales ſunt binarius vel ternarius negatus (-2, vel -3.) Ubi præter molem numeri conſideratur quoque, utram in partem eadem abeat à nihilo, affirmatam an negatam. Ita quoque ſive uniſonum (vel æqualitatis rationem, quæ quantitate caret, atque ideò rectè componitur nihilo) ponas in phthongo graviori (vel in binario), indéque aſcendas ad phthongum intervallo diapente acutiorem (vel ad ternarium); ſive contrà ponas uniſonum in acutiori (vel æqualitatis rationem in ternario), indéque deſcendas ad phthongum intervallo diapente graviorem (vel ad binarium): certè eadem eſt utrobique quantitas intervalli Muſici

(atque

(atque idem numerus ratiuncularum intercedentium), licèt ab uniſono (vel ab æqualitatis ratione, tanquam nihilo) in diverſas planè partes abeat. Unde ſi moles ſola, aut quantitas rationis æſtimetur, diſſimulando utram in partem (majoriſne, an minoris inæqualitatis) vergat ab æqualitate; nihilo major eſt ratio ternarii ad binarium, quàm binarii ad ternarium. Sed ſi cum mole unà includas quoque conſiderationem proceſſus à majori termino ad minorem, vel contrà; non eo inficias, minorem eſſe quamvis rationem minoris inæqualitatis non modò quâvis ratione majoris inæqualitatis; ſed & quâvis aliâ minoris inæqualitatis, quæ ab æqualitate minùs abſit. Ita ratio antecedentis 5 ad conſequentem 8, non modò minor eſt ratione antecedentis 8 ad conſequentem 6 (vel 5), ſed eadem quoque minor eſt ratione antecedentis 6 (vel 7) ad conſequentem 8: licèt ſepoſitâ vel neglectâ notione antecedentis & conſequentis, eadem ſit moles rationis $\frac{5}{8}$ atque $\frac{8}{5}$. Diſtinguemus igitur deinceps inter quantitates mole-majores, ut affectione-majores: ita ut in rationibus notio inæqualitatis majoris vel minoris nil niſi affectionem innuat. Eas porrò rationes appellamus mole-majores, quarum major terminus diviſus per minorem, dat quotum majorem; & vice verſâ. Præterea majoris inæqualitatis rationum quæcunque mole, eædem & affectione majores ſunt; at minoris inæqualitatis rationes quò ſunt mole majores, eò affectione minores ſunt. Quibus præmiſſis, digeremus ea, quæ reſtant, in propoſitiones.

PROPOSITIO I.

Si duæ quantitates ejuſdem affectionis auferantur ab invicem (affirmata ſc. ab affirmata, vel negata à negata), ſitque quantitas reliqua ejuſdem affectionis cum duabus ab initio datis; quantitas ablata mole-minor eſt quantitate ex qua auferebatur. Sin quantitas reliqua diverſæ ſit affectionis à duabus initiò datis; quantitas ablata mole-major eſt quantitate ex qua auferebatur. Sit exempli gr.

ratio ſubducenda	ratio ex qua	ratio reliqua
$\frac{5}{8}$)	$\frac{3}{5}$	($\frac{24}{25}$ }

tres ſcilicet rationes ejuſdem affectionis, putà minoris inæqualitatis ſingulæ; ergo, inquam, ratio ſubducta $\frac{5}{8}$ mole-minor eſt ratione ex quâ $\frac{3}{5}$. Sit rurſus

ratio ſubducenda	ratio ex qua	ratio reliqua
$\frac{8}{5}$)	$\frac{3}{2}$	($\frac{15}{16}$ }

quarum priores duæ ſunt ejuſdem affectionis, nimirum majoris inæqualitatis ambæ, at tertia $\frac{15}{16}$ diverſæ eſt affectionis, putà inæqualitatis minoris; ergo, inquam, ratio ſubducta $\frac{8}{5}$ mole-major eſt ratione ex quà $\frac{3}{2}$.

PROPOSITIO II.

Si ſint quotcunque rationes continuæ & terminorum æquidifferentium, v. gr. $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, & deinceps, faciendo ſcilicet antecedentem cujuſque ex poſterioribus rationibus æqualem conſequenti proximè præcedentis, & à minoribus progrediendo ad majores: Erit quælibet præcedentium rationum mole-major qualibet ſequente; ſed & differentiarum inter ipſas rationes tam primarum, quàm ſecundarum, tertiarum, cæterarumque adeò omnium in infinitum, ſemper præcedens mole-major eſt ſequente. Sin à majoribus terminis progrediare ad minores; contrarium eveniet.

Patet ex collatione ſequentis tabellæ cum propoſitione prima.

Rationes.	Diff. primæ.	Differentiæ ſecundæ.
a		
$a+b$		
	$aa+2ab+bb$	
	$aa+2ab$	
$a+b$		$a^4+6a^3b+12aabb+8ab^3$
$a+2b$		$a^4+6a^3b+12aabb+10ab^3+3b^4$
	$aa+4ab+4bb$	
	$aa+4ab+3bb$	
$a+2b$		$a^4+10a^3b+36aabb+54ab^3+27b^4$
$a+3b$		$a^4+10a^3b+36aabb+56ab^3+32b^4$
	$aa+6ab+9bb$	
	$aa+6ab+8bb$	
$a+3b$		$a^4+14a^3b+72aabb+56ab^3+128b^4$
$a+4b$		$a^4+14a^3b+72aabb+162ab^3+135b^4$
	$aa+8ab+16bb$	
	$aa+8ab+15bb$	
$a+4b$		
$a+5b$		

PROPOSITIO III.

Si quotcunque rationum continuarum, quarum termini ſint æquidifferentes, prima eademque minima vocetur a, differentia autem primæ & ſecundæ rationis vocetur b, tum ex differentiis ſecundis prima vocetur c; atque ex tertiis prima vocetur d; & ſic porrò: Aio ſecundam rationem fore $a+b$, tertiam $a+2b+c$, quartam $a+3b+3c+d$, quintam $a+4b+6c+4d+e$; atque ita deinceps, componendo ſingulas rationes ex prima & tot differentiis, quot quæque locis abeſt à prima, ipſis autem differentiis jungendo coëfficientes numeros figuratos, primis quidem radices, ſecundis trigonales, tertiis pyramidales, atquè ita porrò, ſingulosque adeò, prout naturali ſerie ordinantur in ſubjecta tabella:

A a Unitates

Unitates.	radices.	trigo-nales.	pyrami-dales.	trigono-trigona-les.	trigono-pyrami-dales.	pyrami-di-pyra-midales.	trigono-trigono-pyramid.	trigono-pyrami-di-pyra-mid.	pyrami-di-pyra-midi-py-ramida.
1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45	
1	4	10	20	35	56	84	120		
1	5	15	35	70	126	210			
1	6	21	56	126	252				
1	7	28	84	210					
1	8	36	120						
1	9	45							
1	10								

Nimirum eodem modo, quo iidem figurati numerant complementa poteſtatum à radice binomia genitarum; obſervato aſcenſu obliquo à ſiniſtra dextrorſum.

Sic undecima ratio conſtat ex $a + 10b + 45c + 120d + 210e + 252f + 210g + 120h + 45i + 10k + l$; ſumtis ordine numeris imam tabellæ baſin occupantibus.

Exhibet autem tabella non modò quotam velis rationem, ſed & ſummam quotcunque continuè ſequentium, pari fermè negotio methodoque. Sic ſumma quinque rationum eſt $5a + 10b + 10c + 5d + e$, obſervato aſcenſu obliquo, ut antè.

Rationes.

Demonſtratio.

Rationes.	Diff. Primæ.	Secundæ.	Tertiæ.	Quartæ.
a				
	b			
$a+b$		c		
	$b+c$		d	
$a+2b+c$		$c+d$		e
	$b+2c+d$		$d+e$	
$a+3b+3c+d$		$c+2d+e$		
	$b+3c+3d+e$			
$a+4b+6c+4d+e$				

Cùm enim per præcedentem, progrediendo à majoribus terminis ad minores (vel, quod idem eſt, à minoribus rationibus ad majores), non modò ſecunda ratio excedat primam, ſed & primarum, ſecundarum, cæterarumque differentiarum ſecunda quæque excedat primam; licebit ſanè iſtos exceſſus vocare b, c, d, & deinceps.

Cùm a differentiarum tertiarum prima ſit	d	Ex hypotheſi;
Et quartarum prima (quâ ſecunda tertiarum excedit primam)	e	
Erit ſanè ſecunda tertiarum	$d+e$	
Rurſus cùm ſecundarum differentiarum prima ſit	c	Ex hypotheſi;
Et tertiarum prima (quâ altera ſecundarum excedit primam)	d	
Erit ſanè altera ſecundarum	$c+d$	
Sed 2da 3tiarum (quâ tertia ſecundarum excedit alteram) erat	$d+e$	
Ergo tertia ſecundarum erit	$c+2d+e$	
Porrò cùm primarum differentiarum prima ſit	b	Ex hypotheſi;
Et ſecundarum prima (quâ ſecunda primarum excedit primam)	c	
Erit ſanè ſecunda primarum	$b+c$	
Sed altera ſecundarum (quâ tertia primarum excedit 2dam) erat	$c+d$	
Ergo tertia primarum	$b+2c+d$	
Sed & 3tia 2darum (quâ quarta primarum excedit 3tiam) erat	$c+2d+e$	
Ergo quarta primarum erit	$b+3c+3d+e$	
Denique cùm prima ratio ſit	a	Ex hypotheſi;
Et differentia inter primam & ſecundam rationem	b	
Erit ſanè ſecunda ratio	$a+b$	
Et cùm differentiarum primarum ſecunda foret	$b+c$	
Erit tertia ratio	$a+2b+c$	
Sed & differentiarum primarum tertia erat	$b+2c+d$	
Ergo quarta ratio	$a+3b+3c+d$	
Tandem differentiarum primarum quarta erat	$b+3c+3d+e$	
Ergo quinta ratio	$a+4b+6c+4d+e$	

$$\text{Poſtremò rationes}\left\{\begin{array}{ll}\text{Prima} & a \\ \text{Secunda} & a+b \\ \text{Tertia} & a+2b+c \\ \text{Quarta} & a+3b+3c+d \\ \text{Quinta} & a+4b+6c+4d+e\end{array}\right\}\text{addantur,}$$

ſit ſumma quinque rationum $5a+10b+10c+5d+e$.

 Rationes

Rationes vel Magnitudines.

Prima	a
$\overline{2}$	$a + b$
$\overline{3}$	$a + 2b + c$
$\overline{4}$	$a + 3b + 3c + d$
$\overline{5}$	$a + 4b + 6c + 4d + e$
$\overline{6}$	$a + 5b + 10c + 10d + 5e + f$
$\overline{7}$	$a + 6b + 15c + 20d + 15e + 6f + g$
$\overline{8}$	$a + 7b + 21c + 35d + 35e + 21f + 7g + h$
$\overline{9}$	$a + 8b + 28c + 56d + 70e + 56f + 28g + 8h + i$
$\overline{10}$	$a + 9b + 36c + 84d + 126e + 126f + 84g + 36h + 9i + k$

PROPOSITIO IV.

Si quotcunque rationum continuarum, quarum termini ſint æquidifferentes, prima eadémque maxima vocetur a, differentia autem primæ & ſecundæ vocetur b, tum differentiarum ſecundarum prima vocetur c, tertiarum prima d, & ſic deinceps; aio ſecundam rationem fore $a - b$. tertiam $a - 2b + c$, quartam $a - 3b + 3c - d$, quintam $a - 4b + 6c - 4d + e$; atque ita deinceps, alternatis ſemper ſignis affirmatis & negatis. Demonſtratur ut præcedens.

PROPOSITIO V.

Datæ rationis multiplicem invenire prope verum.

Conſtructio. Differentiam terminorum datæ rationis duc in denominatorem multiplicis dati, & à facto aufer ipſam differentiam, reliqui ſemiſſem adde termino majori, & detrahe minori; ita prodibunt duo termini exprimentes rationem paulò minorem quæſitâ. Tum ſi termini prodeuntes ſint fortè numeri mixti ex integris & fractis; reducantur ad purè fractos, quorum denominatoribus omiſſis, ratio quæſita cenſebitur in numeratoribus integris & à fractione liberis. v. gr. Quæratur rationis $\frac{25}{28}$ quadruplum. Differentia terminorum 3 ducta in 4 exhibet 12, unde ablatis tribus reſtant 9, cujus ſemis $4\frac{1}{2}$ additus termino majori 28, facit $32\frac{1}{2}$, detractus autem minori 25, relinquit $20\frac{1}{2}$; erit igitur ratio $20\frac{1}{2}$ ad $32\frac{1}{2}$ paulò major quadruplo rationis $\frac{25}{28}$. Reductis terminis $20\frac{1}{2}$ & $32\frac{1}{2}$ ad purè fractos, fiunt $\frac{41}{2}$ & $\frac{65}{2}$, omiſſiſque denominatoribus, erit ratio $\frac{41}{65}$ paulò major quæſitâ.

Demonſtratio hujus & ſequentis propoſitionis conſtabit ex propoſitione VII.

PROPOSITIO VI.

Datæ rationis partem imperatam invenire prope verum.

Conſtructio. Differentiam terminorum datæ rationis divide in partes totidem, quot denominator partis quæſitæ conſtat unitatibus, atque ex iis partibus exemtâ unâ, reliquarum ſemiſſem adde termino minori, & detrahe quoque majori; ita pro-

prodibunt duo termini exprimentes rationem paulò minorem quæſitâ: Tum ſi termini prodeuntes ſint fortè numeri mixti ex integris & fractis; reducantur ad purè fractos, quorum denominatoribus omiſſis, ratio quæſita cenſebitur in numeratoribus integris, & à fractione liberis. v. gr. Oporteat rationis $\frac{3}{5}$ invenire partem quintam. Differentia terminorum 2 diviſa quinquefariam exhibet $\frac{2}{5}$, quæ eſt una pars quinta, eximenda ex integra ſumma quinque partium, quæ erat 2, & reſtant $\frac{8}{5}$, quarum ſemis $\frac{4}{5}$ additus termino minori 3, facit $3\frac{4}{5}$; detractus verò ex majori 5, reliquum facit $4\frac{1}{5}$; erit igitur ratio $3\frac{4}{5}$ ad $4\frac{1}{5}$ paulò minor, quàm pars quinta rationis $\frac{3}{5}$. Reductis terminis $3\frac{4}{5}$ & $4\frac{1}{5}$ ad purè fractos, fiunt $\frac{19}{5}$ & $\frac{21}{5}$, omiſſiſque denominatoribus, erit ratio $\frac{19}{21}$ paulò minor quæſitâ. Rurſus inveniendus ſit rationis $\frac{8}{11}$ ſemis. Differentia terminorum 3 bipartita exhibet $1\frac{1}{2}$, qui eſt unus ſemis, eximendus ex integra ſumma duarum partium 3, & reſtat $1\frac{1}{2}$, cujus ſemis $\frac{3}{4}$ additus termino minori 8, facit $8\frac{3}{4}$; detractus autem ex majori 11, relinquit $10\frac{1}{4}$; erit igitur ratio $8\frac{3}{4}$ ad $10\frac{1}{4}$ paulò minor ſemiſſe rationis $\frac{8}{11}$. Reductis terminis $8\frac{3}{4}$ & $10\frac{1}{4}$ ad purè fractos, fiunt $\frac{35}{4}$ & $\frac{41}{4}$, omiſſiſque denominatoribus, erit ratio $\frac{35}{41}$ paulò minor quæſitâ.

PROPOSITIO VII.

Invenire, quantum pars rationis imperata, quæ per præcedentem invenitur, deficiat ab exactiori.

Conſtructio. Primò, ſi partis imperatæ denominator ſit numerus impar; ſume rationes, quæ ſunt rationi per præcedentem inventæ utrinque vicinæ & æquidifferentes, ita habebis tres rationes, quarum minimam aufer à media, & mediam à maxima, prodibunt duæ differentiæ, quarum differentiam denuò inveſtigabis, tantiſper aſſervandam. Deinde partis imperatæ denominatori unitatem detrahe, reliqui ſemiſſem in tabula figuratorum inſertâ prop. III, quære inter radices, & invento congruentem numerum trigonalem excerptum tripartire, ſic invenies, quoties ſumenda ſit differentiarum differentia ſuprà aſſervata, ut acquiras particulam, quâ pars imperata, quæ per præcedentem inveniebatur, deficit ab exactiori. v. g. Scire velim, ratio $\frac{19}{21}$ per præcedentem inventa quantum deficiat ab exactiori quinta parte rationis $\frac{3}{5}$. Rationi $\frac{19}{21}$ utrinque vicinæ & æquidifferentes ſunt $\frac{17}{19}$ & $\frac{21}{23}$. Differentia minimæ à media $\frac{437}{441}$ & mediæ à maxima $\frac{357}{361}$, & harum differentiarum differentia $\frac{157437}{157757}$ aſſervanda. Tum partis imperatæ denominatori 5 detraho 1, reſtant 4, cujus ſemiſſi 2 inter radices invento congruit trigonalis numerus 3, cujus triens eſt 1, indicans differentiarum differentiam ſupra aſſervatam $\frac{157437}{157757}$ ſemel ſumtam exhibere particulam, quâ ratio $\frac{19}{21}$ deficit ab exactiori quinta parte rationis $\frac{3}{5}$, adeo ut hujus exactior pars quinta ſit ratio $\frac{19}{21}$ + ratione $\frac{157437}{157757}$.

Sin partis imperatæ denominator ſit numerus par; ſume ſemiſſem differentiæ terminorum rationis per præcedentem inventæ, quem ejuſdem termino minori detrahes, & majori addes pariter ac detrahes; ita obtinebis quatuor rationes continuas terminorem æquidifferentium, ex quibus minorum duarum differentiam auferes ex majorum duarum differentia, & emergentem differentiarum differentiam aſſervabis. Deinde partis imperatæ denominatorem bipartire, & invento

invento ſemiſſi congruentes in tabella propoſitioni III ſubjuncta ſpecies excerpere, ſaltim uſque ad c ſpeciem, poſitoque $a = \frac{1}{2}$, $b = 2$, $c = 1\frac{1}{3}$; duc cujuſque ſpeciei valorem in ſuum coëfficientem, collectiſque omnibus in unam ſummam, habebis, quot vicibus ſumenda ſit differentiarum differentia ſuprà aſſervata, ut acquiras particulam, quâ pars imperata, quæ per præcedentem inveniebatur, deficit ab exactiori. Ex. gr. Rationis $\frac{8}{11}$ octans per præcedentem inventus ſit $\frac{149}{155}$; ſcire velim, quantum is deficiat ab exactiori. Differentia terminorum eſt 6, cujus ſemis 3 detractus minori termino, relinquit 146; additus autem majori, facit 158; & detractus majori, relinquit 152. Sunt ergo quatuor rationes continuæ terminorum æquidifferentium $\frac{146}{149}$, $\frac{149}{152}$, $\frac{152}{155}$, $\frac{155}{158}$. Differentia duarum minorum rationum $\frac{24016}{24025}$ ablata à differentia duarum majorum $\frac{22192}{22201}$ relinquit differentiarum differentiam $\frac{1753825}{1753879}$ aſſervandam. Partis imperatæ denominator eſt 8, cujus ſemiſſi 4 congruunt in tabella propoſitioni III ſubjuncta ſpecies iſtæ: $a + 3b + 3c$; ſed $a = \frac{1}{2}$, & $3b = 6$, & $3c = 4$. quæ juncta faciunt $10\frac{1}{2}$. Ergo differentiarum differentiæ $\frac{1753825}{1753879}$ ſuprà ſervatæ ſumendum eſt decuplum cum ſemiſſe, ut acquiramus particulam, quâ octans per præcedentem inventus deficit ab exactiori. Atqui rationis $\frac{1753825}{1753879}$ decuplum per V hujus eſt $\frac{1753582}{1754122}$ vel $\frac{876791}{877061}$, & ſemis per VI eſt $\frac{3507677}{3507731}$; adeò ut rationis $\frac{8}{11}$ octans exactior præter rationem per præcedentem inventam $\frac{149}{155}$ contineat etiamnum ratiunculas $\frac{876791}{877061}$, & $\frac{3507677}{3507731}$.

DEMONSTRATIO.

Cùm per III hujus ſumma trium rationum continuarum, terminis æquidifferentibus contentarum, ſit $3a + 3b + c$

Erit ejuſdem ſummæ triens	$a + b + \frac{1}{3}c$ (α)
Rurſus differentia primæ & ſecundæ rationis eſt	b
ſecundæ autem & tertiæ	$b + c$
Ergo differentiarum differentia	c
Cujus triens	$\frac{1}{3}c$
Quem ſi addas mediæ trium rationum	$a + b$
Erit ſumma	$a + b + \frac{1}{3}c$

Æqualis nempe trienti trium rationum ſuprà notato literâ α. Ergo diſcerptâ ratione quavis in tres continuas terminorum æquidifferentium, ut jubet propoſitio VI, erit media ex iis paulò minor triente totius diſcerptæ, & quidem tantò minor, quantus eſt triens differentiæ differentiarum intercedentium inter rationem primam & ſecundam, nec non inter ſecundam & tertiam, quod innuit propoſitio VII.

Sic

Sic quoque per III hujus, ſumma ſedecim rationum continuarum, terminis æquidifferentibus contentarum, eſt	$16a + 120b + 560c + 1820d$
Et ejuſdem ſummæ octans	$2a + 15b + 70c + 227\frac{1}{2}d\ (\beta)$
Tum per tabellam propoſitioni III ſubjunctam, quatuor mediæ ex iſtis ſedecim, nimirum 7^{ma}, 8^{va}, 9^{na}, 10^{ma}, ſunt	$\left\{\begin{array}{l} a + 6b + 15c + 20d \\ a + 7b + 21c + 35d \\ a + 8b + 28c + 56d \\ a + 9b + 36c + 84d \end{array}\right.$
Differentia duarum priorum	$b + 6c + 15d$
poſteriorum	$b + 8c + 28d$
Differentiarum differentia	$2c + 13d$
Hujus decuplum	$20c + 130d$
& ſemis	$c + 6\frac{1}{2}d$
Unà cum ſumma duarum ex quatuor iſtis mediar.	$2a + 15b + 49c + 91\frac{1}{2}d$
Facit	$2a + 15b + 70c + 228d$

Æqualem octanti ſedecim rationum ſuprà notato literâ β. Ergo ſi ratio data diſcerpatur in partes ſedecim, erunt duæ mediæ ex iis ſimul, paulò minores octante totius diſcerptæ; & quidem tantò minores, quantum eſt differentiæ differentiarum (intercedentium inter rationes ex ſedecim iſtis ſeptimam & octavam, nec non inter 9^{nam} & 10^{mam}) decuplum cum ſemiſſe. q. e. d.

PROPOSITIO VIII.

Rationes terminorum æquidifferentium ſunt propemodum, ut reciprocè ipſorum terminorum media arithmetica.

Explicatio. Sumatur per VI hujus rationis cujuſvis, v. gr. $\frac{8}{9}$, pars quævis, v. gr. ſemis $\frac{33}{35}$, tum pars quævis alia, v. gr. triens $\frac{25}{26}$, & ut fiant terminorum æquidifferentium, pro $\frac{8}{9}$ ſumatur $\frac{16}{18}$, & pro $\frac{25}{26}$ æquipollens $\frac{50}{52}$. Medium arithmeticum terminorum rationis totius eſt 17, ſemiſſis 34, trientis 51. Liquet igitur, ut tota ratio $\frac{16}{18}$ eſt ad ſemiſſem ſuum $\frac{33}{35}$; ita reciprocè ſemiſſis medium arithmeticum 34 eſſe ad medium totius 17: & ut tota ratio $\frac{16}{18}$ eſt ad trientem ſuum $\frac{50}{52}$; ita reciprocè trientis medium arithmeticum 51 eſſe ad medium totius 17: ideòque etiam, ut ſemis $\frac{33}{35}$ eſt ad trientem $\frac{50}{52}$; ita reciprocè trientis medium 51 eſſe ad medium ſemiſſis 34: Tantum porrò hanc analogiam abire à vero, quantum ſemiſſes, trientes, parteſve aliæ rationum per VI hujus inventæ deficiunt ab exactis. Quamobrem id agendum, ne defectus ille inſtituto noſtro officiat. Cæterum minor erit defectus, minùſque adeò officiat, quò rationes in analogiam adſcitæ minores fuerint. Cùm enim ſecundum demonſtrationem præcedentis, octans exactus ſedecim rationum foret $2a + 15b + 70c + 227\frac{1}{2}d$, at ſumma duarum ex ſedecim iſtis mediarum $2a + 15b + 49c + 91\frac{1}{2}d$, qui eſt octans per VI inventus; patet differentiam horum octantium conſiſtere in contentiori parte ſecundarum & tertiarum differentiarum. Atqui rationum continuarum minores habent differentias primas minores, ac proinde differentiarum ſecundarum & tertiarum partem exiliorem multò etiamnum minorem. Sed exemplo res fiet illuſtrior. Nam rationis

$\frac{99}{101}$ semis, per VII hujus, est $\frac{199}{201} + \frac{7999399}{7999401}$; ubi ratio $\frac{199}{201}$ ab exactiori semisse deficit ratiunculâ superbipartiente $\frac{7999399}{7999401}$. Rursus rationem $\frac{199}{201}$ (quæ priùs assumtæ $\frac{99}{101}$ propemodum semis est) si denuò bipartiamur per VII hujus, habebimus $\frac{399}{401} + \frac{127997599}{127997601}$; ubi ratio $\frac{399}{401}$ ab exactiori semisse deficit ratiunculâ superbipartiente $\frac{127997599}{127997601}$. Minor est igitur defectus, cùm bipartimur rationem minorem $\frac{199}{201}$, quàm si bipartiamur majorem $\frac{99}{101}$, quantò scilicet ratiuncula superbipartiens $\frac{127997599}{127997601}$ minor est superbipartiente $\frac{7999399}{7999401}$, hoc est propemodum, quantò 8 milliones minores sunt 128 millionibus, nimirum sedecim vicibus. Sed & ejusdem rationis quò minor pars sumetur per VI hujus, eò minus deficiet à vero. Sic rationis $\frac{99}{101}$ semis, per VII hujus, erat $\frac{199}{201} + \frac{7999399}{7999401}$, & ejusdem triens per eandem est $\frac{299}{301} + \frac{8099459197}{8099460797}$; ubi quidem triens $\frac{299}{301}$ (qualis per VI invenitur) minùs deficit ab exactiori, quàm semis $\frac{199}{201}$ (per eandem inventus); quantò ratiuncula $\frac{8099459197}{8099460797}$ minor est alterâ $\frac{7999399}{7999401}$. Unde sequitur, cùm bipartitâ ratione $\frac{99}{101}$ secundum VI hujus, non nisi binarium quasi perdamus in octo millionibus, vel unitatem in quatuor millionibus; futurum, ut istius semisse $\frac{199}{201}$ (sive $\frac{99|5}{100|5}$) diminuto quovis modo per analogiam VI^{tae} hujus superstructam, minùs etiam perdamus; adeóque à ratione $\frac{99|5}{100|5}$ nos analogicè argumentari posse ad quamvis minorem terminorum æquidifferentium, ita ut minùs quàm unitatem perdamus in quatuor millionibus; quóque ratio, ad quam argumentamur, minor fuerit, eò jacturam fore minorem.

PROPOSITIO IX.

Datâ mensurâ rationis $\frac{99|5}{100|5}$ in particulis, qualium decupla continet 1,0000000; invenire mensuram cujusvis minoris rationis terminorum æquidifferentium, in particulis similibus.

Rationis $\frac{99|5}{100}$ mensura suprà inventa fuit 21769|3, & rationis $\frac{100}{100|5}$ mensura ibidem 21659|7, quarum summa exhibet rationem $\frac{99|5}{100|5} = 43429|0$. Dehinc oporteat nos invenire mensuram rationis $\frac{100}{101}$. Ergo per præcedentem dic:

Ut medium Arithmeticum terminorum $\frac{100}{101}$ (nimirum 100|5) ad medium Arithmeticum terminorum $\frac{99|5}{100|5}$ (nimirum 100); ita mensura hujus rationis

(putà

(putà 43429) ad menſuram iſtius 43213. Tot igitur particulis logarithmus abſoluti 101 excedit logarithmum abſoluti 100. Quare, cùm logarithmus abſoluti 100 ſit 2,0000000; oportet, ut logarithmus abſoluti 101 ſit 2,0043213.

Porrò invenienda ſit menſura rationis $\frac{101}{102}$. Dic:

Ut medium arithmeticum terminorum $\frac{101}{102}$ (nimirum 101⌊5) ad medium arithmeticum terminorum $\frac{99\lfloor 5}{100\lfloor 5}$ (nimirum 100); ita menſura hujus rationis (putà 43429) ad menſuram iſtius 42787. Tot igitur particulis logarithmus abſoluti 102 excedit logarithmum abſoluti 101. Quare, cùm logarithmus abſoluti 101 foret 2,0043213; oportet, ut logarithmus abſoluti 102 ſit 2,008600.

Cùm autem in omnibus hiſce analogiis terminus ſecundus ſit 100, & tertius 43429; liquet, ad inveniendam menſuram cujuſvis rationum ſequentium nihil amplius reſtare, quàm ut dividamus numerum 43429 per medium arithmeticum terminorum rationis datæ. Cæterùm invenimus nos quidem rationis $\frac{99\lfloor 5}{100\lfloor 5}$ menſuram 43429, quæ fortè debebat eſſe unitate major, putà 43430; ſed facilè intelligit quivis, ſi pro ratione $\frac{99\lfloor 5}{100\lfloor 5}$ aſſumſiſſemus $\frac{999\lfloor 5}{1000\lfloor 5}$, & in cumulandis horum terminorum poteſtatibus calculum ad plures locos extendiſſemus, ad majorem utique præciſionem perveniri potuiſſe, adeóque huic methodo ad accuratam facilitatem nihil quicquam deeſſe. At non deeſt modus etiam hoc ipſo facilior, qui poſt acquiſitos paucos logarithmos ſolâ additione rem peragit, & præterea probam ſuam ſecum fert, quem propoſitionibus ſequentibus breviter exponemus.

PROPOSITIO X.

Rationum duarum continuarum differentia eſt ad aliarum duarum continuarum differentiam; ut harum communis termini quadratum, ad iſtarum communis termini quadratum; dummodo ſingularum termini ſint æquidifferentes.

Sint duæ rationes continuæ $\frac{a}{a+b}$, & $\frac{a+b}{a+2b}$, quarum terminus communis eſt $a+b$, & hujus quadratum $aa+2ab+bb$; ſint verò & aliæ duæ continuæ $\frac{a+3b}{a+4b}$, & $\frac{a+4b}{a+5b}$, quarum communis terminus eſt $a+4b$, & hujus quadratum $aa+8ab+16bb$. Singularum termini differunt communi exceſſu b. Differentia duarum priorum eſt $\frac{aa+2ab}{aa+2ab+bb}$, & poſteriorum duarum $\frac{aa+8ab+15bb}{aa+8ab+16bb}$. Atque hæ differentiæ ſunt quoque terminorum æquidifferentium. Ergo per VIII hujus ſunt propemodum, ut reciprocè ipſorum terminorum media arithmetica; hoc eſt, ut prior differentia $\frac{aa+2ab}{aa+2ab+bb}$, ad poſteriorem $\frac{aa+8ab+15bb}{aa+8ab+16bb}$: ita horum terminorum medium arithmeticum

$aa+$

$aa + 8ab + 15\frac{1}{2}bb$, ad medium arithmeticum istorum $aa + 2ab + \frac{1}{2}bb$; hoc est propemodum, ut communium terminorum quadrata, nimirum $aa + 8ab + 16bb$ ad $aa + 2ab + bb$; quæ ab istis mediis arithmeticis non nisi quantitate $\frac{1}{2}bb$ differunt, exiguâ sanè, & in rationibus minoribus (ubi sc. differentia terminorum b ad ipsos terminos exiguum instar habet) facilè contemnendâ.

PROPOSITIO XI.

Rationum trium continuarum differentiarum differentia, est ad aliarum trium continuarum differentiarum differentiam; ut cubus medii arithmetici mediæ ex his, ad cubum medii arithmetici mediæ ex istis; dummodo singularum rationum termini sint æquidifferentes.

Sint tres rationes continuæ $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, quarum differentia differentiarum est $\frac{a^4 + 6a^3b + 12aabb + 8ab^3}{a^4 + 6a^3b + 12aabb + 10ab^3 + 3b^4}$, & mediæ illarum medium arithmeticum est $a + \frac{3b}{2}$, cujus cubus $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{35b^3}{8}$; sint verò & aliæ tres continuæ $\frac{a+2b}{a+3b}$, $\frac{a+3b}{a+4b}$, $\frac{a+4b}{a+5b}$, quarum differentia differentiarum est $\frac{a^4 + 14a^3b + 72aabb + 160ab^3 + 128b^4}{a^4 + 14a^3b + 72aabb + 162ab^3 + 135b^4}$, & mediæ illarum medium arithmeticum $a + \frac{7b}{2}$, cujus cubus $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$. Cæterùm singulæ rationes differunt communi excessu b; at differentiæ differentiarum non sunt terminorum æquidifferentium, siquidem differentia terminorum prioris est $2ab^3 + 3b^4$, at posterioris $2ab^3 + 7b^4$: secùs ac in præcedenti propositione. Quare cum illic res expediretur regulâ proportionum simplici inversâ, hic opus est duplici inversâ; nimirum:

Ut medium arithmeticum prioris differentiæ differentiarum (nimirum $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$) ductum in differentiam terminorum posterioris ($2ab^3 + 7b^4$) ad medium arithmeticum posterioris differentiæ differentiarum (nimirum $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$) ductum in differentiam terminorum prioris ($2ab^3 + 3b^4$): Ita differentia differentiarum prior ad posteriorem: Ita quoque cubus medii arithmetici mediæ trium priorum rationum (nimirum $a^3 + \frac{9aab}{2} + \frac{27aab}{4} + \frac{35b^3}{8}$) ad cubum medii arithmetici mediæ trium posteriorum rationum (nimirum $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$).

Quæ analogia vera esse deprehendetur, si productum extremorum æquale sit producto mediorum.

Atqui primus terminus $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$ in $2ab^3 + 7b^4 = 2a^5b^3 + 19a^4b^4 + 66a^3b^5 + 102aab^6 + 66ab^7 + \frac{21b^8}{2}$, si ducatur in

quartum

quartum $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$; productum est $2a^8b^3 + 40a^7b^4 + 297a^6b^5 + 1180a^5b^6 + 2991\frac{1}{8}a^4b^7$, &c.

Rursus secundus terminus $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$ in $2ab^3 + 3b^4 = 2a^5b^3 + 28a^4b^4 + 144a^3b^5 + 322aab^6 + 263ab^7 + \frac{789b^4}{2}$, si ducatur in tertium $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{35b^3}{8}$; productum est $2a^8b^3 + 40a^7b^4 + 339a^6b^5 + 1593a^5b^6 + 4558\frac{1}{8}a^4b^7$, &c.

Hoc igitur productum cùm consentiat cum isto, non modò in primis & secundis speciebus, sed & in maxima parte tertiarum & quartarum; aio analogiam in propositione memoratam, veram esse. Nam defectus, qui hîc apparet in productis terminorum, in ipsis terminis longè minor erat, quippe qui multiplicando crevit. Ut taceam in minoribus rationibus differentias secundas & tertias nullius ferè momenti esse *.

Simili modo ostendetur, differentias tertias rationum continuarum & terminorum æquidifferentium, esse ut quadrato-quadrata; quartas, ut quadrato-cubos terminorum, qui singulis in tabellâ propositioni 11 subjunctâ, è regione opponuntur. Atque ita deinceps.

PROPOSITIO XII.

Numerorum in progressione arithmetica ordinatorum quadrata conveniunt in differentiis secundis, cubi in tertiis, quadrato-quadrata in quartis, & sic deinceps.

Patet ex inspectione tabellarum subjectarum:

Numeri	Quadrata	diff. 1.	diff. 2.
1	1		
		3	
2	4		2
		5	
3	9		2
		7	
4	16		

Numeri	Cubi	diff. 1.	diff. 2.	diff. 3.
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		
5	125			

* See the note at the end of this tract, p. 196.

Numeri	Quadrato-quadrata	diff. 1.	diff. 2.	diff. 3.	diff. 4.
1	1				
		15			
2	16		50		
		65		60	
3	81		110		24
		175		84	
4	256		194		24
		369		108	
5	625		302		
		671			
6	1296				

Hinc patet, datis v. gr. cubis quatuor, vel quadrato-quadratis quinque, quo pacto cæteri continuâ additione ſuccenturiari poſſint. Sint enim dati

Cubi	diff. 1.	diff. 2.	diff. 3.
$a = 8$			
	$e = 19$		
$b = 27$		$h = 18$	
	$f = 37$		$k = 6$
$c = 64$		$i = 24$	
	$g = 61$		$k = 6$
$d = 125$		$l = ..$	
	$m = ..$		
$n = ...$			

Dic: $k + i = l$, $l + g = m$, $m + d = n$.

PROPOSITIO XIII.

Logarithmos quotvis locorum continuâ ac ſolâ additione producere ita, ut ultimo exiſtente probo, cæteri omnes ſint probi.

Conſtructio. Cùm ſit $\frac{a}{b-c} = \frac{a}{b} + \frac{ca}{bb} + \frac{cca}{b^3} + \frac{c^3a}{b^4}$, & deinceps continuando progreſſionem in infinitum; ſi ponamus $a = 100 =$ medio arithmetico rationis $\frac{99\llcorner 5}{100\llcorner 5}$ & $b = 100000$, & $c = 0\llcorner 5$, adeò ut $b - c$ ſit $99999\llcorner 5$ $=$ medio arithmetico rationis datæ $\frac{99999}{100000}$, cujus menſuram invenire oportet, erit $\frac{a}{b-c}$ (nimirum $\frac{100}{99999\llcorner 5}$) $= \frac{a}{b}$ (ſive $\frac{100}{100000}$ vel $0\llcorner 001$) $+ \frac{ca}{bb}$ (ſive $\frac{0\llcorner 5 * 100}{10000000000}$ vel $\frac{50}{10000000000}$ vel $0\llcorner 000000005$) $+ \frac{cca}{b^3}$ (ſive $\frac{0\llcorner 25 * 100}{1000000000000000}$ vel $\frac{25}{1000000000000000}$ vel $0\llcorner 00000000000000025$) $+ \frac{c^3a}{b^4}$ (ſive $\frac{0\llcorner 125 * 100}{100000000000000000000}$ vel $\frac{1\llcorner 25}{100000000000000000000}$ vel $0\llcorner 0000000000000000000125$) $=$

0⌊001000005000025000125 : ſin manentibus a & b, ut antè, ponatur $c = 1⌊5$; erit $\frac{a}{b-c} =$ 0⌊001000015000225003375 : vel ſi rurſus manentibus a & b, ponatur $c = 2⌊5$; erit $\frac{a}{b-c} =$ 0⌊001000025000625015625. Liquet ex præcedenti, quo pacto datis numeri 5 quadrato & cubo, ſequentium quoque numerorum 15, 25, cæterorumque quadrata & cubi additione continuâ inveniri poſſint. Iſtis igitur numeris cum ſuis quadratis & cubis ritè diſpoſitis ita ; ut poſt unitatem in 0⌊001 ſextum locum occupet ultima figura numeri 5 (vel 15, vel 25) ; & ſextum ab hac locum ultima figura quadrati 25 (vel 225, vel 625) ; & ab hac rurſus ſextum locum ultima figura cubi 125 (vel 3375, vel 15625) ; quemadmodum exempla ſuperiora oſtendunt : conflati erunt numeri, qui in tertium regulæ aureæ terminum 43429 ducendi ſunt ſinguli, ut prodeat menſura rationum datarum $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$. Atque hæc multiplicatio, ut ſolâ quoque additione perficiatur, digerendus eſt tertius terminus 43429 in tabellam ſubſidiariam, hoc modo :

1	43429
2	86858
3	130287
4	173716
5	217145
6	260574
7	304003
8	347432
9	390861

Acquiſitâ hoc modo rationum omnium menſurâ à $\frac{99999}{100000}$ uſque ad $\frac{10000}{10001}$ in particulis, qualium decupla continet 1,0000000 ; mox addendo has ordine retrogrado concinnabimus logarithmos ſingulorum abſolutorum à 10000 ad 100000 ; quorum ultimus ſi ſit probus, præcedentes omnes erunt probi ; niſi fortè error poſterior quaſi ex condicto corrigat priorem, quod in hac non magis quàm omnibus aliis probis, nec niſi rariſſimè uſuvenire poteſt.

ATQUE ita expoſitâ methodo conſtruendi logarithmos novâ, accuratâ, & facili, haud ſcio, an opus ſit monere lectorem, ſi ad praxin accedere luberet, non requiri compoſitorum numerorum logarithmos ; ideoque omnes pares primùm excludendos, deinde omnes à quinario productos ; ita ut reſtent ſoli logarithmi numerorum in unitatem 3^{tium}, 7^{rium}, 9^{tium} exeuntium, atque horum quoque tertium quemque, cùm ſit ex ternario compoſitus, omitti poſſe : ſic acquiſitis logarithmis abſolutorum 10003 & 10013, omiſſo logarithmo abſoluti 10023 ex ternario compoſiti quærendus eſt logarithmus abſoluti 10033, utpote triceſimi ab abſoluto 10003 ; tum logarithmus abſoluti 10043, triceſimi ab abſoluto 10013 : tum rurſus omiſſo logarithmo numeri 10053 quærantur logarithmi

garithmi numerorum 10063, 10073; atque ita deinceps. Quare dicendum per regulam propositione VIII traditam:

Ut 10048, nimirum medium arithmeticum inter 10033 & 10063,

ad 10018, nimirum medium arithmeticum inter 10003 & 10033:

Ita 13006, nimirum differentia logarithmorum congruentium absolutis 10003 & 10033,

ad 12967, differentiam logarithmorum competentium absolutis 10033 & 10063.

Dico:

$$\text{Ut} \left\{ \begin{matrix} 10048 \\ 10078 \\ 10108 \\ \&c. \end{matrix} \right\} \text{ ad } 10018; \text{ ita } 13006, \text{ ad } \left\{ \begin{matrix} 12967 \\ \&c. \end{matrix} \right.$$

Item:

$$\text{Ut} \left\{ \begin{matrix} 10058 \\ 10088 \\ 10118 \\ \&c. \end{matrix} \right\} \text{ ad } 10028; \text{ ita } 12992, \text{ ad } \left\{ \begin{matrix} 12953 \\ \&c. \end{matrix} \right.$$

At tertius qui sequitur ordo, nimirum:

$$\text{Ut} \left\{ \begin{matrix} 10068 \\ 10098 \\ \&c. \end{matrix} \right\} \text{ ad } 10038; \text{ ita } 12979, \text{ ad } \left\{ \begin{matrix} 12940 \\ \&c. \end{matrix} \right.$$

hic ipse est, quem omittendum indigeto.

Pari modo tertius quisque in unitatem 7^{riam}, 9^{rium} exeuntium omittetur. Itaque fiet, omissis paribus lucrifaciamus semissem operæ, & detractis quinariis rursus partem decimam, denique excluso tertio quoque in 1, 3, 7, 9, desinentium, trientem laboris residui; unde non nisi $\frac{4}{15}$, nonaginta chiliadum, quæ sunt à 10000 ad 100000, industriam nostram expectant. Hoc est, de 90 chiliadibus restant solùm 24 concinnandæ. Cæteros compositos à 7^{rio}, 11^{rio}, aliisve primis genitos, non est operæ pretium secernere in methodo tam proclivi; præsertim cùm probando calculo inservire possint.

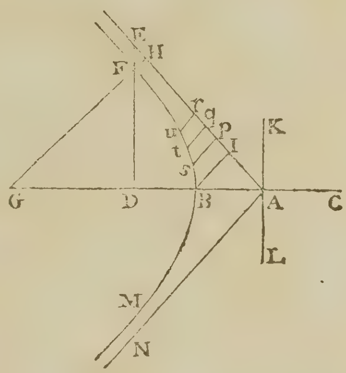
Cæterum ex iis, quæ hactenus disseruimus, satis liquet, naturam logarithmorum geometriæ nullo modo obnoxiam esse; sed verius ac liquidius ex proprio suo fonte manare. Intercedit tamen utrisque cognatio suavis, & contemplatione dignissima, quam deinceps paucis exponere non gravabor.

PRO-

PROPOSITIO XIV.

Sit hyperbole MBF, cujus latus rectum KL æquale sit transverso BC; erunt asymptoti AN & AE ad angulos rectos, & quadratum DF æquale rectangulo CDB per XXI. Conicor. I. Ex B & F cadant perpendiculares ad asymptoton BI & FH. Dico, AH esse ad AI, ut BI ad FH.

DEMONSTRATIO.



Sit $\mathrm{AB} = 1 = \mathrm{AC}$

$\mathrm{BC} = \mathrm{AB} + \mathrm{AC} = 2$

$\mathrm{BD} = a$

$\mathrm{AD} = \mathrm{AB} + \mathrm{BD} = 1 + a = \mathrm{DE}$

$\mathrm{CD} = \mathrm{BC} + \mathrm{BD} = 2 + a$

$\mathrm{CD} \times \mathrm{BD} = \overline{2 + a}$ in $a = 2a + aa = \mathrm{Q}:\mathrm{DF}$

$\mathrm{DF} = \sqrt{2a + aa} = \mathrm{DG}$

$\mathrm{AG} = \mathrm{AD} + \mathrm{DG} = 1 + a + \sqrt{2a + aa}$

$\sqrt{2} \,.\, 1 :: \mathrm{AG} \,.\, \mathrm{AH}$

$$\mathrm{AH} = \frac{1 + a + \sqrt{2a + aa}}{\sqrt{2}}$$

$\mathrm{EF} = \mathrm{DE} - \mathrm{DF} = 1 + a - \sqrt{2a + aa}$

$\sqrt{2} \,.\, 1 :: \mathrm{EF} \,.\, \mathrm{FH}$

$$\mathrm{FH} = \frac{1 + a - \sqrt{2a + aa}}{\sqrt{2}}$$

Ducatur AH in FH;

ponendo $1 + a = c$, & $\sqrt{2a + aa} = d$

erit $1 + 2a + aa = cc$

& $2a + aa = dd$ } subtrahe

$1 = cc - dd$

$$\frac{cc - dd}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2} = \mathrm{AH} \times \mathrm{FH} \qquad \alpha$$

$\sqrt{2} \,.\, 1 :: \mathrm{AB} \,.\, \mathrm{AI}$

$$\mathrm{AI} = \frac{1}{\sqrt{2}}$$

$$\mathrm{AI} \times \mathrm{BI} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \tfrac{1}{2} \qquad \beta$$

Ergo per α & β, $\mathrm{AH} \times \mathrm{FH} = \mathrm{AI} \times \mathrm{BI}$

& $\mathrm{AH} \,.\, \mathrm{AI} :: \mathrm{BI} \,.\, \mathrm{FH}$ q. e. d.

PROPOSITIO XV.

In diagrammate præcedenti, positâ $\mathrm{AI} = \mathrm{BI} = 1$, & $\mathrm{HI} = a$; oporteat invenire FH.

Dic per præcedentem: ut AH ad AI, ita BI ad FH; hoc eſt, $1 + a . 1 :: 1 . \frac{1}{1+a}$; nimirum FH æqualis eſt unitati diviſæ per $1 + a$. Perficitur autem diviſio ipſo opere ſic:

$$\begin{array}{lllll|l}
\overset{b}{1} + \overset{c}{a}) & \overset{d}{1} & & & & \overset{e}{1}\\
& \underline{\overset{f}{1} + \overset{g}{a}} & & & & \\
& \overset{h}{0} - \overset{i}{a} & & & & \overset{k}{-a}\\
& & \underline{\overset{l}{-a} - \overset{m}{aa}} & & & \\
& & \overset{n}{0} + \overset{o}{aa} & & & \overset{p}{+aa}\\
& & & \underline{\overset{q}{aa} + \overset{r}{a^3}} & & \\
& & & \overset{s}{0} - \overset{t}{a^3} & & \overset{u}{-a^3}
\end{array}$$

Applica $\overset{b}{1} + \overset{c}{a}$ ad $\overset{d}{1}$, oritur $\overset{e}{1}$; tum $\overset{e}{1}$ in $\overset{b}{1} + \overset{c}{a}$ producit $\overset{f}{1} + \overset{g}{a}$ ſubducendum ex $\overset{d}{1}$, & reſtat $\overset{h}{0} - \overset{i}{a}$. Rurſus $\overset{b}{1} + \overset{c}{a}$ applicetur ad $\overset{h}{0} - \overset{i}{a}$, oritur $\overset{k}{-a}$; tum $\overset{k}{-a}$ in $\overset{h}{0} - \overset{i}{a}$ producit $\overset{l}{-a} - \overset{m}{aa}$, ſubducendum ex $\overset{h}{0} - \overset{i}{a}$, & reſtat $\overset{n}{0} - \overset{o}{aa}$. Ad hoc applica $\overset{b}{1} + \overset{c}{a}$, oritur $\overset{p}{+aa}$; quod ductum in $\overset{b}{1} + \overset{c}{a}$ gignit $\overset{q}{aa} + \overset{r}{a^3}$, ſubducendum ex $\overset{n}{0} + \overset{o}{aa}$, & reſtat $\overset{s}{0} - \overset{t}{a^3}$. Atque ita continuatâ operatione, deprehenditur $\frac{1}{1+a} = 1 - a + aa - a^3 + a^4$ (&c.) $=$ FH.

PROPOSITIO XVI.

Quovis numero in partes æquales innumeras diſcerpto; invenire ſummam quarumvis poteſtatum ab innumeris iſtis numeris genitarum.

Numeri dati poteſtas proximè ſuperior poteſtatibus quæſitis, ſi dividatur per exponentem ſuum, extabit ſumma poteſtatum quæſita.

V. gr. Numerus datus ſit 21; hic ſi diſcerpatur in partes innumeras, continebit non modò hos numeros 20, 19, 18, 17, &c. ſed & innumeros interjectos, quorum quiſque intelligitur ductus in unam partem infinitiſſimam numeri 21. Horum igitur omnium productorum ſummam ſi quæras; quoniam ipſa producta ſunt poteſtates primæ (ſive lineæ); erit poteſtas proximè ſuperior quadratica; & ejus exponens 2. Ergo dati numeri 21 quadratum 441, ſi dividatur per exponentem

exponentem 2, extabit ſumma omnium primarum poteſtatum, genitarum ab innumeris iſtis numeris, qui in dato numero 21 continentur, nimirum 220|5. Rurſus quævis poteſtas prima intelligatur ducta in ſeipſam, & oporteat nos invenire ſummam omnium iſtorum quadratorum. Poteſtas proximè ſuperior eſt cubica, & ejus exponens 3. Ergo dati numeri 21 cubus 9261, ſi dividatur per exponentem 3, extabit ſumma omnium quadratorum 3087. Horum quadratorum quodvis ducatur in ſuum latus, & oporteat nos invenire ſummam omnium iſtorum cuborum. Poteſtas proximè ſuperior eſt quadrato-quadratica, & ejus exponens 4. Ergo dati numeri 21 quadrato-quadratum 194481, ſi dividatur per exponentem 4, extabit ſumma omnium cuborum 48620|25.

Demonſtratio. Summa omnium ab unitate imparium æqualis eſt quadrato numeri terminorum; ſic numerus terminorum omnium imparium ab unitate uſque ad 21 eſt 11, cujus quadratum 121 æquale eſt ſummæ omnium horum imparium; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21. At idem quadratum 121 duplicatum, nimirum 242, excedit ſummam omnium eorundem imparium unà cum paribus incluſis ipſo numero terminorum 11; deficit autem à ſumma omnium parium æque ac imparium eodem numero terminorum 11. Ergo quadratum duplicatum numeri terminorum imparium non poteſt excedere vel deficere à ſumma omnium, tam parium quàm imparium, pluſquam ipſo numero terminorum imparium, hoc eſt (ſi termini ſint innumeri) eodem numero terminorum ſive dimidio termini maximi, ducto in partem infinitiſſimam numeri dati. Quod productum ſi quis putet, adhuc rationem aliquam obtinere ad ſummam omnium terminorum; nondum utique diviſus eſt numerus datus in partes innumeras, quod eſt contra hypotheſin. Ergo quadratum dimidii numeri terminorum (tam parium quàm imparium) duplicatum; vel, quod idem eſt, dimidium quadrati numeri omnium terminorum (tam parium quàm imparium) æquale eſt ſummæ omnium terminorum.

Rurſus; numerus pyramidalis ultimi ab unitate imparium, æqualis eſt ſummæ omnium quadratorum ab iiſdem imparibus factorum. Sic numeri 21, tanquam ultimi imparium, pyramidalis 1771, æqualis eſt ſummæ omnium quadratorum factorum ab his imparibus; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21. Unde haud ſecùs, ac modò, conficietur eundem pyramidalem duplicatum (ſi termini ſint innumeri); vel, quod idem eſt, trientem cubi facti à numero dato, æqualem eſſe ſummæ omnium quadratorum ab imparibus æquè ac paribus factorum.

Item; ultimi cujuſvis trigonus in ſe ductus, æqualis eſt ſummæ omnium cuborum ab imparibus æquè ac paribus factorum. Sed trigonus iſte, ſive ſumma terminorum, ſuprà æqualis erat $\frac{\text{Quadrato}}{2}$, ergo ejuſdem. Trigoni quadratum æquale eſt $\frac{\text{Quadrato quadrato}}{4} =$ ſummæ omnium cuborum. Atque ita deinceps,

PROPOSITIO XVII.

Quadrare Hyperbolam.

In diagrammate præcedenti, poſita AI $=$ I; intelligatur aſymptoto inde ab I verſus E diviſa in partes æquales innumeras, quæ ſint, v. gr. Ip $=$ pq $=$ qr $= a$. Erit, per XIV & XV hujus, ps $= 1 - a + aa - a^3 + a^4$ &c, & qt $= 1 - 2a + 4aa - 8a^3 + 16a^4$ &c, & ru $= 1 - 3a + 9aa - 27a^3 + 81a^4$ &c. Sed ps $+$ qt $+$ ru $=$ areæ BIru $=$

$$= \left\{\begin{array}{l} 1 - a + aa - a^3 + a^4 \\ 1 - 2a + 4aa - 8a^3 + 16a^4 \\ 1 - 3a + 9aa - 27a^3 + 81a^4 \end{array}\right\} \&c =$$
$$= 3 - 6a + 14aa - 36a^3 + 98a^4,$$

hoc eſt, $=$ numero terminorum contentorum in linea Ir, minus ſummâ eorundem terminorum, plus ſummâ quadratorum ab iiſdem, minus ſummâ cuborum, plus ſummâ quadrato-quadratorum, &c.

Hinc poſito, ut antè, IA $=$ I; ſed Ip $=$ 0⌊1 $=$ numero terminorum: invenio, per XV & XVI hujus, aream BIps $=$ numero terminorum $=$ 0⌊1, minus ſumma eorundem terminorum $=$ 0⌊005, plus ſumma quadratorum ab iiſdem $=$ 0⌊000333333, minus ſumma cuborum $=$ 0⌊000025, plus ſumma quadrato-quadratorum $=$ 0⌊000002, minus ſumma quadrato-cuborum $=$ 0⌊0000000166, plus ſumma cubo-cuborum $=$ 0⌊000000014, &c.

$$\begin{array}{rl|rl} + & \left\{\begin{array}{l} 0\lfloor 1 \\ 0\lfloor 000333333 \\ 0\lfloor 000002 \\ 0\lfloor 000000014 \end{array}\right. & - & \left\{\begin{array}{l} 0\lfloor 005 \\ 0\lfloor 000025 \\ 0\lfloor 000000166 \end{array}\right. \\ \hline + & 0\lfloor 100335347 & - & 0\lfloor 005025166 \\ - & 0\lfloor 005025166 & & \\ \hline + & 0\lfloor 095310181 = \text{areæ BIps.} & & \end{array}$$

Sic poſito Iq $=$ 0⌊21 $=$ numero terminorum: invenio, per XV & XVI hujus, aream BIqt $=$ numero terminorum $=$ 0⌊21, minus ſumma eorundem terminorum $=$ 0⌊02205, plus ſumma quadratorum ab iiſdem $=$ 0⌊003087, minus ſumma cuborum $=$ 0⌊000486202, plus ſumma quadrato-quadratorum $=$ 0⌊000081682, minus ſumma quadrato-cuborum $=$ 0⌊000014294, plus ſumma cubo-cuborum $=$ 0⌊000002572, minus ſumma quadrato-quadrato cuborum $=$ 0⌊000000472, plus ſumma quadrato-cubo-cuborum $=$ 0⌊000000088.

$$
\begin{array}{rl|rl}
+ & \left\{\begin{array}{l} 0\lfloor 21 \\ 0\lfloor 003087 \\ 0\lfloor 00008168 2 \\ 0\lfloor 000002572 \\ 0\lfloor 000000088 \\ 0\lfloor 000000003 \end{array}\right. & - & \left\{\begin{array}{l} 0\lfloor 02205 \\ 0\lfloor 000486202 \\ 0\lfloor 000014294 \\ 0\lfloor 000000472 \\ 0\lfloor 000000016 \end{array}\right. \\
\hline
+ & 0\lfloor 213171345 & - & 0\lfloor 022550984 \\
- & 0\lfloor 022550984 & & \\
\hline
+ & 0\lfloor 190620361 = \text{areæ BIqt.} & &
\end{array}
$$

Unde apparet, ut ratio AI ad AD (1 ad 1⌊1 eſt dimidiata rationis AI ad Aq (1 ad 1⌊21); ita aream BIps eſſe dimidiam areæ BIqt. Cæterum proclive eſt hunc calculum extendere ad quotvis loca, quod mihi tentanti, prodiit area BIps = 0⌊09531017980432486004395212328076509222060534; & area BIqt = 0⌊19062035960864972008790424656153018444121072, quam cùm exactè duplam deprehenderem iſtius ſuperioris ſcivi inde me calculum rectè poſuiſſe.

PROPOSITIO XVIII.

Comparare Areolas hyperbolicas cum Ratiunculis abſolutorum Æquidifferentium.

In diagrammate præcedenti, poſitâ AI = 1, & aſymptoto inde ab I verſus B diviſâ in partes æquales innumeras, quæ ſint, v. gr. Ip, pq, qr: erit areola BIps menſura ratiunculæ, quam AI obtinet ad Ap; & areola spqt menſura ratiunculæ, quam Ap obtinet ad Aq; & areola tqru menſura ratiunculæ, quam Aq obtinet Ar, &c. Atque areolæ iſtæ ſupputantur prorſus eodem modo, quo ſuprà propoſitione VIII & IX rationes terminorum æquidifferentium. Id quod paucis indicare oportunum duxi.

PROPOSITIO XIX.

Invenire ſummam Logarithmorum.

In eodem diagrammate, poſitâ AI = 1, & aſymptoto inde ab I verſus E diviſâ in partes æquales innumeras, quæ ſint, v. gr. Ip, pq, qr, oportet invenire ſummam areolarum; BIps + BIqt + BIru (&c) — ſumma logarithmorum = ſolido, conſtanti ex areola BIps perpendiculariter inſiſtente lineæ ps, & areola BIqt perpendiculariter inſiſtente lineæ qt, & areola BIru perpendiculariter inſiſtente lineæ ru &c, ductis nimirum ſingulis in unam infinitiſſimam lineæ datæ.

Conſtructio hujus problematis congruit cum conſtructione propoſitionis XVII, ſubſtituendo nimirum, pro numero terminorum, ſummam eorundem; & pro ſumma terminorum, ſummam quadratorum; & pro ſumma quadratorum, ſummam cuborum; &c. Sic poſitâ AI = 1, & Ip = 0⌊1, oporteat nos invenire ſummam omnium logarithmorum inde ab I ad 0⌊1.

$$
\begin{array}{rl|rl}
+ & \left\{\begin{array}{l} 0\lfloor 005 \\ 0\lfloor 000008333 \\ 0\lfloor 000000033 \end{array}\right. & - & \left\{\begin{array}{l} 0\lfloor 000166666 \\ 0\lfloor 0000005 \\ 0\lfloor 000000002 \end{array}\right. \\
\hline
+ & 0\lfloor 005008366 & - & 0\lfloor 000167168 \\
- & 0\lfloor 000167168 & & \\
\hline
 & 0\lfloor 004841198 = \text{ſummæ omnium logarithmorum.} & &
\end{array}
$$

Hinc patet, quomodo productum continuum omnium à 0 ad numerum datum arithmeticè progredientium inveniri queat. Nam ſumma logarithmorum, eſt logarithmus producti continui.

Patet quoque ex præcedentibus, quo pacto problema Merſennianum, ſi non geometricè ſaltem in numeris, ad quotvis uſque locos ſolvi poſſit. Atque hîc jam filum abrumpere cogor, tantiſper dum otium pertexendi reliqua largiatur Deus.

Note to prop. XI, referred to in p. 187.

* In demonſtratione hujus propoſitionis undecimæ auctor noſter in plures calculi errores lapſus eſt. Etenim dicit cubum quantitatis binomiæ $a + \frac{3b}{2}$ eſſe $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{35b^3}{8}$, cum reverà iſte cubus ſit $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{27b^3}{8}$; & dicit cubum quantitatis binomiæ $a + \frac{7b}{2}$ eſſe $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$, cum reverà iſte cubus ſit $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$. Hi autem errores non ſunt neceſſarii ad ſtabilienda auctoris ratiocinia, quæ aliquanto propiùs ad veritatem accedent, ſi hi errores corrigantur. His autem correctis, demonſtratio tota erit ut ſequitur.

Sint tres rationes continuæ $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, quarum differentia differentiarum eſt $\frac{a^4 + 6a^3b + 12aabb + 8ab^3}{a^4 + 6a^3b + 12aabb + 10ab^3 + 3b^4}$, & mediæ illarum medium arithmeticum eſt $a + \frac{3b}{2}$, cujus cubus $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{27b^3}{8}$; ſint verò & aliæ tres continuæ $\frac{a+2b}{a+3b}$, $\frac{a+3b}{a+4b}$, $\frac{a+4b}{a+5b}$, quarum differentia differentiarum eſt $\frac{a^4 + 14a^3b + 72aabb + 160ab^3 + 128b^4}{a^4 + 14a^3b + 72aabb + 162ab^3 + 135b^4}$, & mediæ illarum medium arithmeticum $a + \frac{7b}{2}$, cujus cubus $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$. Cæterùm ſingulæ rationes differunt communi exceſſu b; at differentiæ differentiarum non ſunt terminorum æquidifferentium, ſiquidem differentia terminorum prioris eſt $2ab^3 + 3b^4$, at poſterioris $2ab^3 + 7b^4$: ſecùs ac in præcedenti propoſitione. Quare cum illic res expediretur regulâ proportionum ſimplici inversâ, hic opus eſt duplici inversâ; nimirum:

Ut medium arithmeticum prioris differentiæ differentiarum (nimirum $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$) ductum in differentiam terminorum poſterioris ($2ab^3 + 7b^4$) ad medium arithmeticum poſterioris differentiæ differentiarum (nimirum $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$) ductum in differentiam terminorum prioris ($2ab^3 + 3b^4$): Ita differentia differentiarum prior ad poſteriorem: Ita quoque cubus medii arithmetici mediæ trium priorum rationum (nimirum $a^3 + \frac{9aab}{2} + \frac{27aab}{4} + \frac{27b^3}{8}$) ad cubum medii arithmetici mediæ trium poſteriorum rationum (nimirum $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$).

Quæ analogia vera eſſe deprehendetur, ſi productum extremorum æquale ſit producto mediorum.

Atqui primus terminus $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$ in $2ab^3 + 7b^4 = 2a^5b^3 + 19a^4b^4 + 66a^3b^5 + 102aab^6 + 66ab^7 + \frac{21b^8}{2}$ ſi ducatur in quartum $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$; productum eſt $2a^8b^3 + 40a^7b^4 + 339a^6b^5 + 1579a^5b^6 + 4377\frac{1}{8}a^4b^7$, &c.

Rurſùs ſecundus terminus $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$ in $2ab^3 + 3b^4 = 2a^5b^3 + 31a^4b^4 + 186a^3b^5 + 538aab^6 + 746ab^7 + \frac{789b^4}{2}$, ſi ducatur in tertium $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{27b^3}{8}$; productum eſt $2a^8b^3 + 40a^7b^4 + 339a^6b^5 + 1591a^5b^6 + 4527a^4b^7$, &c.

Hoc igitur productum cum conſentiat cum iſto, non modò in primis & ſecundis & tertiis ſpeciebus, ſed & in maximâ parte quartarum & etiàm quintarum; aio analogiam in propoſitione memoratam eſſe veram. Nam defectus, qui hîc apparet in productis terminorum, in ipſis terminis longè minor erat; quippe qui multiplicando crevit. Ut taceam in minoribus rationibus differentias ſecundas & tertias nullius ferè momenti eſſe.

MICHAELIS ANGELI RICCII

EXERCITATIO GEOMETRICA

DE MAXIMIS ET MINIMIS.

ABBATI
STEPHANO GRADIO
MICHAEL ANGELUS RICCIUS, S. P. D.

SCRIPTIONEM hanc meam argumenti, ut vides, inter Mathematica difficillimi, ſed æquè ad difficiliora quæque Problematum efficienda, & obſcuriora Theorematum cognoſcenda utiliſſimi, cùm in Geometrico, tùm in Analytico pulvere, ad te mittendam duxi, vir ornatiſſime, Stephane Gradi, quem ego unum omnium hujus Civitatis plurimi facio, ob egregias animi laudes, præſertim verò propter acre his de rebus exiſtimandi judicium, quotidianis graviſſimarum inter nos diſputationum experimentis mihi perſpectum & cognitum. Lege quæſo illam diligenter, & ubi diù exactiſſimæ tuæ cenſuræ ſubjectam habueris, ecquid reſpondeat ſolitæ tuæ de meis hoc in genere cogitationibus, opinioni, pronuncia. Nam ſi hoc aſſequar, ut tibi cæteriſque Amicis earundem Diſciplinarum intelligentibus probetur, minus erit in poſterum quam ob rem humaniſſimis tuis hortationibus obluctер, cùm autor mihi eſſe perſeverabis edendi alia quæ tecum jampridem communicavi, de præceptis univerſæ Artis analyticæ, geometricâ methodo breviter & expeditè demonſtratis, unà cum animadverſione erratorum quæ in ipſis tradendis magni nominis Auctores erraſſe deprehendi; faciliuſque obtinebis ne diutiùs premam apud me quæcumque de Geometria in genere diſputata & literis conſignata in certas propoſitiones redegi; & ex his illam præcipuè à Torricellio, & à

te quoque tantopere commendatam, quæ integram doctrinam triginta propositionum Archimedis, Lucæ Valerii, & aliorum, una complectitur; duasque præterea, quibus totam penè Jo. Caroli de la Faille de centro gravitatis partium circuli & ellipseos doctrinam [justo volumine ab ipso explicatam] absolvo. Statui autem pauca aliquot hujus scripti exemplaria typis imprimere, quò commodiùs possint ad peritos hujusmodi scientiarum Amicos, tùm per Italiam, tùm exteras apud gentes pervenire, accenso potiùs ea in re tuo studio obsecutus, quàm ingenio meo. Neque enim is ego sum, cui nomen famæ per ambitionem ingerere libeat; aut quem non magis indagatæ veritatis cognitio, quàm cognitæ ostentatio delectet. Interim hunc amicitiæ nostræ jampridem institutæ, & literario præcipuè commercio nunquàm coli intermissæ, fructum jucundissimum feram, ut quæ hac in re de me sentis amicè, hoc est [ut Euripidi placet] liberè, te loquentem audiam; eoque, quid cæteri & sentiant & loquantur, securus fiam. Vale. Romæ, octavo Idus Julii 1666.

MICHAELIS

MICHAELIS ANGELI RICCII

GEOMETRICA EXERCITATIO.

DEFINITIONES.

1 POTESTATEM quamlibet, ejusque radicem, voco dignitatem.

2 Si dignitas in dignitatem ducatur, ut A^2 in B^3, fiet productum A^2 in B^3; cui producto illud simile dicimus, quod gignitur ex dignitatibus graduum eorumdem. Ita, in facta hypothesi, productum E^2 in C^3, ex quadrato & cubo, simile est producto A^2 in B^3.

3 Homogenea producta sunt quæ ad eundem gradum pertinent; ut duo rectangula, quippe quæ ad secundum gradum pertinent; & duo solida, quæ ad tertium.

4 Terminos cùm dico, intelligi volo duos numeros seu æquales seu inæquales, vel numerum & unitatem, vel duas unitates. Terminos inæquales appello duos numeros inæquales, vel numerum & unitatem. Terminos autem æquales, duos æquales numeros, vel duas unitates.

5 Productum in linea fieri secundum terminos datos, aut positos, dicimus, quum illud fit ex duabus dignitatibus, quarum exponentes sunt ipsi termini dati vel positi; radices verò segmenta illius rectæ lineæ sectæ in proportione terminorum eorumdem.

Sit, verbi causa, quæpiam recta linea, cujus majus segmentum ad minus sit in ratione 3 ad 2; productum ex cubo segmenti majoris in quadratum minoris erit factum in linea data secundum terminos positos 3, & 2; quia segmenta quæ sunt dignitatem radices habent rationem numeri 3 ad 2, & exponentes earumdem dignitatum sunt etiam 3, & 2.

Rursus esto, quemadmodum segmentum majus ad minus ejusdem lineæ, sic 3 ad 1, productum ex cubo majoris segmenti in segmentum ipsum minus, erit productum in linea factum secundum terminos positos, numerum & unitatem.

 tem.

tem. Ita, A^3 in B^1 [ſi A vocetur majus ſegmentum, B verò minus] eſt productum factum in linea $A + B$ ſecundum terminos 3, & unitatem, quia radices A & B ſic ſunt, ut eſt numerus 3 ad unitatem; & dignitatis A^3 exponens eſt, 3, numerus datus; dignitatis B^1 exponens eſt, unitas, item data.

LEMMA PRIMUM.

Si duæ rectæ in eadem ratione ſecentur, producta ſimilia facta ex ſegmentis tanquam ex radicibus, erunt proportionalia productis homogeneis quæ fient ex totis.

Sint AB, DE, rectæ, in punctis C, & F ita ſectæ, ut quam rationem AC ad CB habet, eandem habeat DF ad FE, & fiant ex illarum ſegmentis producta AC^2 in CB^3, & DF^2 in FE^3, quæ ſunt ſimilia per ſecundam definitionem; iiſque homogenea producta fiant ex totis AB, DE, nimirum AB^5, DE^5 per tertiam definitionem. Dico AC^2 in CB^3 eandem rationem habere ad AB^5, ac DF^2 in FE^3 ad DE^5. Quia rationes ex quibus ratio producti AC^2 in CB^3 ad AB^5 componitur, eædem ſunt, ac componentes rationem producti DF^2 in FE^3 ad DE^5, ob ſectionem linearum proportionalem, & inde proportionales dignitates ex quibus producta illa reſultant. Quod, &c.

LEMMA SECUNDUM.

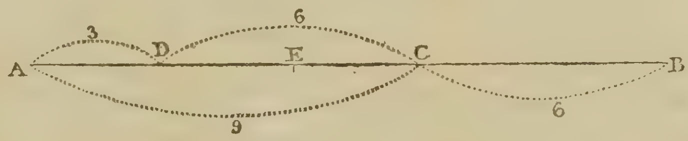
Iiſdem poſitis, dico, AC^2 in CB^3 fuerit maximum omnium ſimilium productorum ex binis ſegmentis rectæ AB, etiam DF^2 in FE^3 fore maximum productorum ſimilium ex binis ſegmentis rectæ DE, tanquam ex radicibus.

Singulis enim productis ex ſegmentis rectæ DE alia reſpondent orta ex ſegmentis rectæ AB in eadem proportione ſectæ; & illa ad homogeneum ſuum DE^5 eandem rationem habent, atque iſta ad ſuum AB^5, ex primo lemmate. Ratio quidem AC^2 in CB^3 ad AB^5 ex hypotheſi eſt eadem ac ratio DF^2 in FE^3 ad DE^5: cæterorum verò productorum ex ſegmentis ipſius DE ad DE^5, eadem eſt atque ratio productorum ſibi reſpondentium, quæ fiunt ex ſegmentis rectæ AB, ad AB^5. Cùm igitur ratio AC^2 in CB^3 [quod maximum eſſe ponitur] ad AB^5 ſit major, per octavam quinti Elem. ratione cæterorum productorum ſibi ſimilium ad AB^5; major etiam erit ratio DF^2 in FE^3 ad DE^5, quàm ratio cæterorum ſimilium productorum ex ſegmentis rectæ DE ad DE^5; ac proinde ipſum DF^2 in FE^3 per decimam quinti Elem. eſt maximum. Quod, &c.

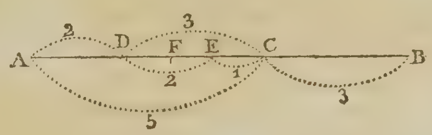
LEMMA TERTIUM.

Si data recta linea ſecetur in ratione terminorum inæqualium, & dividendo, fiat

fiat ſegmentorum differentia ad minus ſegmentum, ut differentia terminorum ad minorem terminum; hæc inventa proportionalitas vel ipſa erit proportionalitas æqualitatis, vel alia, in quam incidemus, iterum dividendo, & ſic deinceps; & in eâ terminorum differentia æquabitur minori termino, & differentia ſegmentorum ſegmento minori.



Eſto AC ad CB, ut 9 ad 6, & AD differentia ſegmentorum AC, CB: erit dividendo, 3 ad 6, ut AD ad CB, vel ad ſegmentum ſibi æquale, DC: Quoniam verò hæc proportio non eſt proportio æqualitatis, fiat DE differentia ſegmentorum AD, & DC; 3, differentia numerorum 6 & 3; & dividendo, erit ut 3 ad 3, ſic DE ad AD, proportio æqualitatis.



Rurſus AC ſit ad CB, ut 5 ad 3; & AD ſegmentorum differentia; dividendo erit, AD ad CB, ſeu ad ſibi æquale ſegmentum DC, ut 2 ad 3. Et iterum dividendo [ſegmentorum AD & DC, eſto, differentia, EC], 1 ad 2 ut EC ad AD ſeu DE; & tertiò [facta FE terminorum DE & EC differentia] dividendo inveniemus, ut 1 ad 1, ita FE ad EC. Quod, &c.

Ratio lemmatis eſt, quod duorum quorumcumque numerorum differentia, vel differentia numeri & unitatis, ſemper eſt numerus aut unitas, ut per ſe patet: & nos dividendo, ſemel atque iterum, ac ſæpius, demimus ſemper minorem terminum diviſæ proportionalitatis qui eſt numerus vel unitas, de majori termino ſeu numero, utimurque deinceps reſiduo tantùm [quod eſt eorum terminorum differentia] & comparamus illud cum minori termino proportionalitatis diviſæ: at non poſſumus ſic demendo progredi in infinitum, quia unitates in terminis ſunt finitæ, ſed exhauritur tandem omnis differentia, reſiduumque majoris termini proportionalitatis diviſæ æquatur termino minori. Ita fit proportio æqualitatis, in qua unitas ad unitatem, vel numerus ad ſibi æqualem numerum, eſt ut ſegmentum ad aliud æquale ſegmentum. Quod oſtendere oportebat.

Quod ſi ab ea proportione æqualitatis, in qua deſitum eſt, rurſus incipiamus, dico nos componendo gradatim, venturos per veſtigia diviſionis ad terminos primæ proportionalitatis, in qua ſegmenta datæ lineæ erant in ratione inæqualium terminorum. Cujus propoſitionis rationem facilè intelliget Geometra, quem latere non poteſt, in Geometria omnia quæ dividendo concluduntur, ex

 contrario

contrario converti posse, & componendo concludi illud ipsum, quod ponebatur ante divisionem, ut in quinto Elementorum ostenditur. Exempli gratia, sit majus segmentum datæ rectæ ad minus, ut 2 ad 1. Igitur dividendo 1 ad 1, est ut differentia segmentorum ad minus segmentum. Ex hac porrò æqualitatis proportione componendo redimus ad primam proportionem, in qua segmenta erant in ratione 2 ad 1. Quod, &c.

LEMMA QUARTUM.

Si duo quælibet producta orta sint ex duabus dignitatibus ductis in aliam communem dignitatem; quam rationem habent illæ duæ dignitates inter se, eandem habent duo producta. Sic productum AB^3 in BC^5 eam rationem habet ad productum AB^3 in EP^5, quam habet dignitas BC^5 ad dignitatem EF^5; in quas duas dignitates ducta communis dignitas AB^3 illa producta effecit.

Ex definitione multiplicationis probatur hoc lemma, quod alii in numeris demonstrarunt.

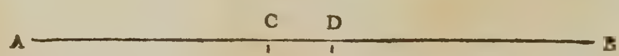
LEMMA QUINTUM.

Datis quatuor quantitatibus, quarum prima ad secundam habeat minorem rationem, quam tertia ad quartam, productum quod gignitur ex duabus extremis est minus producto ex mediis.

Augeatur prima donec fiant quatuor geometricè proportionales; tunc prima in quartam ducta efficiet productum æquale producto ex mediis. Igitur productum quod efficiebat antequàm augeretur, erat productum minus eodem producto ex quantitatibus mediis. Quod, &c.

THEOREMA PRIMUM.

Productum in aliqua recta linea factum secundum positos terminos æquales, maximum est omnium similium productorum, quæ fieri possunt ex binis lineæ datæ segmentis tanquam ex radicibus.

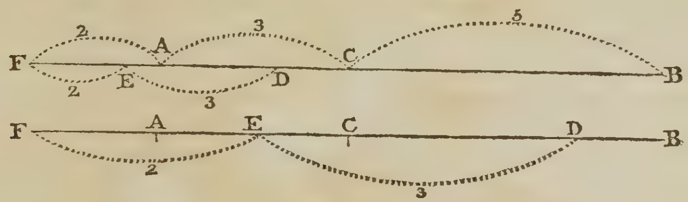


Recta linea AB secetur æqualiter in puncto C, & sit AC ad CB ut 3 ad 3 [termini æquales positi] dico productum AC^3 in CB^3, quod fit in linea AB secundum positos terminos, esse omnium similium productorum maximum. Sumpto quolibet alio puncto D, faciamus aliud simile productum AD^3 in DB^3. Cùm autem sint quatuor lineæ arithmeticè proportionales cum excessu CD, nimirum AD, AC, CB, & BD, minor est ratio maximæ AD ad AC, quàm CB ad BD; & triplicata ratio ipsius AD ad AC [seu ratio AD^3 ad AC^3] minor est, quàm triplicata ipsius CB ad BD [seu CB^3 ad BD^3] & per quintum lemma, productum ex mediis quantitatibus, AC^3, in CB^3, majus est producto AD^3 in BD^3 facto

facto ex duabus extremis. Eodem pacto demonstratur AC^3 in CB^3 esse alio quocumque simili producto majus, & consequenter omnium similium maximum. Quod, &c.

THEOREMA SECUNDUM.

Si duo rectæ lineæ segmenta fuerint in ratione terminorum inæqualium, &, per consequens, dividendo, sit differentia segmentorum ad minus segmentum, ut differentia terminorum ad minorem terminum; quoties ex dignitate differentiæ segmentorum ducta in dignitatem minoris segmenti fit productum maximum, toties fit etiam maximum ex eadem dignitate minoris segmenti ducta in dignitatem majoris; atque ita, si dignitates segmentorum pro exponentibus habeant terminos positos, & dignitas differentiæ, differentiam terminorum.



Sit AB recta linea inæqualiter secta in puncto C, & BC ad AC, ut 5 ad 3, qui sint termini positi. Producatur BA in F, donec æquetur FC ipsi CB, & AF erit differentia segmentorum BC & AC. Quoniam vero segmentum majus BC sic est ad minus CA, ut 5 est ad 3, erit dividendo AF ad CA, ut est 2 ad 3. Nunc fiant duo producta qualia diximus, primum FA^2 in AC^3, ex dignitate ipsius FA, differentiæ segmentorum, ducta in dignitatem minoris segmenti. AC. Secundum AC^3 in CB^5, ortum ex eadem dignitate minoris segmenti ducta in dignitatem majoris. Prima dignitatis FA^2 habet pro exponente, 2, differentiam datorum terminorum, reliquæ habent 3 & 5, terminos positos, ut imperabatur. Dico, si productum primum est maximum omnium similium ex binis segmentis rectæ FC [esse autem ejusmodi supponamus], etiam secundum fore productum maximum omnium similium ex binis segmentis rectæ positæ AB.

Sumatur in AB alius punctus præter punctum C, & esto D; qui accipi à nobis potest infra punctum C, vel supra. In utroque casu, FA nequit habere eam rationem ad AC, quam habet ad AD, sed majorem aut minorem habebit, atque adeò FD non est secta in puncto A secundum rationem ipsius FA ad CA: fiat porrò FE ad ED, ut FA ad AC, & productum FE^2 in ED^3, per secundum lemma, erit maximum [æque ac productum FA^2 in AC^3] & consequenter majus simili producto FA^2 in AD^3, facto ex segmentis ejusdem rectæ FD. Quod maximum FE^2 in ED^3 habet eandem rationem ad FD^5, dignitatem sibi homogeneam, quam FA^2 in AC^3 ad FC^5, ut ex duobus primis lemmatibus colligitur; igitur FA^2 in AD^3 [quod diximus esse minus producto FE^2 in ED^3] minorem rationem habet ad FD^5, quàm FE^2 in ED^3 ad idem FD^5, seu minorem, quàm FA^2 in AC^3 ad FC^5; & permutando, FA^2 in AD^3 minorem habet rationem ad FA^2 in AC^3

[seu,

[ſeu, per lemma quartum, AD^3 minorem habet ratio ad AC^3] quàm FD^5 ad FC^5, & longè minorem, quàm CB^5 ad BD^5. Quippe ſunt rectæ DB, CB, FC, & FD, arithmeticè proportionales cum exceſſu, DC; ac propterea in primo caſu, FD maxima, in ſecundo caſu, FD minima, eſt ad FC in minori ratione quàm CB ad DB, & quintuplicata ratio FD ad FC, nempe ratio ipſius FD^5 ad FC^5, eſt minor quintuplicatâ ratione CB ad DB, ſeu CB^5 ad DB^5.

Igitur cùm quatuor quantitatum, AD^3, AC^3, CB^5, & DB^5, prima ad ſecundam habeat minorem rationem, quàm tertia ad quartam, per quintum lemma, productum AD^3 in DB^5 factum ex duabus extremis erit minus producto AC^3 in CB^5 ex mediis. Similiter oſtendes, aliud quodcumque productum ſimile minus eſſe producto AC^3 in BB^5, quia punctus D ad libitum ſumitur. Ergo AC^3 in CB^5 productum eſt maximum omnium. Quod, &c.

Hactenus de recta linea AB inæqualiter ſecta, quum eſt ſegmentum majus ad minus, uti numerus ad numerum. Reſtaret altera pars theorematis, quum eſt quemadmodum majus ſegmentum ad minus, ſic numerus ad unitatem. Hoc tamen conſtructione ac ratione tam ſimilibus modò factis concluditur, ut id ſibi quiſque invenire, explicare ac dilatare facillimè poſſit. Lectoribus autem ſcribimus à Geometria & ab Algebra inſtructioribus, quos hujuſcemodi rerum intellectu facilium explicatione fruſtra defatigaremus; quare pergimus ad reliqua uſum præſtantiſſimum habentia ad inveniendas plurium linearum tangentes, figurarum centra gravitatis & quadraturas, & ad alia item multa, quæ juſto ſervamus operi; ubi dabimus novam ſolidorum conicorum ſeriem, qui ſecti exhibent infinitas, uti vocant, hyperbolas, infinitas parabolas, infinitas ellipſes, &, analogiam ſervando, circulos etiam infinitos. Unde lectoribus manifeſtè apparebit, de conicis me plus multò adinveniſſe, quàm cæteros, eoſque ingenioſiſſimos viros, qui communem tantùm hyperbolen, parabolen, ellipſim, & circulum [figuras conici in noſtra nova ſerie prædicta, ſecundi gradus] agnoverunt: alias tertii & quarti & cæterorum non item; niſi quod de parabolis infinitis per puncta in plano deſcriptis pauca, licèt cognitione digniſſima, tradidere nonnulli, quos inter, duo præcellentes ingenio viri, Fermatius, ac Torricellius, præceptor meus, inventorum præſtantiâ & numero commendabiles, ac veteribus proximi; qui novum inſuper excogitarunt hyperbolarum infinitarum genus. Neque prætereundum puto, quamplures Apollonii propoſitiones atque demonſtrationes aptari ſectionibus noſtris & per omnia congruere, affectaſque multipliciter æquationes harum ſectionum ope reſolvi facillimè, & determinari poſſe. Nunc revertor ad rem.

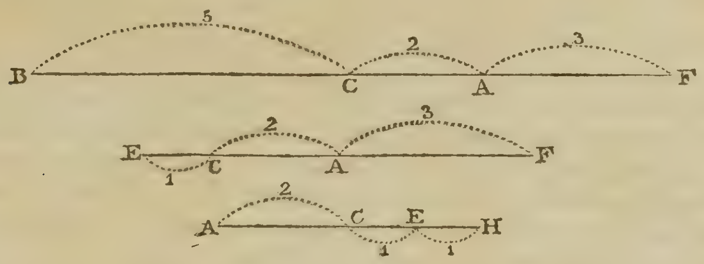
THEOREMA TERTIUM.

Datâ rectâ lineâ, & duobus terminis, ſecundum quos fiat in linea data productum: hoc erit maximum omnium ſimilium productorum, quæ fieri poſſunt ex binis ejuſdem rectæ ſegmentis, velut ex radicibus.

Propoſitionem ſeco in partes duas. Primùm dico, productum, quale deſcripſimus, eſſe omnium ſimilium maximum, quum dantur termini æquales; quod in primo theoremate demonſtravimus.

Deinde

Deinde ſi dantur termini inæquales, ſic rem oſtendo.



Eſto AB recta data, & termini dati 5 & 2. Secetur recta in puncto C ſitque BC ad CA, ut 5 ad 2. Dico productum BC^5 in CA^2 factum in linea data ſecundum terminos datos eſſe maximum. Producatur BA in F, ut AF ſit differentia ſegmentorum, & dividendo primam proportionalitatem, nempe BC ad CA, ut 5 ad 2 [ſicut in tertio lemmate præſcribitur] pergamus uſque dum incidamus in proportionem æqualitatis. In noſtra hypotheſi, primum erit, dividendo, 3 ad 2, ut FA differentia ſegmentorum ad AC minus ſegmentum; quam ſecundam proportionalitatem exhibet ſecunda figura, in qua fiat CE differentia ſegmentorum CA, FA; per conſequens erit, dividendo, 1 ad 2, ut CE ad AC; quam quidem proportionalitatem ſeorſim exhibet tertia figura. Fiat EH differentia ſegmentorum CE & AC, dividendo erit 1 ad 1, ut EH ad EC; quæ eſt demum proportio æqualitatis; ſemper autem minus ſegmentum producimus ut æquemus majori, & ſegmentorum differentiam conſtituamus.

At retrorſum viciſſim, incipiendo à recta EA tertiæ figuræ cujus majus ſegmentum AC eſt 2, minus ſegmentum CE eſt 1, & illorum differentia HE itidem 1. Quoniam productum HE^1 in CE^1 eſt maximum in linea CH, per primum theorema noſtrum, erit proinde, per ſecundum theorema, EC^1 in CA^2 maximum in recta EA.

Deinde in recta FC ſecundæ figuræ, majus ſegmentum AF eſt 3, minus AC eſt 2, & ſegmentorum differentia EC eſt 1; porrò cum EC^1 in CA^2 ſit maximum, erit per ſecundum theorema, etiam maximum in recta FC productum AF^3 in AC^2.

Poſtremò in lineâ AB primæ figuræ, productum AF^3 in AC^2 eſt maximum, ut modò oſtendimus, ergo per ſecundum theorema eſt etiam maximum AC^2 in BC^5. Quod demonſtrandum erat.

Si loco duorum numerorum detur numerus, & unitas, fit ſimilis conſtructio, & demonſtratio.

SCHOLION.

Id quod in ſecundo theoremate ſupponebamus; datâ rectâ lineâ, & datis numeris, 3 & 2, maximum fore productum in ea linea factum ſecundum numeros illos datos; nunc demonſtravimus in theoremate hoc. Erat porrò illius theo-

rematis propositio conditionalis, ex posita illa hypothesi, non absoluta, ut patebit consideranti.

COROLLARIUM.

Si productum genitum ex dignitate ducta in dignitatem quamcumque, maximum fuerit, illarum dignitatum radices & exponentes erunt geometicè proportionales. Quippe in theoremate ostendimus, productum in linea factum secundum terminos datos esse omnium maximum; at productum ejusmodi, ex v definitione nostra, gignitur ex duabus dignitatibus, quarum exponentes rationem eam habent, quam dignitatum earundem radices.

PROBLEMA PRIMUM.

Datam lineam rectam ita secare, ut productum ex dignitatibus segmentorum sit omnium similium maximum.

Sumantur exponentes duarum illarum dignitatum, rectaque dividatur in ratione horum exponentium, & factum erit quod imperatur; quia productum erit in linea data factum secundum terminos positos, nimirum secundum exponentes; ac proinde erit maximum per theorema tertium.

PROBLEMA SECUNDUM.

Æquationem determinare, in qua potestas quæsitæ radicis negatur de homogeneo sub radice data, & dignitate sua parodica, ut B in $A - A^2 \parallel Z^2$: vel B in $A^3 - A^4 \parallel Z^4$, &c.

Oritur hujusmodi æquatio ex dicta parodica dignitate potestatis negatæ ducta in B — A, differentiam datæ & quæsitæ radicis, rem probo. Illa parodica dignitas affirmata, si primùm ducatur in A, radicem quæsitam negatam, gignet potestatem negatam uno gradu altiorem, quàm sit ea parodica dignitas [ut patet ex natura multiplicationis] deinde in + B radicem datam affirmatam ducta, gignet homogeneum affirmatum, sub eadem dignitate parodica & radice data: quæ duo producta, sunt ipsa pars æquationis, de qua in problemate. Pars altera est homogeneum comparationis.

Rursus, per lemma quartum, ratio homogenei ad potestatem negatam est eadem, ac radicis datæ ad quæsitam; sed minor est potestas homogeneo, de quo ipsa negatur & demitur. Ergo etiam radix quæsita minor est datâ; in qua proinde radice datâ nos rectè sumimus segmentum æquale radici quæsitæ A, ut alterum segmentum sit B — A, differentia datæ ac quæsitæ radicis.

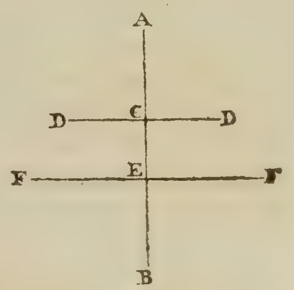
Quoniam igitur prima pars æquationis oritur ex B — A uno radicis datæ segmento, ducto in altero segmentum A, vel in hujus potestatem, efficitur [per tertium theorema] ut inde resultans productum sit maximum omnium similium, quotiescunque A, & B — A, segmenta rationem habent eam, quam exponentes suarum dignitatum. Sic in æquatione B in $A^3 - A^4 \parallel Z^4$; si A, & B — A fuerint ut 3 ad 1, cubus segmenti A in BA ductus gignet partem æquationis B in $A^3 - A^4$; quæ est productum in linea data B, omnium similium maximum; cujus

cujus proinde magnitudinem non poteſt unquam excedere homogeneum comparationis, quod ſemper æquari neceſſe eſt illi alteri æquationis parti. Unde canon pro determinanda problematis æquatione conficitur.

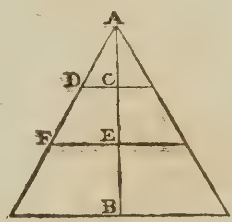
Fiat in radice maximum productum ſecundum terminos, qui ſunt exponentes ejuſdem radicis & parodicæ dignitatis, ſub quibus eſt homogeneum. Illius producti magnitudinem excedere non poteſt homogeneum comparationis.

Idem procedit in alia æquatione orta ex multiplicatione ipſius, A, in B — A, vel in hujus poteſtatem; ſemper enim eſt idem caſus tertii theorematis noſtri, in quo productum factum in linea [ſeu datâ radice B] ſecundum terminos datos eſt maximum, termini verò ſunt exponentes dignitatem ſegmenti A & alterius B — A.

Sed uno vel altero exemplo de geometricis noſtris opuſculis deprompto methodi facilitatem comprobemus.

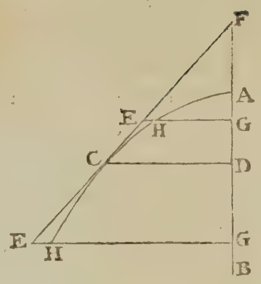


E ſingulis punctis datæ rectæ AB ducantur rectæ CD, EF, &c. rectæ inter ſe parallelæ, cum data AB angulum quemcumque efficientes. Sint autem harum parallelarum dignitates, & dignitates abſciſſarum AC, AE, &c. geometricè proportionales (id quod tripliciter contingere poſſe mox patebit) tranſibit per extrema parallelarum puncta, D, F, &c. perimeter figuræ, cujus diameter aut axis erit AB, vertex A, ordinatim verò ad diametrum applicatæ erunt ipſæ parallelæ.



Nam parallelarum abſciſſarumque dignitates ſi fuerint ejuſdem gradus, exempli gratia, FE^1 ad DC^1, ut AE^1 ad CA^1: vel cubi parallelarum ut cubi abſciſſarum, figura erit triangulum, cujus proprietas notiſſima eſt, non parallelas modò & abſciſſas eſſe geometricè proportionales, ſed parallelarum & abſciſſarum earumdem poteſtates omnes homogeneas; quarum ratio æquè multiplex eſt rationis linearum ſeu radicum ita ut cubi, & quadrato quadrata, &c. abſciſſarum ſint ut cubi, quadrato quadrata, &c. parallelarum; & illorum quoque radices geometricè proportionales.

Sin autem diverſorum graduum fuerint dignitates parallelarum & abſciſſarum, linea deſcripta erit curva, habens ſuum axem, & ad illum ordinatim applicatas, quarum dignitas eſt gradu ſuperior dignitate abſciſſarum: at contrà dignitas applicatarum ordinatim ad rectam [quæ curvam in vertice contingit] ſumptam pro axe, gradu inferior eſt dignitate abſciſſarum tangentis. De quo alibi latius dicam.



Esto igitur AGDB una ex præfatis figuris, ejusque axis AB, & vertex A; in qua quidem gradus dignitatis parallelarum sit altior gradu dignitatis abscissarum; quæratur autem linea recta contingens figuram in puncto dato C. Ducatur ex hoc puncto linea ad axem ordinatim applicata, ut CD, & ponantur exponentes dignitatum, 3 & 2. Erunt consequenter in figura parallelarum cubi ut quadrata abscissarum. Fiat abscissa AD, inter verticem & ordinatim applicatam, ad AF, axem productum, ut minor numerus 2 ad 1, differentiam exponentium, ductaque FC; dico hanc esse tangentem quæsitam.

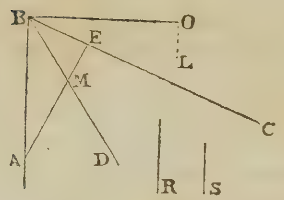
Productum enim FA^1 in AD^2 in linea DF, factum secundum terminos positos 1 & 2, est maximum, per theorema tertium; semperque homogeneum dignitati parallelarum (cùm parallelarum dignitatem exponat major datorum numerorum, maximum verò productum illud oriatur ex dignitatibus quas exponunt minor numerus & differentia numerorum, quæ duo simul efficiunt numerum majorem). Ergo si accipiamus alium punctum G in axe supra D, aut infra, & ducamus ordinatim applicatam GH, quæ secet E rectam FC (ubi opus fuerit productum), productum FA^1 in AG^2 non erit maximum in linea FG, quale est FA^1 in AD^2 in recta FD; propterea quòd major est vel minor ratio ipsius FA ad AG quàm ad AD, & consequenter FG, FD non sunt proportionaliter divisæ. Ergo majorem rationem habet FA^1 in AD^2 ad FD^3 sibi homogeneum, quàm FA^1 in AG^2 ad FG^3, & permutando, majorem rationem habet FA^1 in AD^2 ad FA^1 in AG^2, (vel ex lemmate quarto, AD^2 ad AG^2) quàm FD^3 ad FG^3. Sed AD^2 ad AG^2 ponitur in figura; ut CD^3 ad HG^3: FD^3 ad FB^3, ob similitudinem triangulorum, ut CD^3 ad EG^3. Ergo majorem rationem habet CD^3 ad HG^3, quàm CD^3 ad EG^3; & consequenter CD majorem rationem habet ad HG, quàm ad EG, ac proinde HG recta est minor quàm EG, & punctus E cadet extra datam curvam AHCH. Eodem pacto de singulis punctis ductæ lineæ FC demonstratur illos cadere semper extrà curvam. Ergo FC est illius tangens. Quod, &c.

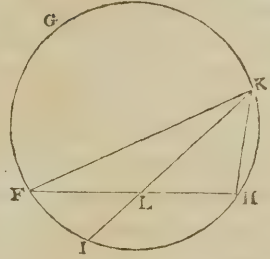
Hæ sunt parabolæ, ut vocant, infinitæ, quarum contingentes lineæ, quo modo ad datum punctum duci possint, ostendimus. Nunc eandem methodum in hyperbolis quoque libet experiri. Præmittimus autem hoc necessarium lemma.

LEMMA SEXTUM.

Dato angulo ABC utcumque secto per rectam BD, & puncto E in alterutro laterum comprehendentium angulum datum; ex eo puncto ducere lineam rectam quæ angulum ABC subtendat, & à recta BD secetur in data ratione R ad S.

Fiat



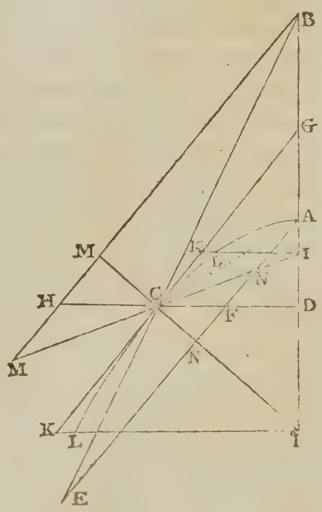


Fiat HGF ſegmentum circuli capiens angulum æqualem dato, & compleatur circulus; deinde ut R ad S, ita fiat FL ad LH; ut angulus ABD ad EBD, ſic arcus FI ad IH; ductaque IL producatur uſque dum pertingat ad K in circumferentia circuli, & connectantur puncta F, K, H. Ad datum punctum E fiat angulus BEA æqualis KHL, & EA ſecet BD in M & BA in puncto A. Dico rectam EA eſſe quæſitam, quæ à BD in M dividitur in ratione data.

Siquidem anguli H & E : K & B ſunt æquales, & hi ſecti proportionaliter [per trigeſimam tertiam ſextî Elementorum] à KLI, & BD. Ergo triangula FHK, ABE ſunt æquiangula, & AE ad EB, ut HF ad HK. Rurſus æquiangula fecimus triangula MBE, LKH, & conſequenter EB eſt ad EM, ut HK ad HL, & ex æqualitate ordinata AE ad EM, ut HF ad HL, & dividendo FL ad LH [ſeu R ad S] ut AM ad ME. Quod, &c.

Quòd ſi punctus datus ſit extrà, ut in O, ducemus BO rectam [punctus autem O ſic detur oportet, ut OB recta cum AB angulum faciat, nec ſit ad lineam poſita] & faciemus angulum BOL æqualem differentiæ angulorum, H, & EBO; & OL producta ſatisfaciet quæſito.

Sit hyperbole ACL, cujus diameter AB, vertex A, & dignitates ordinatim applicatarum habentes eam proportionem quam producta illis dignitatibus homogenea, orta ex dignitate abſciſſæ ducta in dignitatem, abſciſſæ & diametri, ex quibus una recta conflata intelligatur. Exempli gratiâ, quadrato cubi ordinatarum, hoc eſt LI^5 ad CD^5, ſint, ut producta BI^3 in AI^2 ad BD^3 in AD^2, genita ex quadratis abſciſſarum AI, AD, & cubis rectarum BI, BD, quippe quas efficiunt eædem abſciſſæ & diameter.



Detur punctus C, ad quem ducenda ſit tangens, & ordinatim applicetur CD. Porrò ducatur BC, producta ad partes C, quoad oportuerit, & ex lemmate præcedenti, AE [ſecans CD in F, & in F item ſecta pro ratione 2 ad 3, qui numeri exponunt dignitates gignentes producta BI^3 in AI^2, & BD^3 in AD^2, ſubtendens angulum ECA] & tandem GC parallela rectæ AE, occurrens ipſi AB in G. Dico tangentem quæſitam eſſe CG.

 Sumatur

Sumatur in CG alius punctus K ſupra & infra C, & ordinatim applicatis KI ſecantibus hyperbolen in L, ab I puncto ducatur IC incidens in rectam HB in puncto M, & ſecans AE in N; quæ HB ipſi AE parallela in H occurrit DC productæ.

Quoniam verò AE ſecatur in F in ratione 2 ad 3, FA^2 in FE^3, per tertium theorema, eſt productum maximum, & ratio FE^3 ad NE^3, ſeu HB^3 ad MB^3 [propter ſimilitudinem triangulorum HCB, ECF : MBC, CEN] major eſt ratione NA^2 ad AF^2. Ergo per lemma quintum majus eſt HB^3 in AF^2 ipſo MB^3 in NA^2; quæ duo producta ſi comparentur cum CG^5, primum habebit majorem rationem ad CG^5, quàm ſecundum. Sed ratio primi, quod eſt HB^3 in AF^2, ad CG^5 eadem eſt ac ratio BD^3 in AD^2 ad GD^5 [cum HB ad CG ſit ut BD ad GD, ob ſimilitudinem triangulorum HBD, CGD, eandemque proportionem habeant earum linearum cubi: tùm CG^2 ad AF^2, ut GD^2 ad AD^2] ratio ſecundi, ſeu MB^3 in NA^2, ad CG^5, eſt eadem ac ratio BI^3 in AI^2 ad IG^5 [quia ſimilia ſunt triangula MBI, CGI; & MB, CG, BI, IG rectæ earumque cubi proportionales: rurſus ut GI^2 ad IA^2, ſic CG^2 ad AN^2] ergo majorem rationem habet BD^3 in AD^2 ad GD^5, quàm BI^3 in IA^2 ad GI^5, & permutando, BD^3 in AD^2 ad BI^3 in AI^2 [ſeu ex natura hyperboles, CD^5 ad LI^5] majorem rationem habet, quàm DG^5 ad GI^5, ſeu [ob ſimilitudinem triangulorum KGI, CGD] CD^5 ad IK^5, & per decimam quinti Elem. dignitas, LI^5 minor eſt quàm KI^5, & ſua radix, LI recta, minor recta KI; quare punctus K eſt extra curvam. Sic de ceteris punctis oſtendetur cadere extra curvam, atque adeò CG hyperbolen tangere in ſolo C puncto. Quod, &c.

Hæc porrò demonſtratio etiam ad ellipſes, & circulos accommodari poteſt.

Jàm verò quàm latè pateat uſus noſtri theorematis tertii, ex propoſitis exemplis licet intelligere; nec ita multum diſſimili aut difficiliori via centra gravitatis, & quadraturas, quorum problematum paulò ante meminimus, invenimus. Interim, ſi quis Apollonii conſtructionem atque demonſtrationem trigeſimæ quartæ propoſitionis primi Conicorum libri cum noſtris comparabit, nonnihil fortaſſe proficiet in arte dilatandi propoſitiones & demonſtrationes. Nam id quod ille de quadratica tantum hyperbole, ellipſi, & circulo ſtatuit, nos ad omnes porrigimus hyperbolas, ellipſes, circuloſque infinitos. Quam viam placuit indicare, & ſupradicto exemplo confirmare.

AN

EXTRACT

FROM THE THIRD VOLUME OF THE

PHILOSOPHICAL TRANSACTIONS,

Publifhed on the 13th Day of APRIL, 1668,

By Mr. HENRY OLDENBURG, at that Time Secretary of the Royal Society ;

CONTAINING

A Method of fquaring the Hyperbola by an infinite Series of rational Numbers, together with its Demonftration, by the Right Honourable the Lord Vifcount Brouncker.

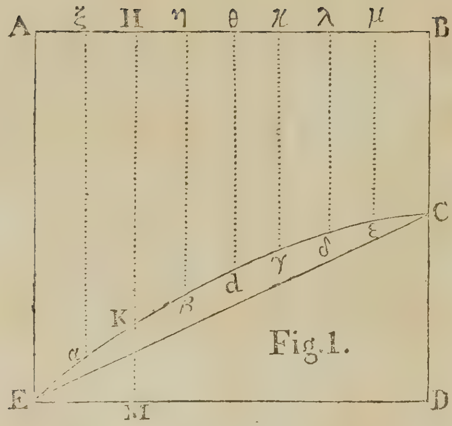


Fig.1.

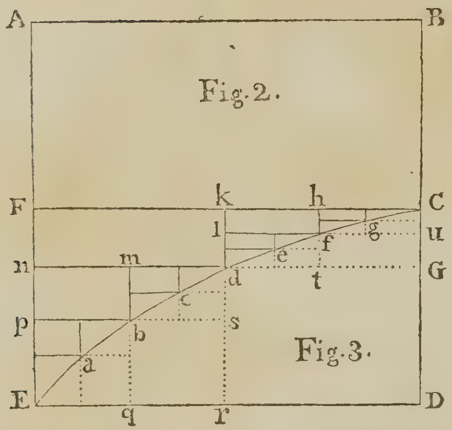


Fig.2.

Fig.3.

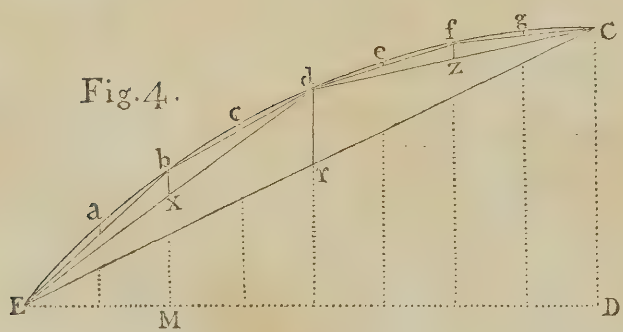


Fig.4.

This Extract is contained in pages 645, 646, 647, 648, and 649 of the ſaid Third Volume of the Philoſophical Tranſactions, and is as follows:

The Squaring of the Hyperbola by an infinite Series of rational Numbers, together with its Demonſtration, by that eminent Mathematician, the Right Honourable the Lord Viſcount Brouncker.

WHAT the acute Dr. John Wallis had intimated, ſome years ſince, in the Dedication of his Anſwer to M. Meibomius de Proportionibus, to wit, "That the world one day would learn from the noble lord Brouncker, the quadrature of the hyperbole;" the ingenious reader may ſee performed in the ſubjoined operation, which its excellent author was now pleaſed to communicate, as followeth in his own words:

My method for ſquaring the hyperbola is this:

Let AB be one aſymptote of the hyperbola EdC; and let AE and BC be parallel to the other: Let alſo AE be to BC as 2 to 1; and let the parallelogram ABDE equal 1. See Fig. 1.

Suppoſing the reader knows, that EA, $\alpha\zeta$, KH, $\varepsilon\eta$, $d\theta$, $\gamma\varkappa$, $\delta\lambda$, $\varepsilon\mu$, CB, &c are in an harmonic ſeries, or a ſeries reciproca primanorum, ſeu arithmetice proportionalium (otherwiſe he is referred for ſatisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. Arithm. Infinitor. Walliſii):

$$\left.\begin{array}{l} \text{I ſay } ABCdEA = \frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \frac{1}{7\times 8} + \frac{1}{9\times 10} \;\&c \\ EdCDE = \frac{1}{2\times 3} + \frac{1}{4\times 5} + \frac{1}{6\times 7} + \frac{1}{8\times 9} + \frac{1}{10\times 11} \;\&c \\ EdCyE = \frac{1}{2\times 3\times 4} + \frac{1}{4\times 5\times 6} + \frac{1}{6\times 7\times 8} + \frac{1}{8\times 9\times 10} \;\&c \end{array}\right\} \text{in infinitum.}$$

For (in Fig. 2 & 3) the Parallelog. And (in Fig. 4) the Triangl.

	$\text{CA} = \frac{1}{1 \times 2}$	$\text{E}dc = \frac{1}{2 \times 3 \times 4} = \frac{\square d\text{D} - \square d\text{F}}{2}$	Note.
$d\text{D} = \frac{1}{2 \times 3}$	$d\text{F} = \frac{1}{3 \times 4}$	$\text{E}bd = \frac{1}{4 \times 5 \times 6} = \frac{\square br - \square bn}{2}$	$\frac{1}{2}\text{CA} = d\text{D} + d\text{F}$
$br = \frac{1}{4 \times 5}$	$bn = \frac{1}{5 \times 6}$	$dfc = \frac{1}{6 \times 7 \times 8} = \frac{\square f\text{G} - \square fk}{2}$	$\frac{1}{2}d\text{D} = br + bn$
$f\text{G} = \frac{1}{6 \times 7}$	$fk = \frac{1}{7 \times 8}$	$\text{E}ab = \frac{1}{8 \times 9 \times 10} = \frac{\square aq - \square ap}{2}$	$\frac{1}{2}d\text{F} = f\text{G} + fk$
$aq = \frac{1}{8 \times 9}$	$ap = \frac{1}{9 \times 10}$	$bcd = \frac{1}{10 \times 11 \times 12} = \frac{\square cs - \square cm}{2}$	$\frac{1}{2}br = aq + ap$
$cs = \frac{1}{10 \times 11}$	$cm = \frac{1}{11 \times 12}$	$def = \frac{1}{12 \times 13 \times 14} = \frac{\square et - \square el}{2}$	$\frac{1}{2}bn = cs + cm$
$et = \frac{1}{12 \times 13}$	$el = \frac{1}{13 \times 14}$	$fg\text{C} = \frac{1}{14 \times 15 \times 16} = \frac{\square gu - \square gb}{2}$	$\frac{1}{2}f\text{G} = et + el$
$gu = \frac{1}{14 \times 15}$	$gb = \frac{1}{15 \times 16}$	&c.	$\frac{1}{2}fk = gu + gb$
&c.	&c.		&c.

And that therefore in the firſt ſeries half the firſt term is greater than the ſum of the two next, and half this ſum of the ſecond and third greater than the ſum of the four next, and half the ſum of thoſe four greater than the ſum of the next eight, &c in infinitum. For $\frac{1}{2}d\text{D} = br + bn$; but $bn > f\text{G}$, therefore $\frac{1}{2}\, d\text{D} > br + f\text{G}$, &c. And in the ſecond ſeries half the firſt term is leſs than the ſum of the two next, and half this ſum leſs than the ſum of the four next, &c in infinitum.

That the firſt ſeries are the even terms, viz. the 2[d], 4[th], 6[th], 8[th], 10[th], &c, and the ſecond, the odd, viz. the 1[st], 3[d], 5[th], 7[th], 9[th], &c of the following ſeries, viz. $\frac{1}{1 \times 2} \cdot \frac{1}{2 \times 3} \cdot \frac{1}{3 \times 4} \cdot \frac{1}{4 \times 5} \cdot \frac{1}{5 + 6} \cdot \frac{1}{6 \times 7}$. &c in infinitum $= 1$. Whereof a being put for the number of terms taken at pleaſure, $\frac{1}{a^2 + a}$ is the laſt, $\frac{a}{a+1}$ is the ſum of all thoſe terms from the beginning, and $\frac{1}{a+1}$ the ſum of the reſt to the end.

That $\frac{1}{4}$ of the firſt term in the third ſeries is leſs than the ſum of the two next, and a quarter of this ſum, leſs than the ſum of the four next, and one fourth of this laſt ſum leſs than the next eight, I thus demonſtrate.

Let $a =$ the 3[d] or laſt number of any term of the firſt column, viz. of diviſors,

$$\frac{1}{a \times a-1 \times a-2} = \frac{1}{a^3 - 3a^2 + 2a} = \frac{16a^3 - 48a^2 + 56a - 24}{16a^6 - 96a^5 + 232a^4 - 288a^3 + 184a^2 - 48a} = \text{A}$$

$$\left.\begin{array}{l} \frac{1}{2a \times 2a-1 \times 2a-2} = \frac{1}{8a^3 - 12a^2 + 4a} \\ \frac{1}{2a-2 \times 2a-3 \times 2a-4} = \frac{1}{8a^3 - 36a^2 + 52a - 24} \end{array}\right\} = \frac{16a^3 - 48a^2 + 56a - 24}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a - 96} = \text{B}$$

$$\frac{64a^6 - 384a^5 + 928a^4 - 1152a^3 + 736a^2 - 192a}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a^2 - 96a} \times \tfrac{1}{4}\text{A} < \text{B}.$$

4 And

And $48a^4 - 192a^3 + 240a^2 - 96a$ = Excefs of the numerator above denominato r.

$$\left.\begin{array}{lrcrl} \text{But the affirm.} & & > & \text{the negat.} & \\ \text{That is,} & 48a^4 + 240a^2 & > & 192a^3 + 96a & \\ \text{Becaufe} & a^4 + 5a^2 & > & 4a^3 + 2a & \\ & a^3 + 5a & > & 4a^2 + 2 & \end{array}\right\} \text{ if } a > 2.$$

Therefore $B > \frac{1}{4}A$.

Therefore $\frac{1}{4}$ of any number of A, or terms, is lefs than their fo many refpective B, that is, than twice fo many of the next terms. Quod, &c.

By any one of which three feries, it is not hard to calculate, as near as you pleafe, thefe and the like hyperbolic fpaces, whatever be the rational proportion of AE to BC. As for example, when AE is to BC, as 5 to 4 (whereof the calculation follows after that where the proportion is, as 2 to 1, and both by the third feries.)

Firft then when (in Fig. 1) AE . BC :: 2 . 1.

$2 \times 3 \times 4)$ 1.	(0,0416666666 —	0,0416666666
$4 \times 5 \times 6)$ 1.	(0,0083333333 —	} 0,0113095237
$6 \times 7 \times 8)$ 1.	(0,0029761904 —	
$8 \times 9 \times 10)$ 1.	(0,0013888888 —	
$10 \times 11 \times 12)$ 1.	(0,0007575757 —	} 0,0029019589
$12 \times 13 \times 14)$ 1.	(0,0004578754 —	
$14 \times 15 \times 16)$ 1.	(0,0002976190 —	
$16 \times 17 \times 18)$ 1.	(0,0002042484 —	
$18 \times 19 \times 20)$ 1.	(0,0001461988 —	
$20 \times 21 \times 22)$ 1.	(0,0001082251 —	
$22 \times 23 \times 24)$ 1.	(0,0000823452 —	} 0,0007306482
$24 \times 25 \times 26)$ 1.	(0,0000641026 —	
$26 \times 27 \times 28)$ 1.	(0,0000508751 —	
$28 \times 29 \times 30)$ 1.	(0,0000410509 —	
$30 \times 31 \times 32)$ 1.	(0,0000336021 —	
$32 \times 33 \times 34)$ 1.	(0,0000278520 —	
$34 \times 35 \times 36)$ 1.	(0,0000233426 —	
$36 \times 37 \times 38)$ 1.	(0,0000197566 —	
$38 \times 39 \times 40)$ 1.	(0,0000168691 —	
$40 \times 41 \times 42)$ 1.	(0,0000145180 —	
$42 \times 43 \times 44)$ 1.	(0,0000125843 —	
$44 \times 45 \times 46)$ 1.	(0,0000109793 —	
$46 \times 47 \times 48)$ 1.	(0,0000096361 —	} 0,0001829939
$48 \times 49 \times 50)$ 1.	(0,0000085034 —	
$50 \times 51 \times 52)$ 1.	(0,0000075415 —	
$52 \times 53 \times 54)$ 1.	(0,0000067193 —	
$54 \times 55 \times 56)$ 1.	(0,0000060125 —	
$56 \times 57 \times 58)$ 1.	(0,0000054014 —	
$58 \times 59 \times 60)$ 1.	(0,0000048704 —	
$60 \times 61 \times 62)$ 1.	(0,0000044068 —	
$62 \times 63 \times 64)$ 1.	(0,0000040002 —	

0,0416666666
0,0113095237
0,0029019589
0,0007306482
3) 0,0001829939(0,0000609980

0,05679179
+ 0,00006100

0,05685279 < Edcy

$$\text{But } \left.\begin{array}{r} 0{,}0007306482 \\ 0{,}0001829939 \\ 0{,}0000458315 \end{array}\right\} \div$$

Therefore 0,05679179
+ 0,00004583
+ 0,00001528

0,05685290 > Edcy

For

For, it has been demonſtrated, that $\frac{1}{4}$ of any term in the laſt column is leſs than the term next after it; and therefore that $\frac{1}{3}$ of the laſt term, at which you ſtop, is leſs than the remaining terms, and that the total of theſe is leſs than $\frac{1}{3}$ of a third proportional to the two laſt.

And therefore ABCyE being $= 0{,}75$ — — — $0{,}75$
and Edcy $> 0{,}05685279$ — and $< 0{,}05685290$

And ABCdE is $< 0{,}69314720$ — and $> 0{,}69314709$

But when AE . BC :: 5 . 4, or as EA to KH, then will the ſpace ABCE, or now, the ſpace AHKE (AH $= \frac{1}{4}$AB), be found as follows.

$8 \times 9 \times 10)\ 1\ (0{,}0013888888$
$16 \times 17 \times 18)\ 1\ (0{,}0002042484$
$18 \times 19 \times 20)\ 1\ (0{,}0001461988$ } $0{,}0003504472$
$32 \times 33 \times 34)\ 1\ (0{,}0000278520$
$34 \times 35 \times 36)\ 1\ (0{,}0000233426$
$36 \times 37 \times 38)\ 1\ (0{,}0000197566$
$38 \times 39 \times 40)\ 1\ (0{,}0000168691$ } $0{,}0000878204$

$0{,}0013888888$
$0{,}0003504472$
$3)0{,}0000878204(0{,}0000292735$

$0{,}0018271564$
$+\ 0{,}0000292735$

$0{,}0018564299 <$ Eab

But $0{,}0003504472$
$0{,}0000878204$
$0{,}00002200737$ } $\div$

Therefore $0{,}0018271564$
$+\ 0{,}0000220074$
$+\ 0{,}0000073358$

$0{,}0018564996 >$ Eab

Therefore EMb (Fig. 4)
being $= 0{,}025$ — $0{,}025$
Eab $> 0{,}0018564299$ — & $< 0{,}0018564996$

EMba (Fig. 4.) or EKM (Fig. 1) $> 0{,}02685643$ — $< 0{,}02685650$
AHKM $< 0{,}22314356$ — $> 0{,}22314349$

Therefore 3ABCdE $= 2{,}07944154$
and AHKE $= 0{,}2231435$

ABCdE (when AE . BC :: 10 . 1) $= 2{,}3025850$

Therefore the logar. of 10 is to the log. of 2, as 2,302585 to 0,693147

ANOTHER

E X T R A C T

FROM THE

PHILOSOPHICAL TRANSACTIONS, N° XXXVIII.

Publiſhed on the 17th Day of AUGUST, 1668.

CONTAINING

Firſt, An Account of Mr. NICHOLAS MERCATOR's TRACT on LOGARITHMS, called LOGARITHMO-TECHNIA, by Dr. JOHN WALLIS, Savilian Profeſſor of Geometry in the Univerſity of Oxford, in a Letter to the Lord Viſcount BROUNCKER; and 2dly, A Method of finding the Sums of Logarithms, by the ſaid Dr. WALLIS, in another Letter to the ſame learned Lord; and, 3dly, An Illuſtration of the ſaid Mr. Nicholas Mercator's Tract aforeſaid, called LOGARITHMO-TECHNIA, by the ſaid Mr. MERCATOR himſelf.

7

This Extract is contained in pages 753, 754, 755, 756, &c.—764, of the Third Volume of the Philosophical Transactions, and is as follows:

LOGARITHMO-TECHNIA

NICOLAI MERCATORIS;

Concerning which we shall here deliver the Account of the judicious Doctor JOHN WALLIS, given in a Letter to the Lord Viscount BROUNCKER, as follows:

INCIDEBAM heri, Illustrissime Domine, in D. Mercatoris Logarithmotechniam, nupèr editam; quæ ita mihi placuit, ut non prius dimiserim quàm perlegissem totam. Et quamquam pauca quædam, phraseologiam quod spectat seu loquendi formulas nonnullas, mutata mallem; sunt tamen ipsa sensu suo sana: Eáque quæ superstruitur doctrina, logarithmos expeditè atque subtilitèr construendi, perspicuè satìs atque ingeniosè traditur.

Quæ huic subjungitur Quadratura Hyperbolæ, elegans admodùm est atque ingeniosa. Nempe ad hunc sensum. V. *Fig.* 1.

Postquam in hyperbolâ MBF, (cujus asymptotæ AN, AH, ad angulum rectum coeunt) ostenderat, prop. XIV, rectangula BIA, FHA, spA, &c. (ductis BI, FH, sp, &c, parallelis asymptotæ AN), invicèm esse æqualia; adeóque latera habere reciprocè proportionalia (quæ nota est hyperbolæ proprietas): Positis AI $=$ BI $= 1$, & HI $= a$: ostendit, prop. XV, FH $= \frac{1}{1+a}$ (nempe propter analogiam AH . AI :: BI . FH, hoc est, $1 + a . 1 :: 1 . \frac{1}{1+a}$. Sed & (quod dividendo 1 per $1 + a$ ostenditur), $\frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4$ &c (continuatis deinceps, ipsius a potestatibus, alternatìm negatis & affirmatis).

$$
\begin{array}{l}
1 + a)\,1\,(\,1 - a^2 + a^3 + a^4, \text{\&c.} \\
\underline{1 + a} \\
\quad - a \\
\quad \underline{- a - a^2} \\
\qquad + a^2 \\
\qquad \underline{+ a^2 + a^3} \\
\qquad\quad - a^3 \\
\qquad\quad \underline{- a^3 - a^4} \\
\qquad\qquad + a^4 \\
\qquad\qquad \text{\&c.}
\end{array}
$$

Cùmque hoc perinde obtineat, ubicunque ultrà punctum I, ponatur H. Positis, ut prius, AI $= 1$; hujusque continuatione qualibet, ut IP $=$ A; quæ intelligatur in

æquales

æquales partes innumeras dividi, quarum quælibet, ut 1p, pq, &c, dicatur a; adeóque 1p, 1q, &c, ſint a, $2a$, $3a$, &c, uſque ad A: Quæ his reſpondent rectæ ps, qt, &c, uſque ad ru, (ſpatium BIru complentes) ſunt,

$$1 - a + a^2 - a^3 + a^4 \text{ \&c.}$$
$$1 - 2a + 4a^2 - 8a^3 + 16a^4 \text{ \&c.}$$
$$1 - 3a + 9a^2 - 27a^3 + 81a^4 \text{ \&c.}$$

& ſic deinceps uſque ad

$$1 - A + A^2 - A^3 + A^4 \text{ \&c.}$$

Cum itaque ſint

$$1 + 1 + 1 \text{ \&c (uſque ad ultimum)} = A$$
$$a + 2a + 3a \text{ \&c (uſque ad A)} = \tfrac{1}{2}A^2$$
$$a^2 + 4a^2 + 9a^2 \text{ \&c (uſque ad } A^2) = \tfrac{1}{3}A^3$$
$$a^3 + 8a^3 + 27a^3 \text{ \&c (uſque ad } A^3) = \tfrac{1}{4}A^4$$

& ſic deinceps:

(quod oſtendit ille prop. XVI, éſtque à me alibi demonſtratum): Rectè colligit, prop. XVII. Expoſitum ſpatium hyperbolicum BIru $= A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \frac{1}{5}A^5$, &c. Adeóque ſi (aſſignato, ipſi $A = 1r$, valore ſuo in numeris, ut res poſtulaverit) diſtribuantur in duas claſſes A, $\frac{1}{3}A^3$, $\frac{1}{5}A^5$, &c, (poteſtates affirmatæ), & $\frac{1}{2}A^2$, $\frac{1}{4}A^4$, &c, (poteſtates negatæ); harúmque aggregatum, ex aggregato illarum, ſubducatur; reſiduum erit ipſum BIru ſpatium hyperbolicum.

Nequis autem operam luſum iri exiſtimet, propter addendorum ſeriem in utrâque claſſe infinitam; adeóque non abſolvendam: Huic incommodo medelam (tacitus) adhibet: ponendo $A = 0{,}1$, vel $A = 0{,}21$, aliíve fractioni decimali æqualem, adeóque minorem quam 1: (hoc eſt, ſumptâ 1r minore quam $AI = 1$). Quo fit, ut poſteriores ipſius A poteſtates tot gradibus infrà integrorum ſedem deſcendant, ut meritò negligi poſſint.

Exempli gratiâ; poſitis $AI = 1$, & $1r = 0{,}21$; erit

A	$= 0{,}21$		
$\frac{1}{3}A^3$	$= 0{,}003087$	$\frac{1}{2}A^2$	$= 0{,}02205$
$\frac{1}{5}A^5$	$= 0{,}000081682$	$\frac{1}{4}A^4$	$= 0{,}000486202$
$\frac{1}{7}A^7$	$= 0{,}000002572$	$\frac{1}{6}A^6$	$= 0{,}000014294$
$\frac{1}{9}A^9$	$= 0{,}000000088$	$\frac{1}{8}A^8$	$= 0{,}000000472$
$\frac{1}{11}A^{11}$	$= 0{,}000000003$	$\frac{1}{10}A^{10}$	$= 0{,}000000016$

$+ 0{,}213171345 - 0{,}022550984 = 0{,}190620361 =$ BIru quæ eſt brevis ſynopſis quadraturæ ſuæ ſatis elegantis.

Diſſimulandum interìm non eſt; ſiquis totius BIHF ſpatii (cujus latus IH longius intelligatur quàm AI) quadraturam poſtulet; rem non ita felicitèr ſucceſſuram: propter medelam, quam modò diximus, malo minùs ſufficientem. Cum enim jam ponenda ſit $A > 1$; manifeſtum eſt, poſteriores ipſius poteſtates, altius in integrorum ſedes penetraturas, adeóque minimè negligendas.

Huic autem incommodo, levi conſtructionis immutatione, facilè ſubvenitur. Vid. Fig. 1.

Cæteris

Cæteris utique ut priùs constructis; quadrandum exponatur HFur spatium; (cujuscunque fuerit longitudinis AH; puta major minórve quàm AI, vel huic æqualis: sumptóque ubivis inter A & H, puncto r; puta ultra citràve punctum I, vel in ipso I puncto): ponantur autem (non, ut priùs, $AI = 1$, & $Ir = A$: sed) $AH = 1$; & $Hr = A$, quæ intelligatur in æquales partes innumeras dividi, quarum quælibet sit a. Erunt itaque, post $AH = 1$, reliquæ deinceps decrescentes $1 - a$, $1 - 2a$, $1 - 3a$, &c, usque ad $Ar = 1 - A$. Item, propter æqualia rectangula FHA, urA, BIA, &c. puta, $= b^2$: Erit $HF = \frac{b^2}{1}$; reliquæque deinceps $\frac{b^2}{1-a}$, $\frac{b^2}{1-2a}$, $\frac{b^2}{1-3a}$, &c. usque ad $ru = \frac{b^2}{1-A}$ spatium HFur complentes. (Quæ omnia ostensa sunt, in meâ Arithmeticâ Infinitorum, prop. 88, 94, 95.)

Factâque divisione; reperietur $\frac{b^2}{1-a} = b^2 + b^2a + b^2a^2 + b^2a^3 + b^2a^4$ &c. Hoc est, b^2 in $1 + a + a^2 + a^3 + a^4$, &c. (sumptis ipsius a potestatibus continuè sequentibus affirmatis omnibus). Cúmque de reliquis idem sit judicium; erunt rectæ omnes, ipsis HF & ru interjectæ,

$$
\begin{array}{l}
1 - a)\ b^2\ (b^2 + b^2a + b^2a^2 + b^2a^3\ \&c\\
\quad \underline{b^2 - b^2a}\\
\qquad + b^2a\\
\qquad \underline{+ b^2a - b^2a^2}\\
\qquad\qquad + b^2a^2\\
\qquad\qquad \underline{+ b^2a^2 - b^2a^3}\\
\qquad\qquad\qquad + b^2a^3\\
\qquad\qquad\qquad \underline{+ b^2a^3 - b^2a^4}\\
\qquad\qquad\qquad\qquad + b^2a^4\\
\qquad\qquad\qquad\qquad \&c.
\end{array}
$$

$$
\left.\begin{array}{l}
1 + a + a^2 + a^3 + a^4\ \&c.\\
1 + 2a + 4a^2 + 8a^3 + 16a^4\ \&c.\\
1 + 3a + 9a^2 + 27a^3 + 81a^4\ \&c.\\
\quad \&\ \text{sic deinceps usque ad}\\
1 + A + A^2 + A^3 + A^4\ \&c.
\end{array}\right\}\ \text{in } b^2
$$

Omniumque aggregatum $A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4 + \frac{1}{5}A^5$ &c, in $b^2 = FHru$. (per Arithm. Infin. prop. 64).

Exempli gratiâ,
Positis $AH = 1$.
$Hr = A = 0{,}21$
$AI = b = 0{,}1$
Adeóque $b^2 = 0{,}01$

$$
\begin{array}{lll}
\text{Erunt } A & = 0{,}21 & \\
\frac{1}{2}A^2 & = 0{,}02205 & \\
\frac{1}{3}A^3 & = 0{,}003087 & \\
\frac{1}{4}A^4 & = 0{,}00048623 & - \\
\frac{1}{5}A^5 & = 0{,}000081682 & + \\
\frac{1}{6}A^6 & = 0{,}000014294 & + \\
\frac{1}{7}A^7 & = 0{,}000002573 & - \\
\frac{1}{8}A^8 & = 0{,}000000473 & - \\
\frac{1}{9}A^9 & = 0{,}000000088 & + \\
\frac{1}{10}A^{10} & = 0{,}0000000017 & - \\
\frac{1}{11}A^{11} & = 0{,}000000003 & + \\
\hline
\text{Horum summa} & = 0{,}235722333 & \\
\text{Ducta in } b^2 & = 0{,}01 & \\
\hline
\text{Exhibet} & 0{,}00235722333 = FHru &
\end{array}
$$

Qua-

Qualium I = ANGH $\left\{\begin{array}{l}\text{Quadrato,}\\ \text{Rhombo,}\end{array}\right\}$ si angulus A sit $\left\{\begin{array}{l}\text{Rectus,}\\ \text{Obliquus.}\end{array}\right.$

Quæ quidem tam absoluta est tamque expedita hyperbolæ quadratura, ut nesciam an melior sperari debeat.

Atque hæc sunt quæ hac de re scribenda duxi. Quæ si D. Mercatori impertiveris; non displicebit, credo, hæc suæ quadraturæ facta accessio.

Posse hæc ad logarithmorum inventionem accomodari, non est quod moneam: sed & ad summam logarithmorum inveniendam (quam inquirit ille prop. XIX). Nempe, positis AH = I, AI = IB = b, (ut prius) planoque BIHF = pl. Erit pl — b^2 + b^3 = BIps + BIqt + BIru &c, usque ad BIHF. Si autem non ab ipsa BI incipiatur; sed ultra citrave, puta à ps: Posita PH = a, & PSFH = pl. Erit (universaliter) pstq + psur &c (usque ad PSFH) = pl — ab²: (qualium I, æquetur cubo ipsius AH). Quod alias, si opus erit, demonstrabitur. Tu interim, illustrissime Domine, vale.

Oxon. d. 8 *Julii*, 1668.

Fig. I.

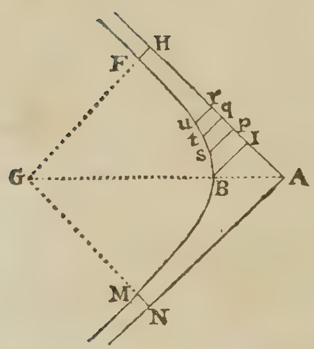
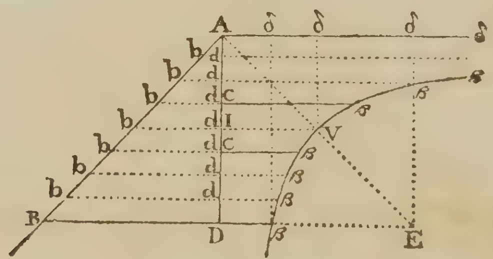


Fig. II.



THE DEMONSTRATION,

Promised at the End of the foregoing Letter, follows in another from the same Author to the same Noble Lord, thus,

PETIS (illustrissime Domine) per literas tuas Aug. 3 datas (quas hesterna nocte accepi) ut demonstrare velim methodum meam logarithmorum summam inveniendi, quam literis meis Julii 8 datis, brevissime insinuaveram.

Quæ quidem, cum sit cum ungularum doctrina (quam alibi trado) connexa; opus erit ut utramque simul exponam: sed & rem totam (quam D. Mercatoris figuræ

figuræ & methodo quantum res ferebat accommodaveram) ad principia mea revocatam ab origine repetam. V. Fig. 2.

Oſtenſum eſt, in mea Arithmetica Infinitorum, prop. XCV, ſpatium hyperbolium ADβßδ (in infinitum continuatum à partè βδ, ſed à parte Dβ ubivis terminatum) figuram eſſe quam ex primanorum reciprocis conflatam appello, prop. LXXXVIII, definitam: Cujus nempe ordinatim — applicatæ dβ, dβ, ſint primanis (ſeu arithmetice proportionalibus) db, db, (triangulum complentibus) adeoque ipſis dA, dA, (ſuis à vertice diſtantiis) reciproce proportionales. Hoc eſt, (poſito $AD = d$; & rectangulo $AD\beta = b^2$; particuliſque infinite exiguis, a, a, &c); ſi à vertice Aδ incipias $\frac{b^2}{0}$, $\frac{b^2}{a}$, $\frac{b^2}{2a}$, $\frac{b^2}{3a}$, &c. uſque ad $\frac{b^2}{d} = D\beta$: vel, ſi à baſe Dβ incipias $\frac{b^2}{d}$, $\frac{b^2}{d-a}$, $\frac{b^2}{d-2a}$, $\frac{b^2}{d-3a}$, &c. uſque ad $\frac{b^2}{d-d} = A\delta$ infinitæ (nempe, ſi ad verticem uſque proceſſum continuaveris); vel, uſque ad $\frac{b^2}{d-A} = C\beta$ (poſito $DC = A$), ſi continuaveris uſque ad Cβ, ubivis intra A & Dβ ſumptam. (Adeoque omnium aggregatum, $\frac{b^2}{d} + \frac{b^2}{d-a} + \frac{b^2}{d-2a} + \frac{b^2}{d-3a}$, &c, eſt ipſum DCββ planum.)

Manifeſtum itaque eſt (& ibidem prop. XCIV oſtenſum), ſi intelligantur ſingulæ dβ, in ſuas à vertice diſtantias Ad, ductæ; hoc eſt, $\frac{B^2}{a}$ in a, $\frac{B^2}{2a}$ in $2a$ (& ſic de reliquis); erunt omnia rectangula Adβ; hoc eſt, rectarum dβ momenta reſpectu Aδ (intellige, facta ex magnitudine in diſtantiam ductâ); ſeu plana ſemiquadrantalem ungulam (cujus acies Aδ) complentia (eiſdem dβ rectis perpendiculariter inſiſtentia); invicem æqualia. Quippe ſingula $= b^2$. (Quorum cum unum ſit AIVδ quadratum, erit $AI = b$.)

Adeoque totius ADββδ (plani infiniti) ſeu omnium dβ illud complentium, momentum reſpectu rectæ Aδ (ut axis æquilibrii); ſeu ungula ſemiquadrantalis eidem ADββδ inſiſtens (aciem habens Aδ); ſunt totidem b^2; hoc eſt, db^2. (Ungula magnitudinis finitæ plano infinitæ magnitudinis inſiſtens.) Ejuſque pars plano ACββδ inſiſtens (propter $AC = d - A$) ſimiliter oſtendetur æqualis ipſi $d - A$ in b^2 ductæ; hoc eſt, $db^2 - Ab^2$. Adeoque pars reliqua, ipſi DCββ inſiſtens, æqualis ipſi Ab^2. Quod itaque eſt ejuſdem DCββ momentum reſpectu Aδ.

Atque hoc momentum per plani DCββ magnitudinem, puta per pl, diviſum; exhibet plani diſtantiam centri gravitatis ab Aδ, $\frac{ab^2}{pl}$: adeoque diſtantiam ejuſdem à Dβ, $d - \frac{ab^2}{pl}$.

Hæc itaque à Dβ diſtantia, in pl (plani magnitudinem) ducta; exhibet $d$pl $- Ab$ ejuſdem DCββ momentum reſpectu Dβ; ſeu ungulam eidem DCββ inſiſtentem, cujus acies ſit Dβ.

Hæc denique ungula (cujus altitudo, in Dβ, nulla ſit, ſed, in Cβ, ipſi DC æqualis): ſi ex planis ipſi DCββ parallelis conflari intelligatur; erunt ea, CDββ, cdββ, & ſic deinceps; hoc eſt, aggregatum omnium cdββ, cdββ, uſque ad CDββ.

Sunt autem ea plana (ut ex Gregorii de Sancto Vincentio, aliorumque post illum, doctrina constat) tanquam logarithmi arithmetice proportionalium cd, cd, &c. usque ad CD; (seu a, $2a$, $3a$, &c, usque ad A.) Ergo ungula ipsa, est eorundem aggregatum. Hoc est (posito $\mathrm{D} = 1$), $d\mathrm{pl} - \mathrm{A}b^2 = \mathrm{pl} - \mathrm{A}b^2$. Quod ostendendum erat.

Porro; cum sit $\frac{b^2}{d-a}$ ($= \mathrm{d}\beta$) $= \frac{b^2}{d} + \frac{ab^2}{d^2} + \frac{a^2b^2}{d^3} + \frac{a^3b^2}{d^4}$ &c. (Quod dividendo b^2 per $d - a$, patebit): vel, posito $d = 1$, (quò ipsius d potestates omnes deleantur), $b^2 + ab^2 + a^2b^2 + a^3b^2$ &c. seu $1 + a + a^2 + a^3$ &c. in b^2; & similiter $\frac{b^2}{d-2a} = \frac{b^2}{d} + \frac{2ab^2}{d^2} + \frac{4a^2b^2}{d^3} + \frac{8a^3b^2}{d^4}$ &c. $= b^2 + 2ab^2 + 4a^2b^2 + 8a^3b^2$ &c. $= b^2$ in $1 + 2a + 4a^2 + 8a^3$ &c. & similiter in reliquis:

Erunt omnes $d\beta$ (spatium DC$\beta\beta$ complentes),

$$\left.\begin{array}{l} 1 + a + a^2 + a^3 + a^4 \text{ \&c,} \\ 1 + 2a + 4a^2 + 8a^3 + 16a^4 \text{ \&c,} \\ 1 + 3a + 9a^2 + 27a^3 + 81a^4 \text{ \&c,} \\ \text{\& sic deinceps usque ad} \\ 1 + \mathrm{A} + \mathrm{A}^2 + \mathrm{A}^3 + \mathrm{A}^4 \text{ \&c,} \end{array}\right\} \text{in } b^2$$

Adeoque (per Arithm. Infin. prop. LXIV), omnium aggregatum, seu ipsum DC$\beta\beta$ spatium, erit ——

$$\mathrm{A} + \tfrac{1}{2}\mathrm{A}^2 + \tfrac{1}{3}\mathrm{A}^3 + \tfrac{1}{4}\mathrm{A}^4 + \tfrac{1}{5}\mathrm{A}^5 \text{ \&c, in } b^2 = \mathrm{pl}.$$

Qualium $1 =$ ADEδ quadrato vel rhombo.

Ideoque, plani DC$\beta\beta$ momentum respectu Dβ; seu semiquadrantalis ungula eidem insistens cujus acies sit Dβ; seu planorum aggregatum ipsam constituentium; seu logarithmorum summa quos ea representant, $d\mathrm{pl} - \mathrm{A}b^2 = pl - \mathrm{A}b^2 = \frac{1}{2}\mathrm{A}^2 + \frac{1}{3}\mathrm{A}^3 + \frac{1}{4}\mathrm{A}^4 + \frac{1}{5}\mathrm{A}^5$ in b^2:

Qualium cubus (seu rhombus solidus) ADEδ sit 1.

Si vero non ponatur $d = 1$, sed cujuscunque magnitudinis: erit saltem

$$\frac{\mathrm{A}}{d} + \frac{\mathrm{A}^2}{2d^2} + \frac{\mathrm{A}^3}{3d^3} + \frac{\mathrm{A}^4}{4d^4} \text{ \&c in } b^2 = pl.$$

Vel (posito $\frac{\mathrm{A}}{d} = e$) erit $e + \frac{1}{2}e^2 + \frac{1}{3}e^3 + \frac{1}{4}e^4$ &c in $b^2 = pl$ qualium $d^2 =$ ADEδ quadrato vel rhombo.

Ungulaque (ut prius) $dpl - \mathrm{A}b^2$ qualium $d^3 =$ ADEδ cubo, vel (si angulus A sit obliquus) rhombo solido.

Cumque A (posito $d = 1$) vel e (quicunque ponatur valor ipsius d) sit minor quam 1 (propter $\mathrm{A} < d$): illius potestates posteriores ita continue decrescunt, ut tandem negligi possint; planique valor pl exhibeatur quantumlibet vero propinquus.

Atque hæc est, Illustrissime Domine, methodi, quam innuebam, ex meis principiis deductio, & demonstratio brevis. Vale. Oxon. d. 5. Aug. 1668.

Some

Some Illuſtration of the Logarithmo-technia of M. Mercator, who communicated it to the Publiſher, as follows:

Si quorum in manus incidit Logarithmo-technia mea, non inviti, opinor, adſpicient paucula hæc exempla, miram iſtius methodi facilitatem cum ſumma præciſione conjunctam oſtendentia.

Exponentes	Unitatis ordo		Binarii ordo	
1	1	1	2	2
2	1	0,5	4	2
3	1	0,333333	8	2,666666
4	1	0,25	16	4
5	1	0,2	32	6,4
6	1	0,166666	64	10,666666
7	1	0,142857	128	18,285714
8	1	0,[illegible]	256	32
9	1	0,111111	512	56,888888
10	1	0,1	1024	102,4

Duo ſunt ordines tabellæ, prior unitatis, alter binarii propago, quorum uterque denorum numerorum primorum logarithmos producit, præter compoſitorum logarithmos, qui & ipſi requiruntur.

Ex primo ordine

1
05
033333333
025
02
016666
01428
0125
011
01

+ 10033534772
− 502516792

$10536051564\frac{9}{10}$
$9531017980\frac{10}{11}$

1
05
0333333
025
02

+ 10000333353
− 500025

$10050335853\frac{99}{100}$
$9950330853\frac{100}{101}$

Pari modo ex eodem ordine procedunt rationes $\frac{999}{1000}$, $\frac{1000}{1001}$, $\frac{9999}{10000}$, $\frac{10000}{10001}$, $\frac{99999}{100000}$, $\frac{100000}{100001}$.

Ex fecundo ordine.

$\dot{2}$
$\dot{2}$
$\dot{2}66666666$
$\dot{4}$
$\dot{6}4$
$1\dot{0}66666$
$18\dot{2}857$
$\dot{3}2$
$\dot{5}689$
$10\dot{2}4$
$18\dot{6}$
$34\dot{1}$
$63\dot{0}$

$+\ 20273255404$
$-\ 2041099724$

$\dot{2}$
$\dot{2}$
$\dot{2}666666$
$\dot{4}$
$\dot{6}4$
$1\dot{0}$

$+\ 20002667306$
$-\ 200040010$

$20202707316\frac{98}{100}$
$19802627296\frac{100}{102}$

Haud fecus ex eodem ordine eliciuntur rationes $\frac{998}{1000}$, $\frac{1000}{1002}$, $\frac{9998}{10000}$, $\frac{10000}{10002}$, $\frac{99998}{100000}$, $\frac{100000}{100002}$, &c.

1		$22314355128\frac{8}{10}$
2		$18232155680\frac{10}{12}$
3	$\dot{1} + \dot{2}$	$40546510808\frac{8}{12} = \frac{2}{3}$
4	ex pc. pag.	$10536051564\frac{9}{10}$
5	$\dot{2} + \dot{4}$	$28768207244\frac{9}{12} = \frac{3}{4}$
6	$\dot{3} + \dot{5}$	$69314718052\frac{2}{4} = \frac{1}{2} = L2^{rii}$
7	$\dot{6} \times \dot{3}$	$207944154156\frac{1}{8} = L8^{rii}$
8	$\dot{1} + \dot{7}$	$230258509284\frac{1}{10} = L10^{rii}$
9	ex pc. pag.	$9531017980\frac{10}{11}$
10	$\dot{8} + \dot{9}$	$239789527264\frac{1}{11} = L11^{rii}$
11	$\dot{3} + \dot{6}$	$109861228860\frac{1}{2} + \frac{2}{3} = L3^{rii}$

Similes ordines à 3^{rio}, 4^{rio}, & quovis alio numero derivari poffunt, fuas quifque rationes exhibiturus.

Acquifito logarithmo 10^{rii}, conficienda eft ftatim tabella reducendorum logarithmorum naturalium ad tabulares, ut quævis ratio, fimul ac inventa eft, reducatur ad menfuram tabularium; ita enim logarithmi compofitorum, quorum ope

ope ad primorum logarithmos defcenditur, fimul fient tabulares abfque reductione.

Fiat igitur, ut logarithmus 10^rii^ non-tabularis 2302585 ad tabularem 10000000, ita 1 ad 4,3429448. Hic numerus bis, ter, quater & pluries fumptus conftituit tabellam reducendorum logarithmorum naturalium ad tabulares, quam hic fubjectam vides.

1	043429448190
2	086858896380
3	130288344570
4	173717792761
5	217147240951
6	260576689141
7	304006137332
8	347435585522
9	390865033712

Hujus igitur ope tabellæ, rationis $\frac{98}{100}$ menfura naturalis 20202707316 reducitur ad tabularem hoc modo:

2	086858896381
0	0
2	0868588964
0	0
2	08685890
7	3040061
0	0
7	30401
3	1303
1	043
6	26
	87739243069

Tum à logarithmo 100^rii^	20000000000000
auferatur rationis $\frac{98}{100}$ menfura	87739243069
reftat	19912260756031 = L 98
unde ablato logarithmo 2^ii^	3010299956640
reftat —	16901960800291 = L 99
cujus femis	8450980400145 = L 7

Item rationis $\frac{100}{102}$ menfura naturalis 19802627296 reducta, fit 86001717619

Ergo jam logarithmo 100^rii^	20000000000000
adde rationis $\frac{100}{102}$ menfuram	86001717619
fit —	20086001717619 = L 102
unde ablato logarithmo 6^rii^	7781512503836
reftat ——	12304489213783 = L 17

Hîc tabula numerorum primorum egregium ufum præftare poteft.

Sed & ejufdem primi 17 logarithmorum abfque ambage invenire datur, dicendo: 20 . 17 :: 10 . 8⌊5; tum differentiæ inter 10 & 8⌊5 (nimirum 1⌊5) fumendo quadrati femiffem, cubi trientem, &c, tractandoque iftum ordinem, ut fuprà, inveniemus fimul logarithmos abfolutorum 23, 197, 203, 1997, 2003, &c.

			15
1	1,5	1,5	1125
2	2,25	1,25	1125
3	3,375	1,125	1265625
4	5,0625	1,265625	1518
5	7,59375	1,51875	189
6	11,390625	1,8984375	22
			+ 15114040
			− 1137845
			$16251885\frac{17}{20}$
			$13976195\frac{20}{23}$

Cæterùm iſthæc omnia, & longè plura ex prop. XIII, XV & XVI Logarithmotechniæ noſtræ apertè derivantur, non magis conſiderando hyperbolam, quàm ſi ea nuſquam in rerum natura extitiſſet. Quare fruſtra ſunt, qui hyperbolam ad conſtructionem logarithmorum vel hilum conferre autumant; imo logarithmorum ope quadrare hyperbolam, verius eſt. Id quod exemplo oſtendere haud pigebit. In diagrammate (Fig. 1) ſit AH 74305816 parium, qualium AI eſt 1, & oporteat invenire aream BIHF.

Opus eſt ad eam rem tabella ſubjecta, quæ continet logarithmos naturales ſuprà acquiſitos, in priori columna ab 1 uſque ad 9, in altera à 10 uſque ad 1000000000.

1	00000000000	02,30258509299
2	69314718052	04,60517018599
3	109861228860	06,90775527898
4	138629436104	09,21034037198
5	160943791232	11,51292546497
6	179175946912	13,81551055796
7	194591014904	16,11809565096
8	207944154156	18,42068074395
9	219722457720	20,72326583695

Tum prima figura numeri dati ſemper diſtinguatur à ſequentibus ſeparatrice, hoc modò; 7,4305816, & ipſi primæ figuræ ſemper adjiciatur 1, ita conflantur, hoc loco, 8. Quærenda eſt nunc rationis 8 ad 7,4305816 menſura naturalis. Id ut fiat commodius, dic: ut 8 ad 7,4305816; ita 1 ad 0,9288227, hunc quartum proportionalem aufer ab 1, reliquum 0,0711773 voco poteſtatem primam, quæ ducenda eſt in ſe ita, ut in facto idem numerus partium extet, qui erat in ipſo 0,0711773; productum 0,0050662 eſt poteſtas ſecunda, quæ rurſus ducatur in primam 0,0711773, ut idem numerus partium exſtet, prodit 0,0003606, quæ eſt tertia poteſtas; & eodem modo invenitur quarta 0,0000256, & quinta 0,0000018. Deinde

Poteſtas

$$\left.\begin{array}{lr} \text{Poteftas prima} & 0{,}0711773 \\ \text{Et fecundæ femis} & 25331 \\ \text{Et tertiæ triens} & 1202 \\ \text{Et quartæ quadrans} & 64 \\ \text{Et quintæ pars quinta} & 4 \end{array}\right\}\ \text{addantur}$$

fumma —— — 0,0738374 eft menfura rationis 8 ad 7,4305816, eadem fcilicet cum ratione 80000000 ad 74305816. Porrò logarithmus abfoluti 80000000 facile acquiritur ex fuperiori tabella; cum enim index prima figuræ numeri 80000000 fit 7, è regione 7^{rii} ex fecunda columna excerpo logarithmum abfoluti 10000000 (hoc eft unitatis feptem cyphris affectæ)

qui reperitur	16,11809565 } addo
cui fubfcribo logarithmum 8^{rii}	2,07944154 }
fumma eft logarithmus abfoluti 80000000	= 18,19753719
ablata menfura rationis 80000000 ad 743005816	= 0,0738374
reftat logarithmus abfoluti 74305816	= 18,1236997, atque

tanta eft area BIHF.

Mantiffæ loco accipe modum facillimum quadrandi quamvis hyperbolæ partem per logarithmos tabulares. Dati numeri 74305816 logarithmus tabularis eft 7,87102278, per fuperioris tabellæ columnam fecundam reducendus ad naturalem, proditque eadem, quæ fupra, area BIHF = 18,123699872.

Poftremo, ne quis hæfitationi locus reftet, accipe, quo pacto ex prop. XIII, XV, XVI, Logarithmot. calculum fuperiorem derivem.

Differentia terminorum rationem quamvis exprimentium fi concipiatur divifa in partes æquales innumeras; compofita erit ratio tota extremorum terminorum ex innumeris ratiunculis terminorum à minimo ad maximum infinitiffima parte ipfius differentiæ fe mutuo excedentium. Sin iidem illi termini innumeri accipiantur pro mediis arithmeticis aliorum terminorum fimili parte infinitiffima diftantium; fumma omnium ratiuncularum pofterioribus hifce terminis intercedentium deficiet à tota ratione extremorum, non nifi femiffe primæ & ultimæ ratiuncularum à prioribus terminis contentarum, id eft, ratiuncula minori, quam quæ ullis numeris exprimi poffit. Quare pofito maximo termino $= 1$, & parte infinitiffima differentiæ $= \dot{1}$, & menfura rationis minimæ itidem $\dot{1}$; erit ut medium arithmeticum terminorum rationis minimam proximè præcedentis, ad medium arithmeticum terminorum ipfius minimæ; ita menfura minimæ, ad menfuram proxime majoris; hoc eft:

$$\left.\begin{array}{l} 1 - \dot{1} \,.\, 1 :: \dot{1} \,.\, \dot{1} + \dot{1}\dot{1} + \dot{1}^3 + \dot{1}^4 \ \&c \ \text{menfuræ ultimæ} \\ 1 - 2\dot{1} \,.\, 1 :: \dot{1} \,.\, \dot{1} + 2\dot{1}\dot{1} + 4\dot{1}^3 + 8\dot{1}^4 \ \&c \ \text{penultimæ} \\ 1 - 3\dot{1} \,.\, 1 :: \dot{1} \,.\, \dot{1} + 3\dot{1}\dot{1} + 9\dot{1}^3 + 27\dot{1}^4 \ \&c \ \text{antepenultimæ} \end{array}\right\}\text{add.}$$

fit fumma ratiuncul. $= 3\dot{1} + 6\dot{1}\dot{1} + 14\dot{1}^3 + 36\dot{1}^4$ &c = numero terminorum, plus fumma eorundem terminorum, plus fumma quadratorum ab iifdem, &c.

Sin minimus terminus ponatur $= 1$, manentibus cæteris ut supra; evadit summa ratiuncularum $= 3\dot{1} - 6\dot{1}\dot{1} + 14\dot{1}^3 - 36\dot{1}^4$, &c.

Hinc data differentia terminorum $= 0\lfloor\dot{1}$, erit numerus terminorum $= 0\lfloor 1$, & per 16 Logarithmot. summa eorundem terminorum $= 0{,}00\dot{5}$, & summa quadratorum $= 0{,}00\dot{0}333$. At data differentia terminorum $= 0\lfloor 01$, numerus terminorum est $= 0{,}0\dot{1}$, & summa eorundem $= 0{,}0000\dot{5}$, & summa quadratorum $= 0{,}0000\dot{0}333$, &c.

Nota. Prop. IV, Logarithmot. signa speciebus intercedentia debebant esse alternatim affirmata & negata: atque ubicunque lector offenderit *infinitissimam*, legat *infinitesimam*.

REMARKS

REMARKS

ON THE

TWO INFINITE SERIESES

$$A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \text{ &c and}$$

$$A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \text{ &c,}$$

Which were found by Mr. Nicholas Mercator and Dr. John Wallis, in the foregoing Tracts, for the Purpose of squaring the Hyperbolick Spaces BI*ru* and FH*ru*.

By FRANCIS MASERES, Esq.
CURSITOR BARON OF THE EXCHEQUER.

REMARKS, &c.

ARTICLE I.

THE former of thefe feriefes, to wit, $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c, gives the logarithm of the ratio of $1 + A$ to 1, whenever A is either equal to, or lefs than, 1; that is, if A be either equal to, or lefs than, 1, fo as to make the terms of the feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c converge, and B be any quantity different from A, and either equal to, or lefs than, 1, fo as to make the terms of the feries $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B^6}{6} +$ &c converge, the feries $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B^6}{6} +$ &c will be to the feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c in the fame proportion as the ratio of $1 + B$ to 1 is to the ratio of $1 + A$ to 1. Mr. Mercator's feries is the logarithm of the ratio of $1 + A$ to 1.

2. In like manner the latter of thefe feriefes, to wit, $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, gives the logarithm of the ratio of 1 to $1 - A$ when A is of any magnitude lefs than 1; that is, if A is of any magnitude lefs than 1, and B be any quantity different from A, but alfo lefs than 1, the feries $B + \frac{B^2}{2} + \frac{B^3}{3} + \frac{B^4}{4} + \frac{B^5}{5} + \frac{B^6}{6} +$ &c will be to the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c in the fame proportion as the ratio of 1 to $1 - B$ to the ratio of 1 to $1 - A$. Dr. Wallis's feries is the logarithm of the ratio of 1 to $1 - A$.

3. By the former of thefe feriefes (which was invented by Mr. Mercator) we may therefore find the logarithm of any ratio not greater than that of $1 + 1$ to 1, or 2 to 1, but not the logarithm of any greater ratio. I mean in a *direct manner:* for *indirectly* we may difcover by it the logarithm of any ratio, how great foever; to wit, by dividing fuch greater ratio into feveral other ratios, that fhall, each of them, be lefs than the ratio of 2 to 1, and computing the logarithms of fuch leffer ratios by the faid feries, and then adding Mercator's feries will exhibit only the logarithms of ratios of majority that are not greater than the ratio of 2 to 1.

adding them together into one fum. Thus, for example, the ratio of the number 1057 to 1 is equal to the fum of the feveral following ratios, to wit, the ratio of 1057 to 1050, the ratio of 1050 to 1000, the ratio of 1000 to 800, the ratio of 800 to 50, (which is equal to the ratio of 16 to 1, or to four times the ratio of 2 to 1, or to 8 times the ratio of $\surd 2$ to 1) and the ratio of 50 to 48, the ratio of 48 to 3, (which is equal to the ratio of 16 to 1, or to 8 times the ratio of $\surd 2$ to 1) the ratio of 3 to $2 + \frac{1}{2}$, the ratio of $2 + \frac{1}{2}$ to 2, the ratio of 2 to $\surd 2$, and the ratio of $\surd 2$ to 1. Therefore, if we were to compute by the faid feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, the logarithms of the ratios of 1057 to 1050, of 1050 to 1000, of 1000 to 800, of $\surd 2$ to 1, of 50 to 48, of 3 to $2 + \frac{1}{2}$, and of $2 + \frac{1}{2}$ to 2, (every one of which ratios is lefs than the ratio of 2 to 1), and then were to add together the logarithms of the ratios of 1057 to 1050, of 1050 to 1000, of 1000 to 800, and eighteen times the logarithm of the ratio of $\surd 2$ to 1, and the logarithm of the ratio of 50 to 48, and the logarithm of the ratio of 3 to $2 + \frac{1}{2}$, and that of the ratio of $2 + \frac{1}{2}$ to 2, the fum would be the logarithm of the propofed ratio of 1057 to 1. And in this indirect manner, it is evident that the logarithm of any other ratio, how great foever, might be obtained by means of this feries of Mercator. But by a direct, or immediate, application of it we cannot compute the logarithm of any ratio that is greater than that of 2 to 1.

But Dr. Wallis's feriefes will exhibit the logarithm of any ratio of majority, how great foever.

4. But by the fecond feries above-mentioned, to wit, the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ (which was invented by Dr. Wallis), it is poffible to find at once the logarithm of any ratio, how great foever. For the ratio of 1 to $1 - A$, of which the faid feries exhibits the logarithm, may, by taking A of a fufficient magnitude, though always lefs than 1, be made to equal any ratio of majority how great foever. Thus for example, if A is $= \frac{999}{1000}$, we fhall have $1 - A = 1 - \frac{999}{1000} = \frac{1000 - 999}{1000} = \frac{1}{1000}$; and confequently the ratio of 1 to $1 - A$ will in this cafe be equal to the ratio of 1 to $\frac{1}{1000}$, or of 1000 to 1, and therefore the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ will in this cafe be equal to the logarithm of the ratio of 1000 to 1. And in the fame manner it may, by increafing the magnitude of A fo as to make it approach ftill nearer to 1 (though it never can be abfolutely equal to it), be made to exhibit the logarithm of any other and greater ratio, how great foever. It may therefore be juftly confidered as a complete fupplement to the former feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, which had been invented by Mercator, and which was capable of exhibiting only the logarithms of fuch ratios as were not greater than the ratio of 2 to 1.

But neither of thefe feriefes is of much ufe in practice, unlefs the ratio,

5. It muft, however, be obferved, that neither of thefe feriefes is of much ufe in practice when A is but a little lefs than 1; as for example, when it is equal to $\frac{999}{1000}$, or $\frac{99}{100}$, or even $\frac{9}{10}$; becaufe the convergency of their terms is in thofe cafes fo exceeding flow, that it would be neceffary to compute a pro-

prodigious number of them in order to obtain the values of thefe feriefes exact to a moderate number of decimal figures. And, when A is abfolutely equal to 1, the feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, (which in this cafe becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c$, and is equal to the logarithm of the ratio of 2 to 1) converges with fuch exceffive flownefs, that it is totally impoffible to obtain its value exact to feven or eight places of figures, by the mere computation and addition and fubtraction of its terms; though by certain compendious methods that have been invented for the purpofe, its value may be found to that, or a ftill greater degree of accuracy. But, when A is equal only to $\frac{1}{9}$ or $\frac{1}{10}$, or any leffer fraction, the terms of thefe two feriefes will converge with great fwiftnefs, and they will be found exceedingly convenient for the purpofe of computing logarithms. And by the computation of thefe feriefes in thefe eafy cafes, in which their terms will decreafe fo fwiftly, it will be poffible, by a little addrefs and management in the choice of the fmall ratios, the logarithms of which we compute by them, to find the logarithms of all other ratios, how great foever; of which I propofe to give a few examples in the fubfequent part of this difcourfe.

of which the logarithm is fought by it, is a fmall ratio, or differs but little from a ratio of equality.

But when that ratio is equal to the ratio of 10 to 9, or that of 11 to 10, or any leffer ratio, thefe feriefes will be found very ufeful.

6. Some learned and ingenious mathematicians have, fince the difcovery of thefe two feriefes by Mr. Mercator and Dr. Wallis, inveftigated fome other infinite feriefes, which converge with a ftill greater degree of fwiftnefs than thefe do, in order to facilitate ftill further the computation of logarithms. But thofe feriefes are lefs fimple in the laws by which their terms are generated than thefe original ones invented by Mercator and Wallis; and there is ufually need of a good deal of care and attention in the application of them to the computation of the logarithm of a given ratio: fo that, upon the whole, I think thefe of Mercator and Wallis deferve to be preferred to them. Nor do I conceive that any more convenient methods of computing the logarithms of given ratios are likely ever to be found out, or indeed need be wifhed for, than thofe which are afforded us by thefe two valuable feriefes.

7. Thefe two feriefes $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, and $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ are nearly of equal ufe for the purpofe of computing the logarithms of given ratios; infomuch that I know not upon what ground to give one of them the preference to the other. For, though the latter will, in theory, exhibit the logarithm of any given ratio of majority how great foever; whereas the former can only exhibit the logarithm of a ratio that is not greater than the ratio of 2 to 1: yet, in practice, they are neither of them (as we have already obferved) fit to exhibit the logarithms of any but fmall ratios, or fuch as differ but little from a ratio of equality, and arife from fuppofing A to be much fmaller than 1, as, for example, to be equal to $\frac{1}{9}$ or $\frac{1}{10}$ or $\frac{1}{11}$, or fome leffer fraction; and thefe logarithms they will enable us to compute with nearly the fame degree of eafe and expedition. It feems therefore to be almoft a matter of indifference which of thefe two feriefes we make ufe of in the bufinefs of computing logarithms; and perhaps it may be beft in fome cafes to make ufe of one of them, and in other cafes to make ufe of the other. Thus, for example, if we wanted to

When the ratios, of which the logarithms are to be computed, are fmall, the two feriefes are nearly equally ufeful.

to find the logarithm of the ratio of 11 to 10 by one of thefe feriefes, it would be rather eafier to find it by means of the former feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c, which was invented by Mercator, than by the latter feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, which was invented by Dr. Wallis. For, if we confider the ratio of 11 to 10 as being equal to the ratio of $\frac{11}{10}$ to $\frac{10}{10}$, or of $\frac{10}{10} + \frac{1}{10}$ to $\frac{10}{10}$, or of $1 + \frac{1}{10}$ to 1, its logarithm will be equal to the feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c, upon a fuppofition that A is $= \frac{1}{10}$, that is to the feries $\frac{1}{10} - \frac{1}{2 \times 10^2} + \frac{1}{3 \times 10^3} - \frac{1}{4 \times 10^4} + \frac{1}{5 \times 10^5} - \frac{1}{6 \times 10^6} +$ &c, or $0,100,000,000 - \frac{0,010,000,000}{2} + \frac{0,001,000,000}{3} - \frac{0,000,100,000}{4} + \frac{0,000,010,000}{5} - \frac{0,000,001,000}{6} +$ &c. But, if we confider the fame ratio of 11 to 10 as being equal to the ratio of $\frac{11}{11}$ to $\frac{10}{11}$, or of $\frac{11}{11}$ to $\frac{11}{11} - \frac{1}{11}$, or of 1 to $1 - \frac{1}{11}$, its logarithm will be equal to the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, upon a fuppofition that A is $= \frac{1}{11}$, that is, to the feries $\frac{1}{11} + \frac{1}{2 \times 11^2} + \frac{1}{3 \times 11^3} + \frac{1}{4 \times 11^4} + \frac{1}{5 \times 11^5} + \frac{1}{6 \times 11^6} +$ &c. And it is evident that this latter feries, which proceeds by the powers of $\frac{1}{11}$, is not quite fo eafy to compute as the feries $\frac{1}{10} - \frac{1}{2 \times 10^2} + \frac{1}{3 \times 10^3} - \frac{1}{4 \times 10^4} + \frac{1}{5 \times 10^5} - \frac{1}{6 \times 10^6} +$ &c, which proceeds by the powers of $\frac{1}{10}$; and therefore in computing the logarithm of this ratio of 11 to 10, it would be rather more convenient to make ufe of the feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c than of the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c. And, on the other hand, in computing the logarithm of the ratio of 10 to 9, it would be rather more convenient to make ufe of the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c than of the feries $A - \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c. For, if we confider the ratio of 10 to 9 as being equal to the ratio of $\frac{10}{9}$ to $\frac{9}{9}$, or of $\frac{9}{9} + \frac{1}{9}$ to $\frac{9}{9}$, or of $1 + \frac{1}{9}$ to 1, its logarithm will be equal to the feries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c, upon a fuppofition that A is $= \frac{1}{9}$, that is, to the feries $\frac{1}{9} - \frac{1}{2 \times 9^2} + \frac{1}{3 \times 9^3} - \frac{1}{4 \times 9^4} + \frac{1}{5 \times 9^5} - \frac{1}{6 \times 9^6} +$ &c, or $\frac{1}{9} - \frac{1}{2 \times 81} + \frac{1}{3 \times 729} - \frac{1}{4 \times 6561} + \frac{1}{5 \times 59,049} - \frac{1}{6 \times 531,441} +$ &c, which proceeds by the powers of $\frac{1}{9}$. But, if we confider the fame ratio of 10 to 9 as being equal to the ratio of $\frac{10}{10}$ to $\frac{9}{10}$, or of $\frac{10}{10}$ to $\frac{10}{10} - \frac{1}{10}$, or of 1 to $1 - \frac{1}{10}$, its logarithm will be equal to the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, upon a fuppofition that A is $= \frac{1}{10}$, that is, to the feries $\frac{1}{10} + \frac{1}{2 \times 10^2} + \frac{1}{3 \times 10^3}$

$+ \frac{1}{4 \times 10^4} + \frac{1}{5 \times 10^5} + \frac{1}{6 \times 10^6} +$ &c, or $0{,}100{,}000{,}000 + \frac{0{,}010{,}000{,}000}{2} + \frac{0{,}001{,}000{,}000}{3} + \frac{0{,}000{,}100{,}000}{4} + \frac{0{,}000{,}010{,}000}{5} + \frac{0{,}000{,}001{,}000}{6} +$ &c, which proceeds by the powers of $\frac{1}{10}$. And it is evident that this latter ſeries, which proceeds by the powers of $\frac{1}{10}$, is rather eaſier to compute than the other ſeries $\frac{1}{9} - \frac{1}{2 \times 9^2} + \frac{1}{3 \times 9^3} - \frac{1}{4 \times 9^4} + \frac{1}{5 \times 9^5} - \frac{1}{6 \times 9^6} +$ &c, which proceeds by the powers of $\frac{1}{9}$: and therefore in computing the logarithm of this ratio of 10 to 9, it would be rather more convenient to make uſe of the ſeries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c than of the ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c. But the advantage of the one of theſe ſerieſes over the other in theſe examples, and in all other caſes of the ſame kind, with reſpect to the facility of computing them, is not very conſiderable.

8. There is, however, one advantage in having both theſe ſerieſes for the purpoſe of computing logarithms rather than only one of them, which may deſerve to be taken notice of; which is, that they ſerve to confirm each other, or as proofs of the truth of the arithmetical operations, by which the logarithm of a given ratio has been obtained. Thus, for example, if we had computed the logarithm of the ratio of 11 to 10 by the ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c to eighteen or twenty places of figures, and were doubtful whether we had not made a ſlip in ſome of the arithmetical operations by which we had obtained it, it would be expedient to compute it alſo by means of the ſeries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c to the ſame number of places of figures. And, if the value obtained for it by this ſecond ſeries agreed exactly with the value found for it by the firſt ſeries, we might conclude with confidence, that the arithmetical operations had been rightly performed, and that the value of the ſaid logarithm was rightly aſſigned.

And they may be uſed as proofs and confirmations of each other.

9. As theſe two ſerieſes $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} +$ &c and $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, are ſo beautiful from their ſimplicity, and at the ſame time ſo fit and convenient for the purpoſe of computing logarithms, it ſeems to be deſirable that the truth of them ſhould be eſtabliſhed in the fulleſt and cleareſt manner poſſible, and by more than one demonſtration. The foregoing demonſtrations of them are derived from the contemplation of the hyperbola, and that of the latter ſeries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, which was invented by Dr. Wallis, is grounded likewiſe on ſome of the propoſitions of that learned author's ARITHMETICA INFINITORUM, which is a work that is now but little read, though formerly it had a great and deſerved reputation. I ſhall therefore here ſubjoin other inveſtigations of both theſe ſerieſes, which have no relation to the hyperbola, or to any other curve-lined figure what-

As theſe two ſerieſes are ſo highly uſeful, it is fit that their truth ſhould be eſtabliſhed by more than one demonſtration.

Other demonſtrations will therefore be given

of them in this diſcourſe.

whatſoever, but are grounded on principles that are purely arithmetical, and derived from the nature of ratios (of which logarithms are nothing more than the meaſures expreſſed in numbers), by the help of the celebrated theorem of Sir Iſaac Newton for exhibiting, in an infinite ſeries of rational quantities, the roots of a binomial quantity. They are, as I believe, the ſame in ſubſtance with the inveſtigation given of theſe ſerieſes by Dr. Edmund Halley in his diſcourſe on this ſubject, in the Philoſophical Tranſactions, N° 216, and were ſuggeſted to my mind by reading that diſcourſe; which, however, is written with ſo great a degree of obſcurity that I am not ſure that I rightly underſtand it. But, whether they expreſs the ideas he meant to convey, and are therefore only a commentary upon his inveſtigation, or whether they are ſomething different from it, I have endeavoured to make them as clear and as eaſy as I could; and, in order to enſure that eſſential object in treating of theſe difficult ſubjects, have not ſcrupled to expoſe myſelf to the danger of being cenſured by ſome of my more acute and learned readers for the much leſs fault of too great prolixity. The inveſtigation of Mercator's ſeries $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, which expreſſes the logarithm of the ratio of $1 + A$ to 1, is as follows:

The propoſition which is here intended to be demonſtrated.

10. The propoſition of which I here mean to eſtabliſh the truth, is this: If there be two different quantities, A and B, that are both of them leſs than 1, and of which, conſequently, the ſeveral powers A^2, A^3, A^4, A^5, A^6, &c, and B^2, B^3, B^4, B^5, B^6, &c, are decreaſing quantities, the two infinite ſerieſes $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ and $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B}{6} + \&c$, (which, it is evident, will be decreaſing progreſſions) will be meaſures, or logarithms of the ratios of $1 + A$ to 1 and of $1 + B$ to 1; or the ſeries $A - \frac{A^3}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ will be to the ſeries $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B^6}{6} + \&c$ in the ſame proportion as the ratio of $1 + A$ to 1 is to the ratio of $1 + B$ to 1.

Or, if we change the notation a little, and ſubſtitute the ſmall letters k and q, inſtead of the capital letters A and B reſpectively (which will be more agreeable to the notation now moſt in uſe in treating of theſe kinds of ſerieſes), the propoſition which we are to demonſtrate the truth of, will be as follows:

If there be two different quantities k and q, that are both of them leſs than 1, and of which conſequently the ſeveral powers k^2, k^3, k^4, k^5, k^6, &c, and q^2, q^3, q^4, q^5, q^6, &c, will be decreaſing quantities, the two infinite ſerieſes $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ and $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, (which, it is evident, will be decreaſing progreſſions), will be meaſures, or logarithms, of the ratios of $1 + k$ to 1 and of $1 + q$ to 1; or the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ will be to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ in the ſame proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1.

In

In order to demonſtrate the truth of this propoſition, it will be expedient to premiſe the two following propoſitions, as lemmas.

LEMMA I.

11. If there be three unequal quantities, as the right lines AD, AE, and AF in the figure hereunto annexed, whereof AD is the leaſt, and AF is the greateſt, and their differences DE, EF, are extremely ſmall in compariſon of the quantities themſelves (as for example, leſs than a millionth part of either of them), the proportion of the ratio of AF (the greateſt of the three quantities) to AD, (the leaſt of them), to the ratio of AE (the middle quantity) to AD (the leaſt quantity) will be very nearly equal to the proportion of the difference, DF, of the terms of the former ratio, to the difference, DE, of the terms of the latter ratio. And the differences DF, DE may be taken ſo very ſmall in compariſon of the leaſt term AD, that the proportion of the ſaid differences ſhall approach as near as we pleaſe to the proportion of the foreſaid ratios, or ſhall differ from it by a quantity, or ratio, that is leſs than any aſſigned ratio whatſoever.

DEMONSTRATION.

Let the ratio of AF, the greateſt of the three quantities, to AD, the leaſt of them, be divided into a very great number of leſſer ratios, all equal to each other, by the inſertion of intermediate proportionals, as AL, AM, AN, &c, between AD and AF.

Then, ſince AM is to AL as AL is to AD, it will follow, *dividendo,* that LM is to AL as DL to AD, and, *permutando,* that LM is to DL as AL to AD. And, in like manner, becauſe AN is to AM as AM is to AL, it will follow, *dividendo,* that MN is to AM as LM is to AL, and, *permutando,* that MN is to LM as AM is to AL. Therefore, when AD, AL, AM, and AN are nearly equal to each other, the differences DL, LM, and MN will alſo be nearly equal to each other. Therefore, when the two extreme terms AD and AF (and conſequently, *à fortiori,* all the intermediate terms, AL, AM, AN, &c, between the ſaid two extreme terms), are nearly equal to each other (as for example, when AF exceeds AD only in the proportion of 1000,001 to 1000,000), all the differences DL, LM, MN, &c contained between the points D and F, will alſo be nearly equal to each other, and more nearly than AD is to AF: ſo that, if AF is to AD as 1000,001 to 1000,000, the common ratio of MN to LM, and of LM to DL (being the ſame with the common ratio of AN to AM, and of AM to AL, and of AL to AD), will be leſs, or nearer to a ratio of equality, than the ratio of 1000,001 to 1000,000.

 Now,

Now, when two lines or other quantities, of unequal magnitudes, are divided into parts of the fame fize, the length or magnitude of the greater of the two will be to the length or magnitude of the leffer in the fame proportion as the number of thofe equal parts contained in the greater to the number of them contained in the leffer. Therefore, when AF is to AD in a proportion that approaches very nearly to a ratio of equality (as for example, when it exceeds AD only in the proportion of 1000,001 to 1000,000, or the excefs of AF above AD is equal to only one millionth part of AD), and confequently the differences DL, LM, and MN, &c, are nearly equal to each other, the line DF will be to the line DE very nearly in the fame proportion as the number of the leffer differences DL, LM, MN, &c, contained between the points D and F, to the number of thofe differences contained between the points D and E, and confequently in the fame proportion as the number of the equal ratios of AL to AD, AM to AL, AN to AM, &c, contained in the ratio of AF to AD, to the number of thofe ratios contained in the ratio of AE to AD, and therefore in the fame proportion as the ratio of AF to AD to the ratio of AE to AD; that is, DF, the difference of the greateft and leaft terms AF and AD, is to DE, the difference of the middle term AE and the leaft term AD, in this cafe very nearly in the fame proportion as the ratio of AF, the greateft term, to AD, the leaft term, is to the ratio of AE, the middle term, to AD, the leaft term.

And, as, by diminifhing continually the difference DF, by making the point F approach nearer and nearer to the point D, the proportion of AF to AD may be made to approach as near as we pleafe to a ratio of equality, it follows that that difference may be taken fo fmall, in comparifon of AD, that the common ratio of the differences DL, LM, MN, &c to each other fhall approach as near as we pleafe to a ratio of equality, and confequently that the proportion of DF to DE fhall be as nearly equal as we pleafe to the proportion of the ratio of AF to AD to the ratio of AE to AD. Q. E. D.

LEMMA II.

Sir Ifaac Newton's binomial theorem in the cafe of roots, expreffed in the plaineft and cleareft manner.

12. If x be any quantity not greater than 1, and n be any whole number whatfoever, the quantity $\overline{1+x}|^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, that is, the n^{th} root of the binomial quantity $1+x$, will be equal to the following infinite feries of decreafing terms, to wit, $1 + \frac{1}{n} \times x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times x^5 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{5n-1}{6n} \times x^6 +$ &c ad infinitum, or (if we put $A = 1$, and $B = \frac{1}{n}$, and $C = \frac{1}{n} \times \frac{n-1}{2n}$, and $D = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and E, F, G, H, I, K, &c for the feve-

 ral

ral following co-efficients of the powers of x in this ſeries), to the ſeries $1 + \frac{1}{n} \times \text{A}x - \frac{n-1}{2n} \times \text{B}x^2 + \frac{2n-1}{3n} \times \text{C}x^3 - \frac{3n-1}{4n} \times \text{D}x^4 + \frac{4n-1}{5n} \times \text{E}x^5 - \frac{5n-1}{6n} \times \text{F}x^6 + \frac{6n-1}{7n} \times \text{G}x^7 - \frac{7n-1}{8n} \times \text{H}x^8 + \frac{8n-1}{9n} \times \text{I}x^9 - \frac{9n-1}{10n} \times \text{K}x^{10}$ + &c ad infinitum; in which ſeries the ſeveral co-efficients of x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, &c, are derived, or generated, from the firſt term 1 by the continual, or ſucceſſive, multiplication of the fractions $\frac{1}{n}$, $\frac{n-1}{2n}$, $\frac{2n-1}{3n}$, $\frac{3n-1}{4n}$, $\frac{4n-1}{5n}$, $\frac{5n-1}{6n}$, $\frac{6n-1}{7n}$, $\frac{7n-1}{8n}$, $\frac{8n-1}{9n}$, $\frac{9n-1}{10n}$, &c, which are therefore called their *generating fractions.*

The generating fractions of the terms.

13. In theſe generating fractions, it is evident that $n - 1$, the numerator of the ſecond fraction $\frac{n-1}{2n}$, is formed from 1, the numerator of the firſt fraction $\frac{1}{n}$ by the addition of $n - 2$, or the addition of n and the ſubtraction of 2; for $1 + n - 2$ is $= n - 1$: And it is evident likewiſe, that the numerators of all the following fractions after the ſecond fraction $\frac{n-1}{2n}$, to wit, the numerators $2n - 1$, $3n - 1$, $4n - 1$, $5n - 1$, $6n - 1$, $7n - 1$, $8n - 1$, $9n - 1$, &c, are formed from the numerator $n - 1$ of the ſaid ſecond fraction, and from each other by the continual addition of n to every preceding numerator. And the denominators of the ſecond fraction $\frac{n-1}{2n}$ and the third fraction $\frac{2n-1}{3n}$ and all the following fractions, to wit, the denominators $2n$, $3n$, $4n$, $5n$, $6n$, $7n$, $8n$, $9n$, $10n$, &c, are formed from n (the denominator of the firſt fraction $\frac{1}{n}$), and from each other by the continual addition of n to the preceding denominator.

The conſtitution, or law, of the ſaid generating fractions.

14. This is Sir Iſaac Newton's famous binomial theorem in the caſe of roots or fractional powers, expreſſed in what I take to be the moſt intelligible and convenient manner poſſible; as by this way of expreſſing it we avoid all mention of negative quantities in the co-efficients of the ſeveral terms of the ſeries, from the uſe of which negative quantities, I have obſerved that a good deal of perplexity often ariſes in other ways of expreſſing it. This expreſſion of the value of $\overline{1+x}|^{\frac{1}{n}}$ is derived from the ſeries $1 + \frac{m}{1}\text{A}x + \frac{m-1}{2}\text{B}x^2 + \frac{m-2}{3}\text{C}x^3 + \frac{m-3}{4}\text{D}x^4 + \frac{m-4}{5}\text{E}x^5 + \frac{m-5}{6}\text{F}x^6$ + &c (which is $= \overline{1+x}|^m$) by ſubſtituting $\frac{1}{n}$ in its terms inſtead of m.

15. For a demonſtration of this celebrated theorem in the caſe of roots, or fractional powers, I refer the reader to a learned treatiſe of Algebra, lately publiſhed at Dublin by Dr. Hales, a fellow of Trinity College in that city, and intitled *Analyſis Æquationum.* In this valuable work (which was publiſhed in the year 1784) the reader will find, amongſt many other curious matters, a de-

Books in which the reader may find demonſtrations of this theorem.

a demonſtration of this theorem of Sir Iſaac Newton, that extends to the caſe of roots, or fractional powers, as well as to the caſe of integral powers. See that treatiſe, pages 33, 34, 35, &c, — 39. And he may likewiſe ſee another and very ſatisfactory demonſtration of this theorem in the caſe of roots, or fractional powers, in the Mathematical Tracts of the very learned Dr. Charles Hutton, Profeſſor of Mathematics at the Royal Military Academy at Woolwich, which were publiſhed at London in the year 1786. See the ſaid Tracts, Tract VI, pages 65, 66, 67, &c, —— 79. On the preſent occaſion I ſhall take the truth of this theorem for granted.

16. Coroll. 1. It follows from the foregoing lemma, that $\overline{1+x}|^{\frac{1}{n}} - 1$, or $\sqrt[n]{1+x} - 1$, or the exceſs of the n^{th} root of the binomial quantity $1+x$ above 1, is equal to the infinite ſeries $\frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \frac{5n-1}{6n}Fx^6 +$ &c; in which A is, as before, equal to 1; and B is $= \frac{1}{n}A$, or the co-efficient of x; and C is $= \frac{n-1}{2n}B$, or the co-efficient of x^2; and D is $= \frac{2n-1}{3n}C$, or the co-efficient of x^3; and the following capital letters E, F, G, &c, are equal to, or ſtand for, the co-efficients of x^4, x^5, x^6, and the other following powers of x.

17. Coroll. 2. If n be a very large number, as for example, the ninth power of a million, or 1 with 54 cyphers annexed to it, or 1,000000,000000,000000, 000000,000000,000000,000000,000000,000000, (which Mr. Locke, in his Eſſay on Human Underſtanding, calls a nonillion), the quantity $\overline{1+x}|^{\frac{1}{n}} - 1$, or the exceſs of the n^{th} root of the binomial quantity $1+x$ above 1, will be very nearly equal to the ſeries $\frac{x}{n} - \frac{x^2}{2n} + \frac{x^3}{3n} - \frac{x^4}{4n} + \frac{x^5}{5n} - \frac{x^6}{6n} +$ &c, *ad infinitum*, or to $\frac{1}{n} \times$ the ſeries $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} +$ &c, *ad infinitum:* and n may be taken of ſo great a magnitude, that the ratio of $\frac{1}{n} \times$ the ſeries $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} +$ &c, to $\overline{1+x}|^{\frac{1}{n}} - 1$ ſhall approach as near to a ratio of equality as we pleaſe.

For, when n is equal to a nonillion, or any ſuch very great number, it is evident that $n-1$ and $2n-1$ and $3n-1$ and $4n-1$ and $5n-1$, &c, will be very nearly equal to n, $2n$, $3n$, $4n$, $5n$, &c, on account of the immenſe magnitudes of the numbers n, $2n$, $3n$, $4n$, $5n$, &c in compariſon of the unit which is ſubtracted from them. Therefore the ſeries $\frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \frac{5n-1}{6n}Fx^6 +$ &c (which is $= \overline{1+x}|^{\frac{1}{n}} - 1$) will in this caſe be very nearly equal to $\frac{1}{n}Ax - \frac{n}{2n}Bx^2 + \frac{2n}{3n}$

Cx^3

$\text{C}x^3 - \frac{3n}{4n}\text{D}x^4 + \frac{4n}{5n}\text{E}x^5 - \frac{5n}{6n}\text{F}x^6 + \&c, = \frac{1}{n}\text{A}x - \frac{1}{2}\text{B}x^2 + \frac{2}{3}$ $\text{C}x^3 - \frac{3}{4}\text{D}x^4 + \frac{4}{5}\text{E}x^5 - \frac{5}{6}\text{F}x^6 + \&c, = \frac{1}{n} \times 1 \times x - \frac{1}{2} \times \frac{1}{n} \times 1 \times x^2 + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^3 - \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^4 + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^5 - \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^6 + \&c, = \frac{1}{n} \times x - \frac{1}{n} \times \frac{x^2}{2} + \frac{1}{n} \times \frac{x^3}{3} - \frac{1}{n} \times \frac{x^4}{4} + \frac{1}{n} \times \frac{x^5}{5} - \frac{1}{n} \times \frac{x^6}{6} + \&c, =$ $\frac{1}{n} \times$ the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.$ Therefore $\overline{1 + x}|^{\frac{1}{n}} - 1$ will in this case be very nearly equal to $\frac{1}{n} \times$ the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c,$ *ad infinitum*. And it is evident, that if any ratio whatsoever be assigned, that differs very little from a ratio of equality, n may be taken of so great a magnitude that the proportion of $\overline{1 + x}|^{\frac{1}{n}} - 1$ to $\frac{1}{n} \times$ the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c,$ shall approach still nearer to a ratio of equality than such assigned ratio. Q. E. D.

18. These lemmas being premised, the main proposition may be stated and demonstrated in the manner following.

THEOREM I.

If k and q are any two quantities less than 1, whereof we will suppose k to be the greater; the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1 in the same proportion as the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ to the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c.$

DEMONSTRATION.

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

Then, by the 2nd corollary of the 2nd of the foregoing lemmas, we shall have $\overline{1 + k}|^{\frac{1}{n}} - 1$, very nearly, $= \frac{1}{n} \times$ the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^5}{6} +$ &c, *ad infinitum*, and $\overline{1 + q}|^{\frac{1}{n}} - 1$, very nearly, $= \frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c.$ And, by increasing the magnitude of n, each of these ratios may be made to come as near to a ratio of equality as we please.

pleaſe. Therefore the proportion of $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$ is in this caſe very nearly the ſame as that of $\frac{1}{n} \times$ the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$, to $\frac{1}{n} \times$ the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, and conſequently, as that of n times $\frac{1}{n} \times$ the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$, to n times $\frac{1}{n} \times$ the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, or as that of the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$, to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$; that is, the proportion of the exceſs of the n^{th} root of the binomial quantity $1 + k$ above 1 to the exceſs of the n^{th} root of the binomial quantity $1 + q$ above 1 is, in this caſe of the very great magnitude of n, very nearly the ſame as that of the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum* to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ *ad infinitum.*

Now the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1 in the ſame proportion as any given part of the former ratio, is to the like part of the latter ratio, and conſequently, as the n^{th}, or nonillionth, part of the former ratio is to the n^{th}, or nonillionth, part of the latter ratio. But the ratio of the n^{th} root of $1 + k$ to 1 is the n^{th} part of the ratio of $1 + k$ to 1; and the ratio of the n^{th} root of $1 + q$ to 1 is the n^{th} part of the ratio of $1 + q$ to 1. Therefore the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1 in the ſame proportion as the very ſmall ratio of the n^{th} root of $1 + k$ to 1 is to the very ſmall ratio of the n^{th} root of $1 + q$ to 1.

But it has been ſhewn above in lemma 1, that when three quantities are very nearly equal to each other, the ratio of the greateſt of the three to the leaſt will be to the ratio of the middle quantity to the leaſt in very nearly the ſame proportion as the exceſs of the greateſt quantity above the leaſt is to the exceſs of the middle quantity above the leaſt: and by diminiſhing the greateſt and middle quantities continually, ſo that they ſhall approach nearer and nearer to an equality with the leaſt, we may make theſe two proportions, to wit, that of the ſaid two ratios and that of the ſaid two exceſſes, approach as near to each other as we pleaſe. Therefore in theſe three quantities, to wit, 1 and the n^{th}, or nonillionth, root of $1 + q$ and the n^{th}, or nonillionth, root of $1 + k$, (in which the exceſſes of the two latter quantities above the firſt quantity 1 are ſo extremely ſmall that they will not appear before the 54^{th} place of decimal fractions), we may conclude, that the ratio of the greateſt, to wit, $\overline{1+k}|^{\frac{1}{n}}$, or the n^{th} root of $1 + k$, to 1, the leaſt, will be to the ratio of the ſecond quantity, to wit, $\overline{1+q}|^{\frac{1}{n}}$, or the n^{th} root of $1 + q$, to 1, the leaſt, very nearly in the ſame

ſame proportion as the exceſs of the greateſt, or $\overline{1+k}|^{\frac{1}{n}}$, above the leaſt, or 1, to the exceſs of the ſecond quantity, or $\overline{1+q}|^{\frac{1}{n}}$, above the leaſt, or 1.

But it has been before ſhewn, that the ratio of $1+k$ to 1 is to the ratio of $1+q$ to 1 in the ſame proportion as the very ſmall ratio of the n^{th} root of $1+k$ to 1 is to the very ſmall ratio of the n^{th} root of $1+q$ to 1.

Therefore the ratio of $1+k$ to 1 will be to the ratio of $1+q$ to 1 very nearly in the ſame proportion as the exceſs of the n^{th} root of $1+k$ above 1 to the exceſs of the n^{th} root of $1+q$ above 1.

But it has been ſhewn above, that the proportion of the exceſs of the n^{th} root of $1+k$ above 1 to the exceſs of the n^{th} root of $1+q$ above 1 is, in this caſe of the very great magnitude of n, very nearly the ſame as that of the ſeries $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ &c *ad infinitum* to the ſeries $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ &c *ad infinitum.*

Therefore the ratio of $1+k$ to 1 will be to the ratio of $1+q$ to 1, in the ſame proportion as the ſeries $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ &c *ad infinitum* to the ſeries $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ &c *ad infinitum.*

Q. E. D.

19. The foregoing demonſtration may be expreſſed in fewer words, as follows: *The foregoing demonſtration expreſſed in fewer words.*

Let n be put as before, for any large number; as, for example, for a nonillion, or the ninth power of a million.

Then it will follow from lemma 2, coroll. 2, that $\overline{1+k}|^{\frac{1}{n}}-1$ will be very nearly equal to $\frac{1}{n}\times$ the infinite ſeries $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ &c, and that $\overline{1+q}|^{\frac{1}{n}}-1$ will, in like manner, be, very nearly, equal to $\frac{1}{n}\times$ the ſeries $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ &c. Therefore $\overline{1+k}|^{\frac{1}{n}}-1$ will be to $\overline{1+q}|^{\frac{1}{n}}-1$, very nearly, in the ſame proportion as $\frac{1}{n}\times$ the ſeries $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ &c to $\frac{1}{n}\times$ the ſeries $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ &c, that is, in the ſame proportion as the n^{th} part of the ſeries $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ &c to the n^{th} part of the ſeries $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ &c, and conſequently in the ſame proportion as the whole ſeries $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ &c to the whole ſeries $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ &c.

Now

Now the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1 in the ſame proportion as the n^{th} part of the former ratio is to the n^{th} part of the latter ratio, and conſequently in the ſame proportion as the ratio of $\overline{1 + k}|^{\frac{1}{n}}$ to 1 is to the ratio of $\overline{1 + q}|^{\frac{1}{n}}$ to 1.

But, becauſe $\overline{1 + k}|^{\frac{1}{n}}$ and $\overline{1 + q}|^{\frac{1}{n}}$ approach extremely near to an equality with 1, it follows from lemma 1, that the ratio of $\overline{1 + k}|^{\frac{1}{n}}$ to 1 will be to the ratio of $\overline{1 + q}|^{\frac{1}{n}}$ to 1, very nearly, in the ſame proportion as the exceſs of $\overline{1 + k}|^{\frac{1}{n}}$ above 1 to the exceſs of $\overline{1 + q}|^{\frac{1}{n}}$ above 1, or as $\overline{1 + k}|^{\frac{1}{n}} - 1$ to $\overline{1 + q}|^{\frac{1}{n}} - 1$.

Therefore the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1, very nearly, in the ſame proportion as $\overline{1 + k}|^{\frac{1}{n}} - 1$ is to $\overline{1 + q}|^{\frac{1}{n}} - 1$.

But it has been ſhewn, that, in this caſe of the very great magnitude of n, the proportion of $\overline{1 + k}|^{\frac{1}{n}} - 1$ to $\overline{1 + q}|^{\frac{1}{n}} - 1$ is, very nearly, the ſame as that of the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c *ad infinitum* to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c *ad infinitum.*

Therefore the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1 in the ſame proportion as the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c *ad infinitum* to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c *ad infinitum.* Q. E. D.

20. And, if ſtill greater brevity of expreſſion be deſired, the foregoing demonſtration may be expreſſed in a ſtill conciſer manner by the help of a little additional notation, as follows:

The ſame demonſtration expreſſed with ſtill greater brevity.

Let R. $\frac{1 + k}{1}$ be uſed to denote the ratio of $1 + k$ to 1, and R. $\frac{1 + q}{1}$ to denote the ratio of $1 + q$ to 1, and R. $\frac{\overline{1 + k}|^{\frac{1}{n}}}{1}$ to denote the ratio of $\overline{1 + k}|^{\frac{1}{n}}$ to 1, and R. $\frac{\overline{1 + q}|^{\frac{1}{n}}}{1}$ to denote the ratio of $\overline{1 + q}|^{\frac{1}{n}}$ to 1, and $\frac{1}{n} \times$ R. $\frac{1 + k}{1}$ to denote the n^{th} part of the ratio of $1 + k$ to 1, and $\frac{1}{n} \times$ R. $\frac{1 + q}{1}$ to denote the n^{th} part of the ratio of $1 + q$ to 1.

With this notation the demonſtration will be as follows:

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

Then

Then will R. $\frac{1+k}{1}$ be to R. $\frac{1+q}{1}$ in the same proportion as $\frac{1}{n} \times$ R. $\frac{1+k}{1}$ to $\frac{1}{n} \times$ R. $\frac{1+q}{1}$, that is (by the nature of roots), as R. $\frac{\overline{1+k}|^{\frac{1}{n}}}{1}$ to R. $\frac{\overline{1+q}|^{\frac{1}{n}}}{1}$, that is, by lemma 1 (on account of the near approach of $\overline{1+k}|^{\frac{1}{n}}$ and $\overline{1+q}|^{\frac{1}{n}}$ to an equality with 1), as $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$, that is, by lemma 2, coroll. 2 (on account of the very great magnitude of the number n), as $\frac{1}{n} \times$ the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c. to $\frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c. and consequently (multiplying both sides by n), as the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c. to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c. Q. E. D.

SCHOLIUM.

21. This proposition is *accurately* true, though, when n is of any finite magnitude, how great soever, some of the intermediate propositions, by means of which it is proved, are only *very nearly* true. For, as these intermediate propositions are not limited in the degree in which they approach to truth or accuracy, but may be made to come as near to being accurately true as we please, by increasing the number n, the conclusion derived from them must be accurately true. For, if it were supposed to be not accurately true, but only to approach within certain limits of the truth, we might increase the magnitude of n till the said conclusion was made to approach nearer to being accurately true than the assigned limits; which would be contrary to the supposition of its having approached only within the said assigned limits of the truth, and consequently would prove that the said supposition was false. Therefore the said conclusion does not only approach within certain limits of the truth, but is accurately true.

22. We have now, I hope, sufficiently established the truth of Mr. Mercator's proposition, to wit, "that if q be made to represent successively several different fractions, or quantities less than 1, the corresponding values of the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c. will be proportional to the several ratios of the corresponding values of $1 + q$ to 1 respectively, or will be the logarithms of those ratios." It remains that we illustrate the use of this series in the business of computing logarithms by applying it to a few examples. I shall therefore now proceed to apply it to the computation of the logarithms of the nine following ratios, to wit, the ratio of 10 to 9, that of 11 to 10, that of 81 to 80, that of 121 to 120, that of 2401 to 2400, that of

169 to 168, that of 289 to 288, that of 361 to 360, and that of 529 to 528; by means of which we ſhall be able to diſcover the logarithms of the ratios of the 23 firſt natural numbers, to wit, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, to 1, or (according to the common abridged and imperfect way of ſpeaking on this ſubject), the logarithms of thoſe numbers. By theſe computations the great utility and excellency of this ſeries will be moſt manifeſt. And as in making theſe computations in the inſtances above-mentioned the numerator of the fraction that is equal to q is always 1, and the denominator of it is a whole number, I think it will be convenient to ſubſtitute $\frac{1}{m}$ inſtead of q in the binomial quantity $1 + q$ and the infinite ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c. whereby the ſaid binomial quantity will be changed into $1 + \frac{1}{m}$, and the ſaid ſeries will be changed into the ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c. or the logarithm of the ratio of $1 + \frac{1}{m}$ to 1 will be the ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ *&c. ad infinitum.*

23. If we ſhould have occaſion to expreſs this ſeries in ſuch a manner as to point out the generating fractions by the ſucceſſive multiplication of which its ſecond and third and other following terms are derived from the firſt term $\frac{1}{m}$ and from each other, it may be done by putting A for the whole firſt term $\frac{1}{m}$, and B for the whole ſecond term $\frac{1}{2m^2}$, and C for the whole third term $\frac{1}{3m^3}$, and D for the whole fourth term $\frac{1}{4m^4}$, and E, F, G, H, I, K, L, M, and the following capital letters of the alphabet, for the whole fifth, ſixth, ſeventh, eighth, ninth, tenth, eleventh, twelfth, and other following terms of the ſeries; by which ſubſtitution the ſaid ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} +$ &c will be converted into the ſeries $\frac{1}{m} - \frac{A}{2m} + \frac{2B}{3m} - \frac{3C}{4m} + \frac{4D}{5m} - \frac{5E}{6m} + \frac{6F}{7m} - \frac{7G}{8m} + \frac{8H}{9m} - \frac{9I}{10m} +$ &c. But the terms of the ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} +$ &c are ſo extremely ſimple and conciſe that there ſeems to be no occaſion to change its form.

EXAMPLES

EXAMPLES

OF THE

COMPUTATION OF THE LOGARITHMS

OF THE

RATIOS OF SEVERAL NUMBERS

Denoted by the binomial quantity $1 + \frac{1}{m}$

TO I.

By means of the infinite series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} + \frac{1}{11m^{11}} - \frac{1}{12m^{12}} + \frac{1}{13m^{13}} - \frac{1}{14m^{14}} + \frac{1}{15m^{15}} - \frac{1}{16m^{16}} + \frac{1}{17m^{17}} - \frac{1}{18m^{18}} +$ &c,

INVENTED BY MR. NICHOLAS MERCATOR.

EXAMPLE I.

24. LET it be required to find by means of the ſaid ſeries the logarithm of the ratio of 10 to 9 (or of 9 + 1 to 9, or of $\frac{9+1}{9}$ to $\frac{9}{9}$), or of $1 + \frac{1}{9}$ to 1.

Here m is = 9, and $\frac{1}{m}$ is $= \frac{1}{9} = 0.111,111,111,111,111,111$, &c.

We ſhall therefore have

$\frac{1}{m^2}$ ($= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{9} = \frac{0.111,111,111,111,111,111, \&c}{9}$) $= 0.012,345,679,012,345,679$;

And $\frac{1}{m^3}$ ($= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{9} = \frac{0.012,345,679,012,345,679}{9}$) $= 0.001,371,742,112,482,853$;

And $\frac{1}{m^4}$ ($= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{9} = \frac{0.001,371,742,112,482,853}{9}$) $= 0.000,152,415,790,275,872$;

And

And $\frac{1}{m^5}$ $(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{9} = \frac{0.000,152,415,790,275,872}{9}) = 0.000,016,935,087,808,430$;

And $\frac{1}{m^6}$ $(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{9} = \frac{0.000,016,935,087,808,430}{9}) = 0.000,001,881,676,423,158$;

And $\frac{1}{m^7}$ $(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{9} = \frac{0.000,001,881,676,423,158}{9}) = 0.000,000,209,075,158,128$;

And $\frac{1}{m^8}$ $(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{9} = \frac{0.000,000,209,075,158,128}{9}) = 0.000,000,023,230,573,125$;

And $\frac{1}{m^9}$ $(= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{9} = \frac{0.000,000,023,230,573,125}{9}) = 0.000,000,002,581,174,791$;

And $\frac{1}{m^{10}}$ $(= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,002,581,174,791}{9}) = 0.000,000,000,286,797,199$;

And $\frac{1}{m^{11}}$ $(= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{9} = \frac{0.000,000,000,286,797,199}{9}) = 0.000,000,000,031,866,355$;

And $\frac{1}{m^{12}}$ $(= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{9} = \frac{0.000,000,000,031,866,355}{9}) = 0.000,000,000,003,540,706$;

And $\frac{1}{m^{13}}$ $(= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{9} = \frac{0.000,000,000,003,540,706}{9}) = 0.000,000,000,000,393,411$;

And $\frac{1}{m^{14}}$ $(= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{9} = \frac{0.000,000,000,000,393,411}{9}) = 0.000,000,000,000,043,712$;

And $\frac{1}{m^{15}}$ $(= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{9} = \frac{0.000,000,000,000,043,712}{9}) = 0.000,000,000,000,004,856$;

And $\frac{1}{m^{16}}$ $(= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{9} = \frac{0.000,000,000,000,004,856}{9}) = 0.000,000,000,000,000,539$;

And $\frac{1}{m^{17}}$ $(= \frac{1}{m^{16}} \times \frac{1}{m} = \frac{1}{m^{16}} \times \frac{1}{9} = \frac{0.000,000,000,000,000,539}{9}) = 0.000,000,000,000,000,059$;

And $\frac{1}{m^{18}}$ $(= \frac{1}{m^{17}} \times \frac{1}{m} = \frac{1}{m^{17}} \times \frac{1}{9} = \frac{0.000,000,000,000,000,059}{9}) = 0.000,000,000,000,000,006$;

And confequently

$\frac{1}{2m^2}$ $(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.012,345,679,012,345,679}{2}) = 0.006,172,839,506,172,839$;

And $\frac{1}{3m^3}$ $(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.001,371,742,112,482,853}{3}) = 0.000,457,247,370,827,617$;

And $\frac{1}{4m^4}$ $(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,152,415,790,275,872}{4}) = 0.000,038,103,947,568,968$;

And $\frac{1}{5m^5}$ $(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,016,935,087,808,430}{5}) = 0.000,003,387,017,561,686$;

And $\frac{1}{6m^6}$ $(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,001,881,676,423,158}{6}) = 0.000,000,313,612,737,193$;

And $\frac{1}{7m^7}$ $(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,209,075,158,128}{7}) = 0.000,000,029,867,879,732$;

And

And $\frac{1}{8m^8}$ $\left(= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,023,230,573,125}{8}\right) = 0.000,000,002,903,821,640$;

And $\frac{1}{9m^9}$ $\left(= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,002,581,174,791}{9}\right) = 0.000,000,000,286,797,199$;

And $\frac{1}{10m^{10}}$ $\left(= \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,286,797,199}{10}\right) = 0.000,000,000,028,679,719$;

And $\frac{1}{11m^{11}}$ $\left(= \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,031,866,355}{11}\right) = 0.000,000,000,002,896,941$;

And $\frac{1}{12m^{12}}$ $\left(= \frac{1}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,003,540,706}{12}\right) = 0.000,000,000,000,295,058$;

And $\frac{1}{13m^{13}}$ $\left(= \frac{1}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,393,411}{13}\right) = 0.000,000,000,000,030,262$;

And $\frac{1}{14m^{14}}$ $\left(= \frac{1}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,043,712}{14}\right) = 0.000,000,000,000,003,122$;

And $\frac{1}{15m^{15}}$ $\left(= \frac{1}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,004,856}{15}\right) = 0.000,000,000,000,000,323$;

And $\frac{1}{16m^{16}}$ $\left(= \frac{1}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,539}{16}\right) = 0.000,000,000,000,000,033$;

And $\frac{1}{17m^{17}}$ $\left(= \frac{1}{m^{17}} \times \frac{1}{17} = \frac{0.000,000,000,000,000,059}{17}\right) = 0.000,000,000,000,000,003$;

And $\frac{1}{18m^{18}}$ $\left(= \frac{1}{m^{18}} \times \frac{1}{18} = \frac{0.000,000,000,000,000,006}{18}\right) = 0.000,000,000,000,000,000$.

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} + \frac{1}{11m^{11}} - \frac{1}{12m^{12}} + \frac{1}{13m^{13}} - \frac{1}{14m^{14}} + \frac{1}{15m^{15}} - \frac{1}{16m^{10}} + \frac{1}{17m^{17}} - \frac{1}{18m^{18}}$ + &c is =

	0;111,111,111,111,111,111,	—	0;006,172,839,506,172,839,
+	;...,457,247,370,827,617,	—	;...,.38,103,947,568,968,
+	;...,..3,387,017,561,686,	—	;...,...,313,612,737,193,
+	;...,...,.29,867,879,732,	—	;...,...,..2,903,821,640,
+	;...,...,...,286,797,199,	—	;...,...,...,.28,679,719,
+	;...,...,...,..2,896,941,	—	;...,...,...,...,295,058,
+	;...,...,...,...,.30,262,	—	;...,...,...,...,..3,122,
+	;...,...,...,...,...,323,	—	;...,...,...,...,...,.33,
+	;...,...,...,...,...,..3,	—	;...,...,...,...,...,...,

= 0.111,571,775,657,104,874, — 0.006,211,259,999,278,572,
= 0.105,360,515,657,826,302. Therefore this number 0.105,360,515,657,826,302 is the logarithm of the ratio of $1 + \frac{1}{9}$ to 1, or of 10 to 9.

Q. E. I.

N. B. This number is true in all the places of figures except the laſt, where there ought to be a unit inſtead of a 2, the more accurate value of this logarithm being 0.105,360,515,657,826,301,227.

EXAMPLE II.

25. Let it be required to find by means of the ſaid ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c the logarithm of the ratio of 11 to 10 (or of 10 + 1 to 10, or of $\frac{10+1}{10}$ to $\frac{10}{10}$) or of $1 + \frac{1}{10}$ to 1.

Here m is $= 10$, and $\frac{1}{m}$ is $= \frac{1}{10} = 0.100,000,000,000,000,000$.

We ſhall therefore have

$\frac{1}{m^2}$ $(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{10} = \frac{0.100,000,000,000,000,000}{10}) = 0.010,000,000,000,000,000$;

And $\frac{1}{m^3}$ $(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{10} = \frac{0.010,000,000,000,000,000}{10}) = 0.001,000,000,000,000,000$;

And $\frac{1}{m^4}$ $(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{10} = \frac{0.001,000,000,000,000,000}{10}) = 0.000,100,000,000,000,000$;

And $\frac{1}{m^5}$ $(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{10} = \frac{0.000,100,000,000,000,000}{10}) = 0.000,010,000,000,000,000$;

And $\frac{1}{m^6}$ $(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{10} = \frac{0.000,010,000,000,000,000}{10}) = 0.000,001,000,000,000,000$;

And $\frac{1}{m^7}$ $(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{10} = \frac{0.000,001,000,000,000,000}{10}) = 0.000,000,100,000,000,000$;

And $\frac{1}{m^8}$ $(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{10} = \frac{0.000,000,100,000,000,000}{10}) = 0.000,000,010,000,000,000$;

And $\frac{1}{m^9}$ $(= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{10} = \frac{0.000,000,010,000,000,000}{10}) = 0.000,000,001,000,000,000$;

And $\frac{1}{m^{10}}$ $(= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{10} = \frac{0.000,000,001,000,000,000}{10}) = 0.000,000,000,100,000,000$;

And $\frac{1}{m^{11}}$ $(= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,100,000,000}{10}) = 0.000,000,000,010,000,000$;

And $\frac{1}{m^{12}}$ $(= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{10} = \frac{0.000,000,000,010,000,000}{10}) = 0.000,000,000,001,000,000$;

And $\frac{1}{m^{13}}$ $(= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{10} = \frac{0.000,000,000,001,000,000}{10}) = 0.000,000,000,000,100,000$;

And $\frac{1}{m^{14}}$ $(= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{10} = \frac{0.000,000,000,000,100,000}{10}) = 0.000,000,000,000,010,000$;

 And

And $\frac{1}{m^{15}}$ $(= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{10} = \frac{0.000,000,000,000,010,000}{10}) = 0.000,000,000,000,001,000$;

And $\frac{1}{m^{16}}$ $(= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{10} = \frac{0.000,000,000,000,001,000}{10}) = 0.000,000,000,000,000,100$;

And $\frac{1}{m^{17}}$ $(= \frac{1}{m^{16}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{10} = \frac{0.000,000,000,000,000,100}{10}) = 0.000,000,000,000,000,010$;

And consequently

$\frac{1}{2m^2}$ $(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.010,000,000,000,000,000}{2}) = 0.005,000,000,000,000,000$;

And $\frac{1}{3m^3}$ $(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.001,000,000,000,000,000}{3}) = 0.000,333,333,333,333,333$;

And $\frac{1}{4m^4}$ $(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,100,000,000,000,000}{4}) = 0.000,025,000,000,000,000$;

And $\frac{1}{5m^5}$ $(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,010,000,000,000,000}{5}) = 0.000,002,000,000,000,000$;

And $\frac{1}{6m^6}$ $(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,001,000,000,000,000}{6}) = 0.000,000,166,666,666,666$;

And $\frac{1}{7m^7}$ $(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,100,000,000,000}{7}) = 0.000,000,014,285,714,285$;

And $\frac{1}{8m^8}$ $(= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,010,000,000,000}{8}) = 0.000,000,001,250,000,000$;

And $\frac{1}{9m^9}$ $(= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,001,000,000,000}{9}) = 0.000,000,000,111,111,111$;

And $\frac{1}{10m^{10}}$ $(= \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,100,000,000}{10}) = 0.000,000,000,010,000,000$;

And $\frac{1}{11m^{11}}$ $(= \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,010,000,000}{11}) = 0.000,000,000,000,909,090$;

And $\frac{1}{12m^{12}}$ $(= \frac{1}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,001,000,000}{12}) = 0.000,000,000,000,083,333$;

And $\frac{1}{13m^{13}}$ $(= \frac{1}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,100,000}{13}) = 0.000,000,000,000,007,692$;

And $\frac{1}{14m^{14}}$ $(= \frac{1}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,010,000}{14}) = 0.000,000,000,000,000,714$;

And $\frac{1}{15m^{15}}$ $(= \frac{1}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,001,000}{15}) = 0.000,000,000,000,000,066$;

And $\frac{1}{16m^{16}}$ $(= \frac{1}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,100}{16}) = 0.000,000,000,000,000,006$;

And $\frac{1}{17m^{17}}$ $(= \frac{1}{m^{17}} \times \frac{1}{17} = \frac{0.000,000,000,000,000,010}{17}) = 0.000,000,000,000,000,000.$

Therefore

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} + \frac{1}{11m^{11}} - \frac{1}{12m^{12}} + \frac{1}{13m^{13}} - \frac{1}{14m^{14}} + \frac{1}{15m^{15}} - \frac{1}{16m^{16}} +$ &c. is =

	0.100,000,000,000,000,000,	— 0.005,000,000,000,000,000,
+	;...,333,333,333,333,333,	;...,.25,000,000,000,000,
+	;...,..2,000,000,000,000,	;...,...,166,666,666,666,
+	;...,...,.14,285,714,285,	;...,...,..1,250,000,000,
+	;...,...,...,111,111,111,	;...,...,...,.10,000,000,
+	;...,...,...,...,909,090,	;...,...,...,...,.83,333,
+	;...,...,...,...,..7,692,	;...,...,...,...,...,714,
+	;...,...,...,...,...,.66,	;...,...,...,...,...,..6,
	= 0.100,335,347,731,075,577,	— 0.005,025,167,926,750,719,

= 0.095,310,179,804,324,858. Therefore this number 0.095,310,179,804,324,858 is the logarithm of the ratio of $1 + \frac{1}{10}$ to 1, or of 11 to 10.

Q. E. I.

EXAMPLE III.

26. Let it be required to find, by means of the foresaid series, the logarithm of the ratio of 81 to 80, (or of 80 + 1 to 80, or of $\frac{80 + 1}{80}$ to $\frac{80}{80}$) or of $1 + \frac{1}{08}$ to 1.

Here m is = 80, and $\frac{1}{m}$ is $\frac{1}{80}$ = 0.012,500,000,000,000,000.

We shall, therefore, in this case have

$$\frac{1}{m^2}\left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{80} = \frac{0.012,500,000,000,000,000}{80}\right) = 0.000,156,250,000,000,000;$$

And $$\frac{1}{m^3}\left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{80} = \frac{0.000,156,250,000,000,000}{80}\right) = 0.000,001,953,125,000,000;$$

And $$\frac{1}{m^4}\left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{80} = \frac{0.000,001,953,125,000,000}{80}\right) = 0.000,000,024,414,062,500;$$

And $$\frac{1}{m^5}\left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{80} = \frac{0.000,000,024,414,062,500}{80}\right) = 0.000,000,000,305,175,781;$$

And $$\frac{1}{m^6}\left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{80} = \frac{0.000,000,000,305,175,781,}{80}\right) = 0.000,000,000,003,814,697;$$

And $$\frac{1}{m^7}\left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{80} = \frac{0.000,000,000,003,814,697}{80}\right) = 0.000,000,000,000,047,683;$$

And $$\frac{1}{m^8}\left(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{80} = \frac{0.000,000,000,000,047,683}{80}\right) = 0.000,000,000,000,000,596;$$

And

And $\frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{80} = \frac{0.000,000,000,000,000,596}{80}) = 0.000,000,000,000,000,007$;

And $\frac{1}{m^{10}} (= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{80} = \frac{0.000,000,000,000,000,007}{80}) = 0.000,000,000,000,000,000$;

And confequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,156,250,000,000,000}{2}) = 0.000,078,125,000,000,000;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,001,953,125,000.000}{3}) = 0.000,000,651,041,666,666;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,024,414,062,500}{4}) = 0.000,000,006,103,515,625;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,305,175,781}{5}) = 0.000,000,000,061,035,156;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,003,814,697}{6}) = 0.000,000,000,000,635,782;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,047,683}{7}) = 0.000,000,000,000,006,811;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,596}{8}) = 0.000,000,000,000,000,074;$$

$$\text{And } \frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,000,000,007}{9}) = 0.000,000,000,000,000,000.$$

Therefore the feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \&c.$

is = 0;012,500,000,000,000,000, — 0;000,078,125,000,000,000,
+ ;...,...,651,041,666,666, — ;...,...,..6,103,515,625,
+ ;...,...,...,.61,035,156, — ;...,...,...,...,635,782,
+ ;...,...,...,...,..6,811, — ;...,...,...,...,...,.74,
= 0.012,500,651,102,708,633, — 0.000,078,131,104,151,481,
= 0.012,422,519,998,557,152. Therefore this number 0.012,422,519, 998,557,152 is the logarithm of the ratio of $1 + \frac{1}{80}$ to 1, or of 81 to 80.

Q. E. I.

EXAMPLE IV.

27. Let it be required to find by means of the fame feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \&c$ the logarithm of the ratio of 121 to 120 (or of 120 + 1 to 120, or of $\frac{120 + 1}{120}$ to $\frac{120}{120}$), or of $1 + \frac{1}{120}$ to 1.

L l Here

Here m is $= 120$, and $\frac{1}{m} = \frac{1}{120} = 0.008,333,333,333,333,333$, &c.

We fhall therefore have,

$$\frac{1}{m^2} \left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{120} = \frac{0.008,333,333,333,333,333}{120}\right) = 0.000,069,444,444,444,444;$$

And $\frac{1}{m^3} \left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{120} = \frac{0.000,069,444,444,444,444}{120}\right) = 0.000,000,578,703,703,703;$

And $\frac{1}{m^4} \left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{120} = \frac{0.000,000,578,703,703,703}{120}\right) = 0.000,000,004,822,530,864;$

And $\frac{1}{m^5} \left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{120} = \frac{0.000,000,004,822,530,864}{120}\right) = 0.000,000,000,040,187,757$

And $\frac{1}{m^6} \left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{120} = \frac{0.000,000,000,040,187,757}{120}\right) = 0.000,000,000,000,334,897$

And $\frac{1}{m^7} \left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{120} = \frac{0.000,000,000,000,334,897}{120}\right) = 0.000,000,000,000,002,790;$

And $\frac{1}{m^8} \left(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{120} = \frac{0.000,000,000,000,002,790}{120}\right) = 0.000,000,000,000,000,023;$

And $\frac{1}{m^9} \left(= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{120} = \frac{0.000,000,000,000,000,023}{120}\right) = 0.000,000,000,000,000,000;$

And confequently

$$\frac{1}{2m^2} \left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,069,444,444,444,444}{2}\right) = 0.000,034,722,222,222,222;$$

And $\frac{1}{3m^3} \left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,578,703,703,703}{3}\right) = 0.000,000,192,901,234,567;$

And $\frac{1}{4m^4} \left(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,004,822,530,864}{4}\right) = 0.000,000,001,205,632,716;$

And $\frac{1}{5m^5} \left(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,040,187,757}{5}\right) = 0.000,000,000,008,037,551;$

And $\frac{1}{6m^6} \left(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,334,897}{6}\right) = 0.000,000,000,000,055,816;$

And $\frac{1}{7m^7} \left(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,002,790}{7}\right) = 0.000,000,000,000,000,398;$

And $\frac{1}{8m^8} \left(= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,023}{8}\right) = 0.000,000,000,000,000,002.$

Therefore

Therefore the ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} +$ &c is =

0.008,333,333,333,333,333,	— 0.000,034,722,222,222,222,
+ ;...,...,192,901,234,567,	— ;...,...,..1,205,632,716,
+ ;...,...,...,..8,037,551,	— ;...,...,...,...,55,816,
+ ;...,...,...,...,...,398,	— ;...,...,...,...,...,..2,
=0.008,333,526,242,605,849,	— 0.000,034,723,427,910,756,

= 0.008,298,802,814,695,093. Therefore this number 0.008,298,802,814, 695,093 is the logarithm of the ratio of $1 + \frac{1}{120}$ to 1, or of 121 to 120.

Q. E. I.

EXAMPLE V.

28. Let it be required to find by means of the ſaid ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c the logarithm of the ratio of 2401 (which is the fourth power of 7) to 2400, (or of 2400 + 1 to 2400 or of $\frac{2400 + 1}{2400}$ to $\frac{2400}{2400}$), or of $1 + \frac{1}{2400}$ to 1.

Here m is = 2400, and $\frac{1}{m}$ is = $\frac{1}{2400}$ = 0.000,416,666,666,666,666.

We ſhall therefore have

$$\frac{1}{m^2}\ \left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{2400} = \frac{0.000,416,666,666,666,666}{2400}\right) = 0.000,000,173,611,111,111;$$

And $$\frac{1}{m^3}\ \left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{2400} = \frac{0.000,000,173,611,111,111}{2400}\right) = 0.000,000,000,072,337,962;$$

And $$\frac{1}{m^4}\ \left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{2400} = \frac{0.000,000,000,072,337,962}{2400}\right) = 0.000,000,000,000,030,140;$$

And $$\frac{1}{m^5}\ \left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{2400} = \frac{0.000,000,000,000,030,140}{2400}\right) = 0.000,000,000,000,000,012;$$

And $$\frac{1}{m^6}\ \left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{2400} = \frac{0.000,000,000,000,000,012}{2400}\right) = 0.000,000,000,000,000,000;$$

And conſequently

$$\frac{1}{2m^2}\ \left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,000,173,611,111,111}{2}\right) = 0.000,000,086,805,555,555;$$

And $$\frac{1}{3m^3}\ \left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,000,072,337,962}{3}\right) = 0.000,000,000,024,112,654;$$

And

And $\frac{1}{4m^4}$ $(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,000,030,140}{4}) = 0.000,000,000,000,007,535$;

And $\frac{1}{5m^5}$ $(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,000,012}{5}) = 0.000,000,000,000,000,002$.

Therefore the feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c is in this cafe $=$ 0.000,416,666,666,666,666, — 0.000,000,086,805,555,555,
$+$;...,...,...,..24,112,654, — ;...,...,...,...,...,..7,535,
$+$;...,...,...,...,...,..2, — ;...,...,...,...,...,...,

0.000,416,666,690,779,322, — 0.000,000,086,805,563,090,
$=$ 0.000,416,579,885,216,232. Therefore this number 0.000,416,579,885, 216,232 is the logarithm of the ratio of $1 + \frac{1}{2400}$ to 1, or of 2401 to 2400.

Q. E. I.

EXAMPLE VI.

29. Let it be required to find, by means of the fame feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c the logarithm of the ratio of 169 (which is the fquare of 13) to 168 (or of 168 + 1 to 168, or of $\frac{168+1}{168}$ to $\frac{168}{168}$), or of $1 + \frac{1}{168}$ to 1.

Here m is $= 168$, and $\frac{1}{m}$ is $= \frac{1}{168} = 0.005,952,380,952,380,952$.

We fhall therefore have

$\frac{1}{m^2}$ $(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{168} = \frac{0.005,952,380,952,380,952}{168}) = 0.000,035,430,839,002,267$;

And $\frac{1}{m^3}$ $(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{168} = \frac{0.000,035,430,839,002,267}{168}) = 0.000,000,210,897,851,203$;

And $\frac{1}{m^4}$ $(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{168} = \frac{0.000,000,210,897,851,203}{168}) = 0.000,000,001,255,344,352$;

And $\frac{1}{m^5}$ $(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{168} = \frac{0.000,000,001,255,344,352}{168}) = 0.000,000,000,007,472,287$;

And $\frac{1}{m^6}$ $(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{168} = \frac{0.000,000,000,007,472,287}{168}) = 0.000,000,000,000,044,477$;

And $\frac{1}{m^7}$ $(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{168} = \frac{0.000,000,000,000,044,477}{168}) = 0.000,000,000,000,000,264$;

And

And $\frac{1}{m^8}(=\frac{1}{m^7}\times\frac{1}{m}=\frac{1}{m^7}\times\frac{1}{168}=\frac{0.000,000,000,000,000,264}{168}) = 0.000,000,000,000,000,001$;

And confequently

$$\frac{1}{2m^2}\ (=\frac{1}{m^2}\times\frac{1}{2}=\frac{0.000,035,430,839,002,267}{2}) = 0.000,017,715,419,501,133;$$

$$\text{And } \frac{1}{3m^3}\ (=\frac{1}{m^3}\times\frac{1}{3}=\frac{0.000,000,210,897,851,203}{3}) = 0.000,000,070,299,283,734;$$

$$\text{And } \frac{1}{4m^4}\ (=\frac{1}{m^4}\times\frac{1}{4}=\frac{0.000,000,001,255,344,352}{4}) = 0.000,000,000,313,836,088;$$

$$\text{And } \frac{1}{5m^5}\ (=\frac{1}{m^5}\times\frac{1}{5}=\frac{0.000,000,000,007,472,287}{5}) = 0.000,000,000,001,494,457;$$

$$\text{And } \frac{1}{6m^6}\ (=\frac{1}{m^6}\times\frac{1}{6}=\frac{0.000,000,000,000,044,477}{6}) = 0.000,000,000,000,007,412;$$

$$\text{And } \frac{1}{7m^7}\ (=\frac{1}{m^7}\times\frac{1}{7}=\frac{0.000,000,000,000,000,264}{7}) = 0.000,000,000,000,000,037;$$

$$\text{And } \frac{1}{8m^8}\ (=\frac{1}{m^8}\times\frac{1}{8}=\frac{0.000,000,000,000,000,001}{8}) = 0.000,000,000,000,000,000.$$

Therefore the feries $\frac{1}{m}-\frac{1}{2m^2}+\frac{1}{3m^3}-\frac{1}{4m^4}+\frac{1}{5m^5}-\frac{1}{6m^5}+\frac{1}{7m^7}-\frac{1}{8m^8}+$

&c is = 0.005,952,380,952,380,952, — 0.000,017,715,419,501,133,
\+ ;. . . ,. . . ,. 70,299,283,734, — ;. . . ,. . . ,. . . ,313,836,088,
\+ ;. . . ,. . . ,. . . ,. . 1,494,457, — ;. . . ,. . . ,. . . ,. . . ,. . 7,412,
\+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 37, — ;. . . ,. . . ,. . . ,. . . ,. . . ,. . . ,

= 0.005,952,451,253,159,180, — 0.000,017,715,733,344,633,
= 0.005,934,735,519,814,547. Therefore this number 0.005,934,735, 519,814,547, is the logarithm of the ratio of $1+\frac{1}{168}$ to 1, or of 169 to 168.

Q. E. I.

EXAMPLE VII.

30\. Let it be required to find by means of the fame feries the logarithm of the ratio of 289 (which is the fquare of 17,) to 288 (or of 288 + 1 to 288, or of $\frac{288+1}{288}$ to $\frac{288}{288}$), or of $1+\frac{1}{288}$ to 1.

Here m is = 288, and $\frac{1}{m}$ is = $\frac{1}{288}$ = 0.003,472,222,222,222,222, &c.

We

We ſhall therefore have

$$\frac{1}{m^2}\left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{288} = \frac{0.003,472,222,222,222,222}{288}\right) = 0.000,012,056,327,160,493;$$

And $$\frac{1}{m^3}\left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{288} = \frac{0.000,012,056,327,160,493}{288}\right) = 0.000,000,041,862,247,085;$$

And $$\frac{1}{m^4}\left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{288} = \frac{0.000,000,041,862,247,085}{288}\right) = 0.000,000,000,145,355,024;$$

And $$\frac{1}{m^5}\left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{288} = \frac{0.000,000,000,145,355,024}{288}\right) = 0.000,000,000,000,504,704;$$

And $$\frac{1}{m^6}\left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{288} = \frac{0.000,000,000,000,504,704}{288}\right) = 0.000,000,000,000,001,752;$$

And $$\frac{1}{m^7}\left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{288} = \frac{0.000,000,000,000,001,752}{288}\right) = 0.000,000,000,000,000,006;$$

And conſequently

$$\frac{1}{2m^2}\left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,012,056,327,160,493}{2}\right) = 0.000,006,028,163,580,246;$$

And $$\frac{1}{3m^3}\left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,041,862,247,085}{3}\right) = 0.000,000,013,954,082,361;$$

And $$\frac{1}{4m^4}\left(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,145,355,024}{4}\right) = 0.000,000,000,036,338,756;$$

And $$\frac{1}{5m^5}\left(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,504,704}{5}\right) = 0.000,000,000,000,100,940;$$

And $$\frac{1}{6m^6}\left(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,001,752}{6}\right) = 0.000,000,000,000,000,292;$$

And $$\frac{1}{7m^7}\left(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,006}{7}\right) = 0.000,000,000,000,000,000.$$

Therefore the ſeries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8}$ + &c is in this caſe =

$$\begin{array}{rrr}
 & 0.003,472,222,222,222,222, & - \ 0.000,006,028,163,580,246, \\
+ & ;\ldots,\ldots,.\ 13,954,082,361, & - \ ;\ldots,\ldots,\ldots,.\ 36,338,756, \\
+ & ;\ldots,\ldots,\ldots,\ldots,100,940, & - \ ;\ldots,\ldots,\ldots,\ldots,\ldots,292, \\
+ & ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots, & - \ ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots, \\
\hline
= & 0.003,472,236,176,405,523, & - \ 0.000,006,028,199,919,294,
\end{array}$$

= 0.003,466,207,976,486,229. Therefore this number 0.003,466,207,976, 486,229 is the logarithm of the ratio of $1 + \frac{1}{288}$ to 1, or of 289 to 288.

Q. E. I.

EXAMPLE VIII.

31. Let it be required to find by means of the same series the logarithm of the ratio of 361 (which is the square of 19,) to 360 (or of 360 + 1 to 360, or of $\frac{360+1}{360}$ to $\frac{360}{360}$ or of $1 + \frac{1}{360}$ to 1.

Here m is $= 360$, and consequently $\frac{1}{m}$ is $= \frac{1}{360} = 0.002,777,777,777,777,777$.

We shall therefore have,

$$\frac{1}{m^2} \left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{360} = \frac{0.002,777,777,777,777,777}{360}\right) = 0.000,007,716,049,382,716;$$

And $\frac{1}{m^3} \left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{360} = \frac{0.000,007,716,049,382,716}{360}\right) = 0.000,000,021,433,470,507;$

And $\frac{1}{m^4} \left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{360} = \frac{0.000,000,021,433,470,507}{360}\right) = 0.000,000,000,059,537,418;$

And $\frac{1}{m^5} \left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{360} = \frac{0.000,000,000,059,537,418}{360}\right) = 0.000,000,000,000,165,381;$

And $\frac{1}{m^6} \left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{360} = \frac{0.000,000,000,000,165,381}{360}\right) = 0.000,000,000,000,000,459;$

And $\frac{1}{m^7} \left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{360} = \frac{0.000,000,000,000,000,459}{360}\right) = 0.000,000,000,000,000,001;$

And consequently

$$\frac{1}{2m^2} \left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,007,716,049,382,716}{2}\right) = 0.000,003,858,024,691,358;$$

And $\frac{1}{3m^3} \left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,021,433,470,507}{3}\right) = 0.000,000,007,144,490,169;$

And $\frac{1}{4m^4} \left(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,059,537,418}{4}\right) = 0.000,000,000,014,884,354;$

And $\frac{1}{5m^5} \left(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,165,381}{5}\right) = 0.000,000,000,000,033,076;$

And $\frac{1}{6m^6} \left(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,459}{6}\right) = 0.000,000,000,000,000,076;$

And $\frac{1}{7m^7} \left(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,001}{7}\right) = 0.000,000,000,000,000,000.$

Therefore

Therefore the feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c is in this cafe =

$$\begin{array}{lll} & 0.002,777,777,777,777,777, & -\ 0.000,003,858,024,691,358, \\ + & ;\ldots,\ldots,\ldots 7,144,490,169, & -\ ;\ldots,\ldots,\ldots,.14,884,354, \\ + & ;\ldots,\ldots,\ldots,\ldots,.33,076, & -\ ;\ldots,\ldots,\ldots,\ldots,\ldots,.76, \\ \hline = & 0.002,777,784,922,301,022, & -\ 0.000,003,858,039,575,788, \end{array}$$

= 0.002,773,926,882,725,234. Therefore this number 0.002,773,926,882, 725,234, is the logarithm of the ratio of $1 + \frac{1}{360}$ to 1, or of 361 to 360.

Q. E. I.

EXAMPLE IX.

32. Let it be required to find by the fame feries the logarithm of the ratio of 529 (which is the fquare of 23,) to 528 (or of 528 + 1 to 528, or of $\frac{528+1}{528}$ to $\frac{528}{528}$), or of $1 + \frac{1}{528}$ to 1.

Here m is = 528, and confequently $\frac{1}{m}$ is = $\frac{1}{528}$ = 0.001,893,939, 393,939,393, &c.

We fhall therefore, in this cafe, have

$\frac{1}{m^2}$ ($= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{528} = \frac{0.001,893,939,393,939,393}{528}$) = 0.000,003,587,006,427,915;

And $\frac{1}{m^3}$ ($= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{528} = \frac{0.000,003,587,006,427,915}{528}$) = 0.000,000,006,793,572,780;

And $\frac{1}{m^4}$ ($= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{528} = \frac{0.000,000,006,793,572,780}{528}$) = 0.000,000,000,012,866,615;

And $\frac{1}{m^5}$ ($= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{528} = \frac{0.000,000,000,012,866,615}{528}$) = 0.000,000,000,000,024,368;

And $\frac{1}{m^6}$ ($= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{528} = \frac{0.000,000,000,000,024,368}{528}$) = 0.000,000,000,000,000,046;

And $\frac{1}{m^7}$ ($= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{528} = \frac{0.000,000,000,000,000,046}{528}$) = 0.000,000,000,000,000,000;

And confequently

$\frac{1}{2m^2}$ ($= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,003,587,006,427,915}{2}$) = 0.000,001,793,503,213,957;

And $\frac{1}{3m^3}$ ($= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,006,793,572,780}{3}$) = 0.000,000,002,264,524,260;

And

$$\text{And } \frac{1}{4m^4} \left(= \frac{1}{m^4} \times \frac{1}{528} = \frac{0.000,000,000,012,866,615}{4}\right) = 0.000,000,000,003,216,653;$$

$$\text{And } \frac{1}{5m^5} \left(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,024,368}{5}\right) = 0.000,000,000,000,004,873;$$

$$\text{And } \frac{1}{6m^6} \left(= \frac{1}{m^5} \times \frac{1}{6} = \frac{0.000,000,000,000,000,046}{6}\right) = 0.000,000,000,000,000,007.$$

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \&c$ is =

$$\begin{array}{rlcl}
 & 0.001,893,939,393,939,393, & - & 0.000,001,793,503,213,957, \\
+ & ;\ldots,\ldots,\ldots2,264,524,260, & - & ;\ldots,\ldots,\ldots,\ldots3,216,653, \\
+ & ;\ldots,\ldots,\ldots,\ldots,\ldots4,873, & - & ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots7, \\
\hline
= & 0.001,893,941,658,468,526, & - & 0.000,001,793,506,430,617,
\end{array}$$

= 0.001,892,148,152,037,909. Therefore this number 0.001,892,148,152, 037,909, is the logarithm of the ratio of $1 + \frac{1}{528}$ to 1, or of 529 to 528.

Q. E. I.

An application of the foregoing nine logarithms of small ratios to the investigation of the logarithms of the ratios of the first 23 natural numbers, (2, 3, 4, 5, &c. to 24 inclusively,) to 1.

33. Having thus found, with no great labour, the logarithms of the nine small ratios of 10 to 9, of 11 to 10, of 81 to 80, of 121 to 120, of 2401 to 2400, of 169 to 168, of 289 to 288, of 361 to 360, and of 529 to 528, by means of the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \&c$, we may, by combining these few logarithms with each other by addition, subtraction, multiplication, and division, easily discover the logarithms of the greater ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 10 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, or (as, for the sake of brevity, they are more frequently called) the logarithms of the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24. This may be done in the manner following.

34. The ratio of 11 to 9 is equal to the sum of the ratios of 11 to 10 and 10 to 9. Therefore the logarithm of the ratio of 11 to 9 is equal to the sum of the logarithms of the two latter ratios, that is, (as appears by Examples first and second,) to the sum of 0.105,360,515,657,826,302, and 0.095,310,179,804,324,858, or to 0.200,670,695,462,151,160.

The ratio of 121 (which is the square of 11) to 81 (which is the square of 9) is double of the ratio of 11 to 9. Therefore the logarithm of the ratio of 121 to 81 is double of the logarithm of 11 to 9, and consequently is equal to 2 × 0.200, 670,695,462,151,160, or to 0.401,341,390,924,302,320.

The

The ratio of 121 to 80 is equal to the fum of the ratios of 121 to 81, and of 81 to 80. Therefore the logarithm of the ratio of 121 to 80 is equal to the fum of the logarithms of the ratios of 121 to 81, and of 81 to 80. But the logarithm of the ratio of 121 to 81 has juft been fhewn to be = 0.401,341,390,924,302,320; and the logarithm of the ratio of 81 to 80 has been fhewn in Example third to be = 0.012,422,519,998,557,152. Therefore the logarithm of the ratio of 121 to 80 is

$$\left.\begin{array}{l} = 0.401,341,390,924,302,320 \\ + 0.012,422,519,998,557,152 \end{array}\right\} = 0.413,763,910,922,859,472.$$

The ratio of 120 to 80 is equal to the excefs of the ratio of 121 to 80 above the ratio of 121 to 120. Therefore the logarithm of the ratio of 120 to 80 is equal to the excefs of the logarithm of the ratio of 121 to 80 above the logarithm of the ratio of 121 to 120. But the logarithm of the ratio of 121 to 80 has juft now been fhewn to be 0.413,763,910,922,859,472; and the logarithm of the ratio of 121 to 120 is fhewn in Example 4 to be 0.008,298,802,814,695,093. Therefore the logarithm of the ratio of 120 to 80 is equal to the excefs of 0.413,763, 910,922,859,472 above 0.008,298,802,814,695,093, or to 0.405,465,108,108, 164,379.

The ratio of 3 to 2 is the fame with the ratio of 120 to 80. Therefore the logarithm of the ratio of 3 to 2 is = 0.405,465,108,108,164,379.

The ratio of 9 (which is the fquare of three) to 4 (which is the fquare of 2) is double of the ratio of 3 to 2. Therefore the logarithm of the ratio of 9 to 4 is double of the logarithm of the ratio of 3 to 2, and confequently is equal to 2 × 0.405,465,108,108,164,379, or to 0.810,930,216,216,328,758.

The ratio of 81 (which is the fquare of 9) to 16 (which is the fquare of 4) is double of the ratio of 9 to 4. Therefore the logarithm of the ratio of 81 to 16 is double of the logarithm of the ratio of 9 to 4, and confequently is equal to 2 × 0.810,930,216,216,328,758, or to 1.621,860,432,432,657,516.

The ratio of 80 to 16 is equal to the excefs of the ratio of 81 to 16 above the ratio of 81 to 80. Therefore the logarithm of the ratio of 80 to 16 is equal to the excefs of the logarithm of the ratio of 81 to 16 above the logarithm of 81 to 80, that is, to the excefs of 1.621,860,432,432,657,516 above 0.012,422,519,998, 557,152, or to 1.609,437,912,434,100,364.

The ratio of 5 to 1 is equal to the ratio of 80 to 16. Therefore the logarithm of the ratio of 5 to 1 is equal to the logarithm of the ratio of 80 to 16, and confequently is 1.609,437,912,434,100,364; or (in the ufual abridged but inaccurate phrafe), the logarithm of the number 5 is 1.609,437,912,434,100,364.

Q. E. I.

35. The ratio of 10 to 2 is the fame with the ratio of 5 to 1. Therefore the logarithm of the ratio of 10 to 2 is 1.609,437,912,434,100,364.

The ratio of 10 to 4 is equal to the fum of the ratios of 10 to 9 and of 9 to 4. Therefore the logarithm of the ratio of 10 to 4 is equal to the fum of the logarithms of the ratios of 10 to 9 and of 9 to 4, that is, to the fum of the logarithm 0.105,

360,

360,515,657,826,302, and the logarithm 0.810,930,216,216,328,758, or to 0.916,290,731,874,155,060.

The ratio of 4 to 2 is equal to the excefs of the ratio of 10 to 2 above the ratio of 10 to 4. Therefore the logarithm of the ratio of 4 to 2 is equal to the excefs of the logarithm of the ratio of 10 to 2 above the logarithm of the ratio of 10 to 4. But we have juft now feen that the logarithm of the ratio of 10 to 2 is 1.609,437, 912,434,100,364, and the logarithm of the ratio of 10 to 4 is 0.916,290,731,874, 155,060. Therefore the logarithm of the ratio of 4 to 2 will be equal to the excefs of 1.609,437,912,434,100,364 above 0.916,290,731,874,155,060, that is, to 0.693,147,180,559,945,304.

The ratio of 2 to 1 is equal to the ratio of 4 to 2. Therefore the logarithm of the ratio of 2 to 1 is equal to the logarithm of the ratio of 4 to 2, and confequently is = 0.693,147,180,559,945,304, or (in the common abridged, but inaccurate, language upon this fubject), the logarithm of the number 2 is = 0.693,147,180, 559,945,304. Q. E. I.

36. The ratio of 4 to 1 is double of the ratio of 2 to 1. Therefore the logarithm of the ratio of 4 to 1 is double of the logarithm of the ratio of 2 to 1; and confequently is equal to 2 × 0.693,147,180,559,945,304, or to 1.386,294,361, 119,890,608; or, in other words, the logarithm of the number 4 is = 1.386, 294,361,119,890,608. Q. E. I.

37. The ratio of 8 to 1 is triple of the ratio of 2 to 1. Therefore the logarithm of the ratio of 8 to 1 is triple of the logarithm of the ratio of 2 to 1; and confequently is equal to 3 × 0.693,147,180,559,945,304, or to 2.079,441,541,679, 835,912; or, in other words, the logarithm of 8 is = 2.079,441,541,679, 835,912. Q. E. I.

38. The ratio of 9 to 1 is equal to the fum of the ratios of 9 to 4 and of 4 to 1. Therefore the logarithm of the ratio of 9 to 1 is equal to the fum of the logarithms of the ratios of 9 to 4 and of 4 to 1. But it has been fhewn that the logarithm of the ratio of 9 to 4 is = 0.810,930,216,216,328,758, and the logarithm of the ratio of 4 to 1 is = 1.386,294,361,119,890,608. Therefore the logarithm of the ratio of 9 to 1 is

$$\left.\begin{array}{l} = 0.810,930,216,216,328,758, \\ + 1.386,294,361,119,890,608, \end{array}\right\} = 2.197,224,577,336,219,366;$$

or the logarithm of the number 9 is 2.197,224,577,336,219,366. Q. E. I.

39. The ratio of 3 to 1 is equal to half the ratio of 9 to 1. Therefore the logarithm of the ratio of 3 to 1 is equal to half the logarithm of the ratio of 9 to 1, and confequently is $= \frac{2.197,224,577,336,219,366}{2}$, or 1.098,612,288,668,109,683; or the logarithm of the number 3 is = 1.098,612,288,668,109,683. Q. E. I.

 40. The

40. The ratio of 6 to 1 is equal to the fum of the ratios of 6 to 3 and of 3 to 1, and confequently to the fum of the ratios of 2 to 1 (which is equal to the ratio of 6 to 3,) and of 3 to 1. Therefore the logarithm of the ratio of 6 to 1 is equal to the fum of the logarithms of the ratios of 2 to 1 and of 3 to 1, that is, to the fum of 0.693,147,180,559,945,304, and 1.098,612,288,668,109,683, or to 1.791, 759,469,228,054,987; or (according to the common way of fpeaking) the logarithm of 6 is = 1.791,759,469,228,054,987. Q. E. I.

41. The ratio of 10 to 1 is equal to the fum of the ratios of 10 to 9 and of 9 to 1. Therefore the logarithm of the ratio of 10 to 1 is equal to the fum of the logarithms of the ratios of 10 to 9 and of 9 to 1. But it has been fhewn that the logarithm of the ratio of 10 to 9 is 0.105,360,515,657,826,302, and that the logarithm of the ratio of 9 to 1 is 2.197,224,577,336,219,366. Therefore the logarithm of the ratio of 10 to 1 is equal to the fum of 0.105,360,515,657,826,302 and 2.197,224,577,336,219,366, or to 2.302,585,092,994,045,668, or (in the ufual way of fpeaking on this fubject) the logarithm of 10 is 2.302,585,092,994, 045,668. Q. E. I.

N. B. This number is true to feventeen places of figures, or in all the figures but the two laft, the more accurate value of the logarithm of 10 being 2.302,585, 092,994,045,684,017.

42. The ratio of 11 to 1 is equal to the fum of the ratios of 11 to 10, and of 10 to 1. Therefore the logarithm of the ratio of 11 to 1 is equal to the fum of the logarithms of the ratios of 11 to 10 and of 10 to 1, that is, to the fum of 0.095,310, 179,804,324,858, and 2.302,585,092,994,045,668, or to 2.397,895,272, 798,370,526; or 2.397,895,272,798,370,526 is the logarithm of the number 11. Q. E. I.

43. The ratio of 24 to 1 is equal to the fum of the ratios of 24 to 8, and of 8 to 1, that is, to the fum of the ratios of 3 to 1 (which is equal to the ratio of 24 to 8) and of 8 to 1. Therefore the logarithm of the ratio of 24 to 1 is equal to the fum of the logarithms of the ratios of 3 to 1 and of 8 to 1, that is, to the fum of 1.098,612,288,668,109,683 and 2.079,441,541,679,835,912, or to 3.178, 053,830,347,945,595.

The ratio of 2400 to 100 is equal to the ratio of 24 to 1. Therefore the logarithm of the ratio of 2400 to 100 is equal to the logarithm of the ratio of 24 to 1, or to 3.178,053,830,347,945,595.

The ratio of 100 to 1 is double of the ratio of 10 to 1. Therefore the logarithm of the ratio of 100 to 1 is double of the logarithm of the ratio of 10 to 1, and confequently is equal to 2 × 2.302,585,092,994,045,668, or to 4.605,170,185, 988,091,336.

The ratio of 2400 to 1 is equal to the fum of the ratios of 2400 to 100 and of 100 to 1. Therefore the logarithm of the ratio of 2400 to 1 is equal to the fum of the logarithms of the ratios of 2400 to 100 and of 100 to 1, that is, to the fum

of

of 3.178,053,830,347,945,595 and 4.605,170,185,988,091,336, or to 7.783, 224,016,336,036,931.

The ratio of 2401 to 1 is equal to the ſum of the ratios of 2401 to 2400 and of 2400 to 1. Therefore the logarithm of the ratio of 2401 to 1 is equal to the ſum of the logarithms of the ratios of 2401 to 2400 and of 2400 to 1. But it has been ſhewn in Example 5th, that the logarithm of the ratio of 2401 to 2400 is 0.000, 416,579,885,216,232; and we have juſt now ſeen that the logarithm of the ratio of 2400 to 1 is 7.783,224,016,336,036,931. Therefore the logarithm of the ratio of 2401 to 1 is equal to 0.000,416,579,885,216,232 + 7.783,224,016, 336,036,931 = 7.783,640,596,221,253,163.

The ratio of 7 to 1 is one fourth part of the ratio of 2401 to 1. Therefore the logarithm of the ratio of 7 to 1 is one fourth part of the logarithm of the ratio of 2401 to 1, and conſequently is $= \frac{7.783,640,596,221,253,163}{4} =$ 1.945,910,149,055, 313,290; or, (according to the uſual way of ſpeaking on this ſubject,) the logarithm of 7 is 1.945,910,149,055,313,290. Q. E. I.

44. The ratio of 14 to 1 is equal to the ſum of the ratios of 14 to 7 and of 7 to 1, or (becauſe the ratio of 14 to 7 is equal to the ratio of 2 to 1) to the ſum of the ratios of 2 to 1 and of 7 to 1. Therefore the logarithm of the ratio of 14 to 1 is equal to the ſum of the logarithms of the ratios of 2 to 1 and of 7 to 1; that is, to the ſum of 0.693,147,180,559,945,304 and 1.945,910,149,055,313,290, or to 2.639,057,329,615,258,594; or 2.639,057,329,615,258,594 is the logarithm of 14. Q. E. I.

45. The ratio of 21 to 1 is equal to the ſum of the ratios of 21 to 7 and of 7 to 1, or (becauſe the ratio of 21 to 7 is equal to the ratio of 3 to 1) to the ſum of the ratios of 3 to 1 and of 7 to 1. Therefore the logarithm of the ratio of 21 to 1 is equal to the ſum of the logarithms of the ratios of 3 to 1 and of 7 to 1; that is, to the ſum of 1.098,612,288,668,109,683 and 1.945,910,149,055,313,290, or to 3.044,522,437,723,422,973; or, in other words, the logarithm of 21 is 3.044,522,437,723,422,973. Q. E. I.

46. The ratio of 16 to 1 is equal to four times the ratio of 2 to 1. Therefore the logarithm of the ratio of 16 to 1 will be equal to four times the logarithm of the ratio of 2 to 1; that is, to four times 0.693,147,180,559,945,304, or to 2.772,588,722,239,781,216. Therefore 2.772,588,722,239,781,216 is the logarithm of 16. Q. E. I.

47. The ratio of 18 to 1 is equal to the ſum of the ratios of 18 to 9 and of 9 to 1, or (becauſe the ratio of 18 to 9 is equal to the ratio of 2 to 1,) to the ſum of the ratios of 2 to 1 and of 9 to 1. Therefore the logarithm of the ratio of 18 to 1 is equal to the ſum of the logarithms of the ratios of 2 to 1 and of 9 to 1; that is, to the ſum of 0.693,147,180,559,945,304 and 2.197,224,577,336,219,366, or to

to 2.890,371,757,896,164,670; or the logarithm of 18 is 2.890,371,757,896, 164,670. Q. E. I.

48. The ratio of 12 to 1 is equal to the ſum of the ratios of 12 to 6 and of 6 to 1, or (becauſe the ratio of 12 to 6 is equal to the ratio of 2 to 1,) to the ſum of the ratios of 2 to 1 and of 6 to 1. Therefore the logarithm of the ratio of 12 to 1 is equal to the ſum of the logarithms of the ratios of 2 to 1 and of 6 to 1; that is, to the ſum of 0.693,147,180,559,945,304 and 1.791,759,469,228,054,987, or to 2.484,906,649,788,000,294; or the logarithm of 12 is 2.484,906,649, 788,000,294. Q. E. I.

49. The ratio of 24 to 1 is equal to the ſum of the ratios of 24 to 12 and of 12 to 1, or (becauſe the ratio of 24 to 12 is equal to the ratio of 2 to 1,) to the ſum of the ratios of 2 to 1 and of 12 to 1. Therefore the logarithm of the ratio of 24 to 1 is equal to the ſum of the logarithms of the ratios of 2 to 1 and of 12 to 1; that is, to the ſum of 0.693,147,180,559,945,304 and 2.484,906,649,788,000, 294, or to 3.178,053,830,347,945,598; or the logarithm of 24 is 3.178,053, 830,347,945,598. Q. E. I.

50. The ratio of 15 to 1 is equal to the ſum of the ratios of 15 to 5 and of 5 to 1, or (becauſe the ratio of 15 to 5 is equal to the ratio of 3 to 1,) to the ſum of the ratios of 3 to 1 and of 5 to 1. Therefore the logarithm of the ratio of 15 to 1 is equal to the ſum of the logarithms of the ratios of 3 to 1 and of 5 to 1; that is, to the ſum of 1.098,612,288,668,109,683 and 1.609,437,912,434,100,364, or to 2.708,050,201,102,210,047; or, in other words, the logarithm of 15 is 2.708,050,201,102,210,047. Q. E. I.

51. The ratio of 20 to 1 is equal to the ſum of the ratios of 20 to 10 and of 10 to 1, or (becauſe the ratio of 20 to 10 is equal to the ratio of 2 to 1,) to the ſum of the ratios of 2 to 1 and of 10 to 1. Therefore the logarithm of the ratio of 20 to 1 is equal to the ſum of the logarithms of the ratios of 2 to 1 and of 10 to 1; that is to the ſum of 0.693,147,180,559,945,304 and 2.302,585,092,994,045, 668, or to 2.995,732,273,553,990,972; or the logarithm of 20 is 2.995,732, 273,553,990,972. Q. E. I.

52. The ratio of 22 to 1 is equal to the ſum of the ratios of 22 to 11 and of 11 to 1, or (becauſe the ratio of 22 to 11 is equal to the ratio of 2 to 1,) to the ſum of the ratios of 2 to 1 and of 11 to 1. Therefore the logarithm of the ratio of 22 to 1 is equal to the ſum of the logarithms of the ratios of 2 to 1 and of 11 to 1; that is, to the ſum of 0.693,147,180,559,945,304 and 2.397,895,272,798,370, 526, or to 3.091,042,453,358,315,830; or the logarithm of 22 is 3.091,042, 453,358,315,830. Q. E. I.

53. To find the logarithm of the ratio of 13 to 1 we muſt proceed as follows. The ratio of 168 to 1 is equal to the ſum of the ratios of 168 to 21 and of 21 to 1, or (becauſe the ratio of 168 to 21 is equal to the ratio of 8 to 1,) to the ſum of the ratios of 8 to 1 and of 21 to 1. Therefore the logarithm of the ratio of 168 to 1 is equal to the ſum of the logarithms of the ratios of 8 to 1 and of 21 to 1; that is, to the ſum of 2.079,441,541,679,835,912 and 3.044,522,437,723,422,973, or to 5.123,963,979,403,258,885.

Further, the ratio of 169 to 1 is equal to the ſum of the ratios of 169 to 168 and of 168 to 1. Therefore the logarithm of the ratio of 169 to 1 is equal to the ſum of the logarithms of the ratios of 169 to 168 and of 168 to 1. But the logarithm of the ratio of 168 to 1 has juſt now been ſhewn to be = 5.123,963,979,403,258,885; and the logarithm of the ratio of 169 to 168 has been found in Example 6th to be = 0.005,934,735,519,814,547. Therefore the logarithm of the ratio of 169 to 1 will be equal to the ſum of 0.005,934,735,519,814,547 and 5.123,963,979,403,258,885, or to 5.129,898,714,923,073,432. But the ratio of 13 to 1 is equal to half the ratio of 169 to 1, becauſe 169 is the ſquare of 13. Therefore the logarithm of the ratio of 13 to 1 will be equal to half the logarithm of the ratio of 169 to 1, and conſequently will be equal to half of 5.129,898,714,923,073,432, or to 2.564,949,357,461,536,716; or, in other words, the logarithm of the number 13 will be 2.564,949,357,461,536,716. Q. E. I.

54. To find the logarithm of the ratio of 17 to 1 we muſt proceed as follows. The ratio of 288 to 1 is equal to the ſum of the ratios of 288 to 18 and of 18 to 1, or (becauſe the ratio of 288 to 18 is equal to the ratio of 16 to 1, 288 being equal to 16 × 18,) to the ſum of the ratios of 16 to 1 and of 18 to 1. Therefore the logarithm of the ratio of 288 to 1 is equal to the ſum of the logarithms of the ratios of 16 to 1 and of 18 to 1; that is, to the ſum of 2.772,588,722,239,781,216 and 2.890,371,757,896,164,670, or to 5.662,960,480,135,945,886.

Further, the ratio of 289 to 1 is equal to the ſum of the ratios of 289 to 288 and of 288 to 1. Therefore the logarithm of the ratio of 289 to 1 is equal to the ſum of the logarithms of the ratios of 289 to 288 and of 288 to 1. But the logarithm of the ratio of 289 to 288 has been found, in Example 7th, to be 0.003,466,207,976,486,229; and the logarithm of the ratio of 288 to 1 has been juſt now ſhewn to be 5.662,960,480,135,945,886. Therefore the logarithm of the ratio of 289 to 1 will be = 0.003,466,207,976,486,229 + 5.662,960,480,135,945,886 = 5.666,426,688,112,432,115. Therefore the logarithm of the ratio of 17 (which is the ſquare-root of 289) to 1 will be $= \frac{5.666,426,688,112,432,115}{2} = 2.833,213,344,056,216,057$; or the logarithm of the number 17 will be 2.833,213,344,056,216,057. Q. E. I.

55. To find the logarithm of the ratio of 19 to 1 we muſt proceed as follows. The ratio of 360 to 1 is equal to the ſum of the ratios of 360 to 20 and of 20 to 1, or (becauſe the ratio of 360 to 20 is equal to the ratio of 18 to 1,) to the ſum of the

the ratios of 18 to 1 and of 20 to 1. Therefore the logarithm of the ratio of 360 to 1 is equal to the ſum of the logarithms of the ratios of 18 to 1 and of 20 to 1; that is, to the ſum of 2.890,371,757,896,164,670 and 2.995,732,273,553,990,972, or to 5.886,104,031,450,155,642.

Further, the ratio of 361 to 1 is equal to the ſum of the ratios of 361 to 360 and of 360 to 1. Therefore the logarithm of the ratio of 361 to 1 is equal to the ſum of the logarithms of the ratios of 361 to 360 and of 360 to 1. But the logarithm of the ratio of 361 to 360 has been found in Example 8th to be 0.002,773,926,882,725,234; and the logarithm of the ratio of 360 to 1 has been juſt now ſhewn to be 5.886,104,031,450,155,642. Therefore the logarithm of the ratio of 361 to 1 will be = 0.002,773,926,882,725,234 + 5.886,104,031,450,155,642 = 5.888,877,958,332,880,876; and conſequently the logarithm of the ratio of 19 (which is the ſquare-root of 361) to 1 will be $= \frac{5.888,877,958,332,880,876}{2} =$ 2.944,438,979,166,440,438; or, in other words, the logarithm of the number 19 will be 2.944,438,979,166,440,438. Q. E. I.

56. It remains that we find the logarithm of the ratio of 23 to 1. Now this may be done as follows.

The ratio of 528 to 1 is equal to the ſum of the ratios of 528 to 22 and of 22 to 1, or (becauſe the ratio of 528 to 22 is equal to the ratio of 24 to 1, 528 being = 22 × 24) to the ſum of the ratios of 24 to 1 and of 22 to 1. Therefore the logarithm of the ratio of 528 to 1 is equal to the ſum of the logarithms of the ratios of 24 to 1 and of 22 to 1, that is, to the ſum of 3.178,053,830,347,945,598 and 3.091,042,453,358,315,830, or to 6.269,096,283,706,261,428.

Further, the ratio of 529 to 1 is equal to the ſum of the ratios of 529 to 528 and of 528 to 1. Therefore the logarithm of the ratio of 529 to 1 is equal to the ſum of the logarithms of the ratios of 529 to 528 and of 528 to 1. But the logarithm of the ratio of 529 to 528 has been found in Example 9th to be = 0.001,892,148,152,037,909; and we have juſt now ſeen that the logarithm of the ratio of 528 to 1 is = 6.269,096,283,706,261,428. It follows therefore that the logarithm of the ratio of 529 to 1 will be = 0.001,892,148,152,037,909 + 6.269,096,283,706,261,428 = 6.270,988,431,858,299,337; and conſequently the logarithm of the ratio of 23 (which is the ſquare-root of 529) to 1 will be equal to half the logarithm 6.270,988,431,858,299,337, or to 3.135,494,215,929,149,668; or, in other words, the logarithm of the number 23 is = 3.135,494,215,929,149,668. Q. E. I.

57. It appears therefore that the logarithms of the ratios of the firſt 23 natural numbers, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, to 1, or (according to the common way of ſpeaking on this ſubject) the logarithms of thoſe numbers are as follows; to wit,

Log. 2. = 0.693,147,180,559,945,304;
Log. 3. = 1.098,612,288,668,109,683;

Log.

Log. 4. = 1.386,294,361,119,890,608;
Log. 5. = 1.609,437,912,434,100,364;
Log. 6. = 1.791,759,469,228,054,985;
Log. 7. = 1.945,910,149,055,313,290;
Log. 8. = 2.079,441,541,679,835,912;
Log. 9. = 2.197,224,577,336,219,366;
Log. 10. = 2.302,585,092,994,045,668;
Log. 11. = 2.397,895,272,798,370,526;
Log. 12. = 2.484,906,649,788,000,294;
Log. 13. = 2.564,949,357,461,536,716;
Log. 14. = 2.639,057,329,615,258,594;
Log. 15. = 2.708,050,201,102,210,047;
Log. 16. = 2.772,588,722,239,781,216;
Log. 17. = 2.833,213,344,056,216,057;
Log. 18. = 2.890,371,757,896,164,670;
Log. 19. = 2.944,438,979,166,440,438;
Log. 20. = 2.995,732,273,553,990,972;
Log. 21. = 3.044,522,437,723,422,973;
Log. 22. = 3.091,042,453,358,315,830;
Log. 23. = 3.135,494,215,929,149,668;
Log. 24. = 3.178,053,830,347,945,598.

Theſe logarithms (if no miſtakes have been made in my computing them) are exact to the ſixteenth place of decimal figures, or in all but the two laſt places of figures. The firſt nine of them have been computed to a few more places of figures, by the learned Mr. Leonard Euler, and are inſerted in his *Introductio in Analyſin Infinitorum*, in two volumes quarto, publiſhed at Lauſanne, in the year 1748. They are as follows,

Log. 2. = 0.693,147,180,559,945,309,417,232,1;
Log. 3. = 1.098,612,288,668,109,691,395,245,2;
Log. 4. = 1.386,294,361,119,890,618,834,464,2;
Log. 5. = 1.609,437,912,434,100,374,600,759,3;
Log. 6. = 1.791,759,469,228,055,000,812,477,3;
Log. 7. = 1.945,910,149,055,313,305,105,463,9;
Log. 8. = 2.079,441,541,679,835,928,251,696,4;

Log. 9. = 2.197,224,577,336,219,382,790,490,5;
Log. 10. = 2.302,585,092,994,045,684,017,991,4.

See Euler's *Introductio in Analysin Infinitorum*, tom. i. page 91.

A recapitulation of the foregoing deductions of the logarithms of the ratios of 2 *to* 1, 3 *to* 1, 4 *to* 1, 5 *to* 1, *and the other following numbers, up to* 24, *to* 1, *from the logarithms of the nine smaller ratios before computed, expressed in a more concise manner.*

58. The foregoing deductions of the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, and the other following numbers, up to 24, to 1, from the logarithms of the nine smaller ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, 2401 to 2400, 169 to 168, 289 to 288, 361 to 360, and 529 to 528 (which were computed by means of the infinite series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c.) may be expressed more concisely as follows:

Let the logarithm of any ratio be denoted by the capital letter L. prefixed to a fraction of which the antecedent of the ratio is the numerator, and the consequent of it is the denominator; so that, for example, the logarithm of the ratio of 10 to 9 shall be denoted by L. $\frac{10}{9}$.

Then we shall have in the 1st place,

$$\text{L. } \frac{10}{9} = 0.105,360,515,657,826,302;$$

$$\text{2dly, L. } \frac{11}{10} = 0.095,310,179,804,324,858;$$

$$\text{3dly, L. } \frac{81}{80} = 0.012,422,519,998,557,152;$$

$$\text{4thly, L. } \frac{121}{120} = 0.008,298,802,814,695,093;$$

$$\text{5thly, L. } \frac{2401}{2400} = 0.000,416,579,885,216,232;$$

$$\text{6thly, L. } \frac{169}{168} = 0.005,934,735,519,814,547;$$

$$\text{7thly, L. } \frac{289}{288} = 0.003,466,207,976,486,229;$$

$$\text{8thly, L. } \frac{361}{360} = 0.002,773,926,882,725,234;$$

$$\text{And 9thly, L. } \frac{529}{528} = 0.001,892,148,152,037,909.$$

These logarithms were all computed by means of the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c. in the above nine examples. And from these original logarithms

logarithms all the others were derived, by reasonings which may be more briefly expressed in the following manner:

59. $L.\frac{11}{9}$ is $= L.\frac{11}{10} + L.\frac{10}{9} = \left\{\begin{array}{r} 0.095,310,179,804,324,858 \\ +\ 0.105,360,515,657,826,302 \end{array}\right\} = 0.200,670,695,462,151,160.$

$L.\frac{121}{81}$ is $= L.\frac{\overline{11}|^2}{\overline{9}|^2} = 2\,L.\frac{11}{9} = 2 \times 0.200,670,695,462,151,160 = 0.401,341,390,924,302,320.$

$L.\frac{121}{80}$ is $= L.\frac{121}{81} + L.\frac{81}{80} = \left\{\begin{array}{r} 0.401,341,390,924,302,320 \\ +\ 0.012,422,519,998,557,152 \end{array}\right\} = 0.413,763,910,922,859,472.$

$L.\frac{120}{80}$ is $= L.\frac{121}{80} - L.\frac{121}{120} = \left\{\begin{array}{r} 0.413,763,910,922,859,472 \\ -\ 0,008,298,802,814,695,093 \end{array}\right\} = 0.405,465,108,108,164,379.$

$L.\frac{3}{2}$ is $= L.\frac{120}{80} = 0.405,465,108,108,164,379.$

$L.\frac{9}{4}$ is $= L.\frac{\overline{3}|^2}{\overline{2}|^2} = 2 \times L.\frac{3}{2} = 2 \times 0.405,465,108,108,164,379 = 0.810,930,216,216,328,758.$

$L.\frac{81}{16}$ is $= L.\frac{\overline{9}|^2}{\overline{4}|^2} = 2 \times L.\frac{9}{4} = 2 \times 0.810,930,216,216,328,758 = 1.621,860,432,432,657,516.$

$L.\frac{80}{16}$ is $= L.\frac{81}{16} - L.\frac{81}{80} = \left\{\begin{array}{r} 1.621,860,432,432,657,516 \\ -\ 0.012,422,519,998,557,152 \end{array}\right\} = 1.609,437,912,434,100,364.$

$L.\frac{5}{1}$ is $= L.\frac{80}{16} = 1.609,437,912,434,100,364$; or, in the usual way of expressing it, the logarithm of the number 5 is $= 1.609,437,912,434,100,364$. Q. E. I.

$L.\frac{10}{2}$ is $= L.\frac{5}{1} = 1.609,437,912,434,100,364.$

$L.\frac{10}{4}$ is $= L.\frac{10}{9} + L.\frac{9}{4} = \left\{\begin{array}{r} 0.105,360,515,657,826,302 \\ +\ 0.810,930,216,216,328,758 \end{array}\right\} = 0.916,290,731,874,155,060.$

$L.\frac{4}{2}$ is $= L.\frac{10}{2} - L.\frac{10}{4} = \left\{\begin{array}{r} 1.609,437,912,434,100,364 \\ -\ 0.916,290,731,874,155,060 \end{array}\right\} = 0.693,147,180,559,945,304.$

$L.\frac{2}{1}$ is $= L.\frac{4}{2} = 0.693,147,180,559,945,304$; or the logarithm of the number 2 is $= 0.693,147,180,559,945,304$. Q. E. I.

$L.\frac{4}{1}$ is $= 2 \times L.\frac{2}{1} = 2 \times 0.693,147,180,559,945,304 = 1.386,294,361,119,890,608$; or the logarithm of the number 4 is $1.386,294,361,119,890,608$. Q. E. I.

L. $\frac{8}{1}$ is $= 3 \times$ L. $\frac{2}{1} = 3 \times 0.693,147,180,559,945,304 = 2.079,441,541,679,835,912$; or the logarithm of the number 8 is 2.079,441,541,679,835,912. Q. E. I.

L. $\frac{9}{1}$ is $=$ L. $\frac{9}{4} +$ L. $\frac{4}{1} = \left\{\begin{array}{r} 0.810,930,216,216,328,758 \\ +\ 1.386,294,361,119,890,608 \end{array}\right\} = 2.197,224,577,336,219,366$; or the logarithm of the number 9 is $= 2.197,224,577,336,219,366$. Q. E. I.

L. $\frac{3}{1}$ is $= \frac{1}{2} \times$ L. $\frac{9}{1} = \frac{2.197,224,577,336,219,366}{2} = 1.098,612,288,668,109,683$; or the logarithm of the number 3 is 1.098,612,288,668,109,683. Q. E. I.

L. $\frac{6}{1}$ is $=$ L. $\frac{6}{3} +$ L. $\frac{3}{1} =$ L. $\frac{2}{1} +$ L. $\frac{3}{1} = \left\{\begin{array}{r} 0.693,147,180,559,945,304 \\ +\ 1.098,612,288,668,109,683 \end{array}\right\} = 1.791,759,469,228,054,987$; or the logarithm of the number 6 is $= 1.791,759,469,228,054,987$. Q. E. I.

L. $\frac{10}{1}$ is $=$ L. $\frac{10}{9} +$ L. $\frac{9}{1} = \left\{\begin{array}{r} 0.105,360,515,657,826,302 \\ +\ 2.197,224,577,336,219,366 \end{array}\right\} = 2.302,585,092,994,045,668$; or the logarithm of the number 10 is $= 2.302,585,092,994,045,668$. Q. E. I.

The more exact value of this logarithm is 2.302,585,092,994,045,684,017.

L. $\frac{11}{1}$ is $=$ L. $\frac{11}{10} +$ L. $\frac{10}{1} = \left\{\begin{array}{r} 0.095,310,179,804,324,858 \\ +\ 2.302,585,092,994,045,668 \end{array}\right\} = 2.397,895,272,798,370,526$; or the logarithm of the number 11 is $= 2.397,895,272,798,370,526$. Q. E. I.

L. $\frac{24}{1}$ is $=$ L. $\frac{24}{8} +$ L. $\frac{8}{1} =$ L. $\frac{3}{1} +$ L. $\frac{8}{1} = \left\{\begin{array}{r} 1.098,612,288,668,109,683 \\ +\ 2.079,441,541,679,835,912 \end{array}\right\} = 3.178,053,830,347,945,595$.

L. $\frac{2400}{100}$ is $=$ L. $\frac{24}{1} = 3.178,053,830,347,945,595$.

L. $\frac{100}{1}$ is $= 2 \times$ L. $\frac{10}{1} = 2 \times 2.302,585,092,994,045,668 = 4.605,170,185,988,091,336$.

L. $\frac{2400}{1}$ is $=$ L. $\frac{2400}{100} +$ L. $\frac{100}{1} =$ L. $\frac{24}{1} +$ L. $\frac{100}{1} = \left\{\begin{array}{r} 3.178,053,830,347,945,595 \\ +4.605,170,185,988,091,336 \end{array}\right\} = 7.783,224,016,336,036,931$.

L. $\frac{2401}{1}$ is $=$ L. $\frac{2401}{2400} +$ L. $\frac{2400}{1} = \left\{\begin{array}{r} 0.000,416,579,885,216,232 \\ +\ 7.783,224,016,336,036,931 \end{array}\right\} = 7.783,640,596,221,253,163$.

L. $\frac{7}{1}$ is $= \frac{1}{4}$ L. $\frac{2401}{1} = \frac{7.783,640,596,221,253,163}{4} = 1.945,910,149,055,313,290$; or the logarithm of the number 7 is $= 1.945,910,149,055,313,290$. Q. E I.

$L.\frac{14}{1}$ is $= L.\frac{14}{7} + L.\frac{7}{1} = L.\frac{2}{1} + L.\frac{7}{1} = \left\{\begin{matrix} 0.693,147,180,559,945,304 \\ +1.945,910,149,055,313,290 \end{matrix}\right\} = 2.639,057,329,615,258,594$; or the logarithm of the number 14 is $= 2.639,057,329,615,258,594$. Q. E. I.

$L.\frac{21}{1}$ is $= L.\frac{21}{7} + L.\frac{7}{1} = L.\frac{3}{1} + L.\frac{7}{1} = \left\{\begin{matrix} 1.098,612,288,668,109,683 \\ +1.945,910,149,055,313,290 \end{matrix}\right\} = 3.044,522,437,723,422,973$; or the logarithm of the number 21 is $= 3.044,522,437,723,422,973$. Q. E. I.

$L.\frac{16}{1}$ is $= 4 \times L.\frac{2}{1} = 4 \times 0.693,147,180,559,945,304 = 2.772,588,722,239,781,216$; or the logarithm of 16 is 2.772,588,722,239,781,216. Q. E. I.

$L.\frac{18}{1}$ is $= L.\frac{18}{9} + L.\frac{9}{1} = L.\frac{2}{1} + L.\frac{9}{1} = \left\{\begin{matrix} 0.693,147,180,559,945,304 \\ +2.197,224,577,336,219,366 \end{matrix}\right\} = 2.890,371,757,896,164,670$; or the logarithm of 18 is 2.890,371,757,896,164,670. Q. E. I.

$L.\frac{12}{1}$ is $= L.\frac{12}{6} + L.\frac{6}{1} = L.\frac{2}{1} + L.\frac{6}{1} = \left\{\begin{matrix} 0.693,147,180,559,945,304 \\ +1.791,759,469,228,054,987 \end{matrix}\right\} = 2.484,906,649,788,000,294$; or the logarithm of 12 is 2.484,906,649,788,000,294. Q. E. I.

$L.\frac{24}{1}$ is $= L.\frac{24}{12} + L.\frac{12}{1} = L.\frac{2}{1} + L.\frac{12}{1} = \left\{\begin{matrix} 0.693,147,180,559,945,304 \\ +2.484,906,649,788,000,294 \end{matrix}\right\} = 3.178,053,830,347,945,598$; or the logarithm of 24 is 3.178,053,830,347,945,598. Q. E. I.

$L.\frac{15}{1}$ is $= L.\frac{15}{5} + L.\frac{5}{1} = L.\frac{3}{1} + L.\frac{5}{1} = \left\{\begin{matrix} 1.098,612,288,668,109,683 \\ +1.609,437,912,434,100,364 \end{matrix}\right\} = 2.708,050,201,102,210,047$; or the logarithm of 15 is 2.708,050,201,102,210,047. Q. E. I.

$L.\frac{20}{1}$ is $= L.\frac{20}{10} + L.\frac{10}{1} = L.\frac{2}{1} + L.\frac{10}{1} = \left\{\begin{matrix} 0.693,147,180,559,945,304 \\ +2.302,585,092,994,045,668 \end{matrix}\right\} = 2.995,732,273,553,990,972$; or the logarithm of 20 is 2.995,732,273,553,990,972. Q. E. I.

$L.\frac{22}{1}$ is $= L.\frac{22}{11} + L.\frac{11}{1} = L.\frac{2}{1} + L.\frac{11}{1} = \left\{\begin{matrix} 0.693,147,180,559,945,304 \\ +2.397,895,272,798,370,526 \end{matrix}\right\} = 3.091,042,453,358,315,830$; or the logarithm of 22 is 3.091,042,453,358,315,830. Q. E. I.

$L.\frac{168}{1}$ is $= L.\frac{168}{21} + L.\frac{21}{1} = L.\frac{8}{1} + L.\frac{21}{1} = \left\{\begin{matrix} 2.079,441,541,679,835,912 \\ +3.044,522,437,723,422,973 \end{matrix}\right\} = 5.123,963,979,403,258,885.$

$L.\frac{169}{1}$ is $= L.\frac{169}{168} + L.\frac{168}{1} = \left\{\begin{matrix} 0.005,934,735,519,814,547 \\ +5.123,963,979,403,258,885 \end{matrix}\right\} = 5.129,898,714,923,073,432.$

$L.\frac{13}{1}$ is $= \frac{1}{2} \times L.\frac{169}{1} = \frac{5.129,898,714,923.073,432}{2} = 2.564,949,357,461,536,716$; or the logarithm of 13 is 2.564,949,357,461,536,716. Q. E. I.

$L.\frac{288}{1}$ is $= L.\frac{288}{18} + L.\frac{18}{1} = L.\frac{16}{1} + L.\frac{18}{1} = \left\{\begin{array}{l} 2.772,588,722,239,781,216 \\ +2.890,371,757,896,164,670 \end{array}\right\} = 5.662,960,480,135,945,886.$

$L.\frac{289}{1}$ is $= L.\frac{289}{288} + L.\frac{288}{1} = \left\{\begin{array}{l} 0.003,466,207,976,486,229 \\ +\ 5.662,960,480,135,945,886 \end{array}\right\} = 5.666,426,688,112,432,115.$

$L.\frac{17}{1}$ is $= \frac{1}{2} \times L.\frac{289}{1} = \frac{5.666,426,688,112,432,115}{2} = 2.833,213,344,056,216,057$; or the logarithm of 17 is 2.833,213,344,056,216,057. Q. E. I.

$L.\frac{360}{1}$ is $= L.\frac{360}{20} + L.\frac{20}{1} = L.\frac{18}{1} + L.\frac{20}{1} = \left\{\begin{array}{l} 2.890,371,757,896,164,670 \\ +2.995,732,273,553,990,972 \end{array}\right\} = 5.886,104,031,450,155,642.$

$L.\frac{361}{1}$ is $= L.\frac{361}{360} + L.\frac{360}{1} = \left\{\begin{array}{l} 0.002,773,926,882,725,234 \\ +5.886,104,031,450,155,642 \end{array}\right\} = 5.888,877,958,332,880,876.$

$L.\frac{19}{1}$ is $= \frac{1}{2} \times L.\frac{361}{1} = \frac{5.888,877,958,332,880,876}{2} = 2.944,438,979,166,440,438$; or the logarithm of 19 is 2.944,438,979,166,440,438. Q. E. I.

$L.\frac{528}{1}$ is $= L.\frac{528}{22} + L.\frac{22}{1} = L.\frac{24}{1} + L.\frac{22}{1} = \left\{\begin{array}{l} 3.178,053,830,347,945,598 \\ +3.091,042,453,358,315,830 \end{array}\right\} = 6.269,096,283,706,261,428.$

$L.\frac{529}{1}$ is $= L.\frac{529}{528} + L.\frac{528}{1} = \left\{\begin{array}{l} 0.001,892,148,152,037,909 \\ +6.269,096,283,706,261,428 \end{array}\right\} = 6.270,988,431,858,299,337.$

$L.\frac{23}{1}$ is $= \frac{1}{2} \times L.\frac{529}{1} = \frac{6.270,988,431,858,299,337}{2} = 3.135,494,215,929,149,668$; or the logarithm of the number 23 is 3.135,494,215,929,149,668. Q. E. I.

Concerning the logarithmick feries invented by Dr. Wallis.

I fhall now proceed to confider the feries $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, which was invented by Dr. Wallis, and which, if A be fuppofed to be lefs than 1, is equal to the logarithm of the ratio of 1 to $1 - A$.

60. The propofition of which I here mean to eftablifh the truth, is this.

If there be two different quantities, A and B, that are, both of them, lefs than 1, and of which, confequently, the feveral powers A^2, A^3, A^4, A^5, A^6, &c, and B^2, B^3, B^4, B^5, B^6, &c are decreafing quantities, the two infinite feriefes $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c and $B + \frac{B^2}{2} + \frac{B^3}{3} + \frac{B^4}{4} + \frac{B^5}{5} + \frac{B^6}{6} +$ &c (which, it is evident, will be decreafing progreffions,) will be meafures, or logarithms, of the ratios of 1 to $1 - A$ and of 1 to $1 - B$; or the feries $A +$

$\frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} +$ &c, will be to the series $B + \frac{B^2}{2} + \frac{B^3}{3} + \frac{B^4}{4} + \frac{B^5}{5} + \frac{B^6}{6} +$ &c in the same proportion as the ratio of 1 to $1 - A$ is to the ratio of 1 to $1 - B$.

Or, if we change the notation a little, and substitute the small letters k and q instead of the capital letters A and B respectively (which will be more agreeable to the notation now most in use in treating of these kinds of serieses), the proposition which we are to demonstrate will be as follows.

If there be two different quantities k and q, that are both of them less than 1, and of which, consequently, the several powers k^2, k^3, k^4, k^5, k^6, &c, and q^2, q^3, q^4, q^5, q^6, &c will be decreasing quantities, the two infinite serieses $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c and $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, (which, it is evident, will be decreasing progressions) will be measures or logarithms, of the ratios of 1 to $1 - k$ and of 1 to $1 - q$, or the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c will be to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c in the same proportion as the ratio of 1 to $1 - k$ to the ratio of 1 to $1 - q$.

In order to demonstrate this proposition it will be necessary to premise the following proposition, as a Lemma.

LEMMA III.

61. If x be any quantity less than 1, and n be any whole number whatsoever, the quantity $\overline{1 - x}|^{\frac{1}{n}}$, or $\sqrt[n]{1 - x}$, that is, the n^{th} root of the residual quantity $1 - x$, will be equal to the following infinite series of decreasing terms, to wit,

$$1 - \frac{1}{n} \times x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2$$
$$- \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3$$
$$- \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4$$
$$- \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times x^5$$
$$- \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{5n-1}{6n} \times x^6 - \&c$$

ad infinitum, or (if we put $A = 1$, and $B = \frac{1}{n}$, and $C = \frac{1}{n} \times \frac{n-1}{2n}$, and $D = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and E, F, G, H, I, K, &c for the several following co-

 efficients

efficients of the powers of x in this feries, refpectively), to the infinite feries $1 - \frac{1}{n} \times Ax - \frac{n-1}{2n} \times Bx^2 - \frac{2n-1}{3n} \times Cx^3 - \frac{3n-1}{4n} \times Dx^4 - \frac{4n-1}{5n} \times Ex^5 - \frac{5n-1}{6n} \times Fx^6 - \frac{6n-1}{7n} \times Gx^7 - \frac{7n-1}{8n} \times Hx^8 - \frac{8n-1}{9n} \times Ix^9 - \frac{9n-1}{10n} \times Kx^{10}$ — *&c ad infinitum*; in which feries all the terms after the firft term 1 are marked with the fign —, or are to be fubtracted from the faid firft term, and the feveral co-efficients of $x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}$, *&c*, are derived, or generated, from the firft term 1 by the continual, or fucceffive, multiplication of the fractions $\frac{1}{n}, \frac{n-1}{2n}, \frac{2n-1}{3n}, \frac{3n-1}{4n}, \frac{4n-1}{5n}, \frac{5n-1}{6n}, \frac{6n-1}{7n}, \frac{7n-1}{8n}, \frac{8n-1}{9n}, \frac{9n-1}{10n}$, *&c*, which are therefore called their *generating fractions*.

62. The law by which all thefe generating fractions after the firft fraction $\frac{1}{n}$ are formed, or derived, from the faid firft fraction $\frac{1}{n}$ and from each other, is as follows. The numerator $n - 1$ of the fecond fraction $\frac{n-1}{2n}$ is formed from 1, the numerator of the firft fraction $\frac{1}{n}$, by the addition of $n - 2$, or by the addition of n and the fubtraction of 2; for $1 + n - 2$ is $= n - 1$. And the numerators of all the following fractions after the fecond fraction $\frac{n-1}{2n}$, to wit, the numerators $2n - 1, 3n - 1, 4n - 1, 5n - 1, 6n - 1, 7n - 1, 8n - 1, 9n - 1$, *&c*, are formed from the numerator $n - 1$ of the faid fecond fraction, and from each other by the continual addition of n to every preceding numerator. And the denominators of the fecond fraction, $\frac{n-1}{2n}$, and the third fraction, $\frac{2n-1}{3n}$, and all the following fractions, to wit, the denominators $2n, 3n, 4n, 5n, 6n, 7n, 8n, 9n$, *&c*, are formed from n, the denominator of the firft fraction, $\frac{1}{n}$ and from each other by the continual addition of n to the preceding denominator.

63. This is another branch of the above-mentioned celebrated theorem of Sir Ifaac Newton for finding the roots of a binomial quantity. For it extends to the roots of a refidual quantity, as $1 - x$, or of the difference of 1 and x, as well as to the roots of a binomial quantity, properly fo called, as $1 + x$, or the fum of 1 and x. And the foregoing manner of expreffing this theorem in this cafe of the n^{th} root of $1 - x$ is, as I apprehend, the moft intelligible and convenient way of expreffing it that can be chofen. This feries is derived from the feries $1 - \frac{m}{1} Ax +$

$\frac{m-1}{2}\,\mathrm{B}\,x^2 - \overline{\frac{m-2}{3}}\,\mathrm{C}\,x^3 + \frac{m-3}{4}\,\mathrm{D}\,x^4 - \overline{\frac{m-4}{5}}\,\mathrm{E}\,x^5 + \frac{m-5}{6}\,\mathrm{F}\,x^6 -$ &c or $1 - \frac{m}{1} \times 1 \times x + \frac{m}{1} \times \frac{m-1}{2} \times x^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times x^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times x^5 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \frac{m-5}{6} \times x^6 -$ &c, (which is $= \overline{1-x}|^{m}$) by ſubſtituting $\frac{1}{n}$ in its terms inſtead of m.

64. Coroll. 1. It follows from the foregoing Lemma, that $1 - \overline{1-x}|^{\frac{1}{n}}$, or the exceſs of 1 above the n^{th} root of the reſidual quantity $1 - x$, is equal to the infinite ſeries $\frac{1}{n}\,\mathrm{A}\,x + \frac{n-1}{2n}\,\mathrm{B}\,x^2 + \frac{2n-1}{3n}\,\mathrm{C}\,x^3 + \frac{3n-1}{4n}\,\mathrm{D}\,x^4 + \frac{4n-1}{5n}\,\mathrm{E}\,x^5 + \frac{5n-1}{6n}\,\mathrm{F}\,x^6 + \frac{6n-1}{7n}\,\mathrm{G}\,x^7 + \frac{7n-1}{8n}\,\mathrm{H}\,x^8 + \frac{8n-1}{9n}\,\mathrm{I}\,x^9 + \frac{9n-1}{10n}\,\mathrm{K}\,x^{10} +$ &c, *ad infinitum*; in which A is, as before, $= 1$; and B is $= \frac{1}{n}\,\mathrm{A}$, or the co-efficient of x; and C is $= \frac{n-1}{2n}\,\mathrm{B}$, or the co-efficient of x^2; and D is $= \frac{2n-1}{3n}\,\mathrm{C}$, or the co-efficient of x^3; and the following capital letters, E, F, G, H, I, K, &c, are equal to, or ſtand for, the co-efficients of x^4, x^5, x^6, x^7, x^8, x^9, and the other following powers of x, and in which all the terms are marked with the ſign $+$, or added to each other.

65. Coroll. 2. If n be a very large number, as, for example, the ninth power of a million, or 1 with 54 cyphers annexed to it, or 1,000,000,000,000,000, 000,000,000,000,000,000,000,000,000,000,000,000,000, (which Mr. Locke, in his Eſſay on Human Underſtanding, calls a *nonillion*,) the quantity $1 - \overline{1-x}|^{\frac{1}{n}}$, or the exceſs of 1 above the n^{th} root of the reſidual quantity $1 - x$, will be very nearly equal to the ſeries $\frac{x}{n} + \frac{x^2}{2n} + \frac{x^3}{3n} + \frac{x^4}{4n} + \frac{x^5}{5n} + \frac{x^6}{6n} +$ &c *ad infinitum*, or to $\frac{1}{n} \times$ the ſeries $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6}$ &c *ad infinitum*. And the number n may be taken of ſo great a magnitude that the ratio of $\frac{1}{n} \times$ the ſeries $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} +$ &c *ad infinitum* to $1 - \overline{1-x}|^{\frac{1}{n}}$ ſhall approach as near to a ratio of equality as we pleaſe.

For, when n is equal to a nonillion, or any ſuch very great number, it is evident that $n - 1$, and $2n - 1$ and $3n - 1$ and $4n - 1$ and $5n - 1$, &c, will be very nearly equal to n, $2n$, $3n$, $4n$, $5n$, &c, on account of the immenſe magnitudes of the numbers n, $2n$, $3n$, $4n$, $5n$, &c, in compariſon of the unit which is ſubtracted from them. Therefore the ſeries $\frac{1}{n}\,\mathrm{A}\,x + \frac{n-1}{2n}\,\mathrm{B}\,x^2 +$

$\frac{2n-1}{3n} \text{C} x^3 + \frac{3n-1}{4n} \text{D} x^4 + \frac{4n-1}{5n} \text{E} x^5 + \frac{5n-1}{6n} \text{F} x^6 +$ &c (which is $= 1 - \overline{1-x}|^{\frac{1}{n}}$) will in this caſe be very nearly $= \frac{1}{n} \text{A} x + \frac{n}{2n} \text{B} x^2 + \frac{2n}{3n} \text{C} x^3 + \frac{3n}{4n} \text{D} x^4 + \frac{4n}{5n} \text{E} x^5 + \frac{5n}{6n} \text{F} x^6 +$ &c. $= \frac{1}{n} \text{A} x + \frac{1}{2} \text{B} x^2 + \frac{2}{3} \text{C} x^3 + \frac{3}{4} \text{D} x^4 + \frac{4}{5} \text{E} x^5 + \frac{5}{6} \text{F} x^6 +$ &c $= \frac{1}{n} \times 1 \times x + \frac{1}{2} \times \frac{1}{n} \times 1 \times x^2 + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^3 + \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^4 + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^5 + \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^6 +$ &c $= \frac{1}{n} x + \frac{1}{2n} x^2 + \frac{1}{3n} x^3 + \frac{1}{4n} x^4 + \frac{1}{5n} x^5 + \frac{1}{6n} x^6 +$ &c $= \frac{1}{n} \times x + \frac{1}{n} \times \frac{x^2}{2} + \frac{1}{n} \times \frac{x^3}{3} + \frac{1}{n} \times \frac{x^4}{4} + \frac{1}{n} \times \frac{x^5}{5} + \frac{1}{n} \times \frac{x^6}{6} +$ &c $= \frac{1}{n} \times$ the ſeries $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} +$ &c. Therefore $1 - \overline{1-x}|^{\frac{1}{n}}$ will in this caſe be very nearly equal to $\frac{1}{n} \times$ the ſeries $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} +$ &c *ad infinitum*. And it is evident that, if any ratio whatſoever be aſſigned that differs very little from a ratio of equality, n may be taken of ſo great a magnitude that the proportion of $1 - \overline{1-x}|^{\frac{1}{n}}$ to $\frac{1}{n} \times$ the ſeries $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} +$ &c ſhall approach ſtill nearer to a ratio of equality than ſuch aſſigned ratio.

Q. E. D.

66. This lemma being premiſed, the main propoſition may be ſtated and demonſtrated in the manner following.

THEOREM II.

If k and q are any two ſmall quantities leſs than 1, whereof we will ſuppoſe k to be the greater; the ratio of 1 to $1-k$ will be to the ratio of 1 to $1-q$ in the ſame proportion as the infinite ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c to the infinite ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c.

DEMONSTRATION.

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

3 Then,

Then, by the 2d Corollary of the foregoing Lemma, we ſhall have $1 - \overline{1-k}|^{\frac{1}{n}}$, very nearly, $= \frac{1}{n} \times$ the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, and $1 - \overline{1-q}|^{\frac{1}{n}}$, very nearly, $= \frac{1}{n} \times$ the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ *&c ad infinitum.* And, by increaſing the magnitude of n, each of theſe ratios may be made to come as near to a ratio of equality as we pleaſe. Therefore the proportion of $1 - \overline{1-k}|^{\frac{1}{n}}$ to $1 - \overline{1-q}|^{\frac{1}{n}}$ is, in this caſe of the very great magnitude of n, very nearly the ſame as that of $\frac{1}{n} \times$ the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c *ad infinitum* to $\frac{1}{n} \times$ the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c *ad infinitum*, and conſequently, as that of n times $\frac{1}{n} \times$ the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c *ad infinitum* to n times $\frac{1}{n} \times$ the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c *ad infinitum*, or as that of the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* to the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ *&c ad infinitum*; that is, the proportion of the exceſs of 1 above the n^{th} root of the reſidual quantity $1 - k$ to the exceſs of 1 above the n^{th} root of the reſidual quantity $1 - q$, is, in this caſe of the very great magnitude of n, very nearly the ſame as that of the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* to the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c ad infinitum.*

Now the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the ſame proportion as any given part of the former ratio is to the like part of the latter ratio, and conſequently, as the n^{th}, or nonillionth, part of the former ratio is to the n^{th}, or nonillionth, part of the latter ratio. But the ratio of 1 to the n^{th} root of $1 - k$ or to $\overline{1-k}|^{\frac{1}{n}}$ is the n^{th} part of the ratio of 1 to $1 - k$; and the ratio of 1 to the n^{th} root of $1 - q$ is the n^{th} part of the ratio of 1 to $1 - q$. Therefore the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the ſame proportion as the very ſmall ratio of 1 to the n^{th} root of $1 - k$, or to $\overline{1-k}|^{\frac{1}{n}}$, is to the very ſmall ratio of 1 to the n^{th} root of $\overline{1-q}|^{\frac{1}{n}}$, or to $\overline{1-q}|^{\frac{1}{n}}$.

But it has been ſhewn above, in Lemma 1, article 11, that when three quantities are very nearly equal to each other, the ratio of the greateſt of the three to the leaſt will be to the ratio of the middle quantity to the leaſt in very nearly the

the fame proportion as the excefs of the greateft quantity above the leaft is to the excefs of the middle quantity above the leaft; and hence it follows, *dividendo*, that the ratio of the greateft of the three to the leaft will be to the excefs of the faid ratio above the ratio of the middle quantity to the leaft, or to the ratio of the greateft quantity to the middle quantity, in the fame proportion as the excefs of the greateft quantity above the leaft is to the difference whereby the faid excefs exceeds the excefs of the middle quantity above the leaft, or to the excefs of the greateft quantity above the middle quantity: or, in other words, if L, M, and N are three quantities that are very nearly equal to each other, whereof L is the greateft and N is the leaft; it appears from Lemma 1, article 11, that the ratio of L to N will be to the ratio of M to N very nearly in the fame proportion as the difference L — N is to the difference M — N; and confequently, *dividendo*, the ratio of L to N will be to the excefs of the ratio of L to N above the ratio of M to N, that is, to the ratio of L to M, very nearly in the fame proportion as the difference L — N is to the excefs of the difference L — N above the difference M — N, that is, to the difference L — M. Therefore in thefe three quantities, to wit, 1 and the n^{th}, or nonillionth, root of $1 - q$, and the n^{th}, or nonillionth, root of $1 - k$, or 1, $\overline{1-q}|^{\frac{1}{n}}$, and $\overline{1-k}|^{\frac{1}{n}}$, (in which the exceffes of the firft quantity 1 above the two latter quantities are fo extremely fmall, that they will not appear before the 54^{th} place of decimal fractions), we may conclude, that the ratio of the greateft, to wit, 1, to the leaft, to wit, $\overline{1-k}|^{\frac{1}{n}}$, will be to the ratio of 1, the greateft, to $\overline{1-q}|^{\frac{1}{n}}$, the middle quantity, very nearly in the fame proportion as $1 - \overline{1-k}|^{\frac{1}{n}}$, the difference between the greateft and the leaft, is to $1 - \overline{1-q}|^{\frac{1}{n}}$, the difference between the greateft and the middle quantity.

But it has been before fhewn, that the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the fame proportion as the very fmall ratio of 1 to $\overline{1-k}|^{\frac{1}{n}}$ is to the very fmall ratio of 1 to $\overline{1-q}|^{\frac{1}{n}}$.

Therefore the ratio of 1 to $1 - k$ will be to the ratio of 1 to $1 - q$ very nearly in the fame proportion as the excefs of 1 above $\overline{1-k}|^{\frac{1}{n}}$ to the excefs of 1 above $\overline{1-q}|^{\frac{1}{n}}$.

But it has been fhewn that, in this cafe of the very great magnitude of n, the excefs of 1 above $\overline{1-k}|^{\frac{1}{n}}$ is to the excefs of 1 above $\overline{1-q}|^{\frac{1}{n}}$, very nearly in the fame proportion as the feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* to the feries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ *&c ad infinitum.*

Therefore

Therefore the ratio of 1 to $1 - k$ will be to the ratio of 1 to $1 - q$, in the ſame proportion as the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c *ad infinitum* to the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c *ad infinitum*.

Q. E. D.

The foregoing demonſtration expreſſed in fewer words.

67. The foregoing demonſtration may be expreſſed in fewer words, as follows:

Let n be put as before, for any very large number; as, for example, for a nonillion, or the ninth power of a million.

Then it will follow from Lemma 3, coroll. 2, that $1 - \overline{1 - k}|^{\frac{1}{n}}$ will be very nearly equal to $\frac{1}{n} \times$ the infinite ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \frac{k^7}{7} +$ &c, and that $1 - \overline{1 - q}|^{\frac{1}{n}}$ will, in like manner, be, very nearly, equal to $\frac{1}{n} \times$ the infinite ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \frac{q^7}{7} +$ &c. Therefore $1 - \overline{1 - k}|^{\frac{1}{n}}$ will be to $1 - \overline{1 - q}|^{\frac{1}{n}}$, very nearly, in the ſame proportion as $\frac{1}{n} \times$ the infinite ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c to $\frac{1}{n} \times$ the infinite ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, that is, in the ſame proportion as the n^{th} part of the infinite ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c to the n^{th} part of the infinite ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, and conſequently in the ſame proportion as the whole ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c *ad infinitum*, to the whole ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c *ad infinitum*.

Now the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the ſame proportion as the n^{th} part of the former ratio is to the n^{th} part of the latter ratio, and conſequently, in the ſame proportion as the ratio of 1 to $\overline{1 - k}|^{\frac{1}{n}}$ is to the ratio of 1 to $\overline{1 - q}|^{\frac{1}{n}}$.

But, becauſe $\overline{1 - k}|^{\frac{1}{n}}$ and $\overline{1 - q}|^{\frac{1}{n}}$ approach extremely near to an equality with 1, it follows from Lemma 1, that the ratio of 1 to $\overline{1 - k}|^{\frac{1}{n}}$ will be to the ratio of

of $\overline{1-q}|^{\frac{1}{n}}$ to $\overline{1-k}|^{\frac{1}{n}}$ in very nearly the same proportion as the excess of 1 above $\overline{1-k}|^{\frac{1}{n}}$ is to the excess of $\overline{1-q}|^{\frac{1}{n}}$ above $\overline{1-k}|^{\frac{1}{n}}$, or (putting R. $\frac{1}{\overline{1-k}|^{\frac{1}{n}}}$ for the ratio of 1 to $\overline{1-k}|^{\frac{1}{n}}$, and R. $\frac{\overline{1-q}|^{\frac{1}{n}}}{\overline{1-k}|^{\frac{1}{n}}}$ for the ratio of $\overline{1-q}|^{\frac{1}{n}}$ to $\overline{1-k}|^{\frac{1}{n}}$) that R. $\frac{1}{\overline{1-k}|^{\frac{1}{n}}}$ will be to R. $\frac{\overline{1-q}|^{\frac{1}{n}}}{\overline{1-k}|^{\frac{1}{n}}}$ very nearly in the same proportion as $1-\overline{1-k}|^{\frac{1}{n}}$ is to $\overline{1-q}|^{\frac{1}{n}}-\overline{1-k}|^{\frac{1}{n}}$. And therefore it follows, *dividendo*, that the first of these four quantities will be to the excess of the first above the second in very nearly the same proportion as the third quantity to the excess of the third above the fourth, that is, that R. $\frac{1}{\overline{1-k}|^{\frac{1}{n}}}$ will be to R. $\frac{1}{\overline{1-k}|^{\frac{1}{n}}}$ − R. $\frac{\overline{1-q}|^{\frac{1}{n}}}{\overline{1-k}|^{\frac{1}{n}}}$ very nearly in the same proportion as $1-\overline{1-k}|^{\frac{1}{n}}$ is to $1-\overline{1-k}|^{\frac{1}{n}}-\overline{\overline{1-q}|^{\frac{1}{n}}-\overline{1-k}|^{\frac{1}{n}}}$, or (because R. $\frac{1}{\overline{1-k}|^{\frac{1}{n}}}$ − R. $\frac{\overline{1-q}|^{\frac{1}{n}}}{\overline{1-k}|^{\frac{1}{n}}}$ is $=$ R. $\frac{1}{\overline{1-q}|^{\frac{1}{n}}}$, and $1-\overline{1-k}|^{\frac{1}{n}}-\overline{\overline{1-q}|^{\frac{1}{n}}-\overline{1-k}|^{\frac{1}{n}}}$ is $=1-\overline{1-k}|^{\frac{1}{n}}-\overline{1-q}|^{\frac{1}{n}}+\overline{1-k}|^{\frac{1}{n}}=1-\overline{1-q}|^{\frac{1}{n}}$,) that R. $\frac{1}{\overline{1-k}|^{\frac{1}{n}}}$ will be to R. $\frac{1}{\overline{1-q}|^{\frac{1}{n}}}$ very nearly in the same proportion as $1-\overline{1-k}|^{\frac{1}{n}}$ to $1-\overline{1-q}|^{\frac{1}{n}}$. Therefore the ratio of 1 to $1-k$ will be to the ratio of 1 to $1-q$ very nearly in the same proportion as $1-\overline{1-k}|^{\frac{1}{n}}$ is to $1-\overline{1-q}|^{\frac{1}{n}}$.

But it has been shewn that, in this case of the very great magnitude of n, the proportion of $1-\overline{1-k}|^{\frac{1}{n}}$ to $1-\overline{1-q}|^{\frac{1}{n}}$ is very nearly the same as that of the series $k+\frac{k^2}{2}+\frac{k^3}{3}+\frac{k^4}{4}+\frac{k^5}{5}+\frac{k^6}{6}+$ &c *ad infinitum* to the series $q+\frac{q^2}{2}+\frac{q^3}{3}+\frac{q^4}{4}+\frac{q^5}{5}+\frac{q^6}{6}+$ &c *ad infinitum*.

Therefore

Therefore the ratio of 1 to $1-k$ will be the ratio of 1 to $1-q$ in the ſame proportion as the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c *ad infinitum* is to the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c *ad infinitum*.

Q. E. D.

The ſame demonſtration expreſſed with ſtill greater brevity.

68. And, if ſtill greater brevity of expreſſion be deſired, the foregoing demonſtration may be expreſſed in a ſtill conciſer manner, as follows.

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

Then will R. $\frac{1}{1-k}$ be to R. $\frac{1}{1-q}$ in the ſame proportion as $\frac{1}{n} \times$ R. $\frac{1}{1-k}$ to $\frac{1}{n} \times$ R. $\frac{1}{1-q}$, that is, (by the nature of roots,) as R. $\frac{1}{\overline{1-k})^{\frac{1}{n}}}$ to R. $\frac{1}{\overline{1-q})^{\frac{1}{n}}}$ that is by Lemma 1, (on account of the near approach of $\overline{1-k})^{\frac{1}{n}}$ and $\overline{1-q})^{\frac{1}{n}}$ to an equality with 1,) as $1 - \overline{1-k}\,^{\frac{1}{n}}$ to $1 - \overline{1-q})^{\frac{1}{n}}$, that is, by Lemma 3[d], coroll. 2, (on account of the very great magnitude of the number n,) as $\frac{1}{n} \times$ the infinite ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6}$ &c to $\frac{1}{n} \times$ the infinite ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, and conſequently (multiplying both ſides by n) as the infinite ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c to the infinite ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c. Q. E. D.

SCHOLIUM.

69. This propoſition is *accurately* true, though, when n is of any finite magnitude, how great ſoever, ſome of the intermediate propoſitions by means of which it is proved, are only *very nearly* true. For, as theſe intermediate propoſitions are not limited in the degree in which they approach to truth or accuracy, but may be made to come as near to being accurately true as we pleaſe, by increaſing the number n, the concluſion derived from them muſt be accurately true. For, if it were ſuppoſed to be not accurately true, but only to approach within certain limits of the truth, we might increaſe the magnitude of n till the ſaid concluſion was made to approach nearer to being accurately true than within the aſſigned limits; which would be contrary to the ſuppoſition of its having approached only as near as

as the faid affigned limits to the truth, and confequently would prove that the faid fuppofition was falfe. Therefore the faid conclufion does not only approach within certain limits of the truth, but is accurately true.

70. We have now, I hope, fufficiently eftablifhed the truth of Dr. Wallis's propofition, to wit, "that, if q be made to reprefent fucceffively feveral different fractions, or quantities lefs than 1, the feveral values of the infinite feries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c will be proportional to the feveral ratios of 1 to the correfponding values of $1 - q$ refpectively, or will be the logarithms of thofe ratios." It remains that we illuftrate the ufe of this feries in the bufinefs of computing logarithms by applying it to a few examples. I fhall therefore now proceed to apply it to the computation of the logarithms of the nine following ratios, to wit, the ratio of 10 to 9, the ratio of 11 to 10, the ratio of 81 to 80, the ratio of 121 to 120, the ratio of 2401 to 2400, the ratio of 169 to 168, the ratio of 289 to 288, the ratio of 361 to 360, and the ratio of 529 to 528; which are the ratios of which we computed the logarithms above in the nine examples of the ufe of the former feries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c, that was invented by Mr. Mercator. See above, art. 24, 25, 26, 27, 28, 29, 30, 31, and 32. By the help of thefe nine logarithms we fhall be able to find the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 10 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, and the logarithms of all greater ratios that are compofed of thefe ratios, by mere addition and fubtraction; which will make the utility of this feries fufficiently manifeft. And, as in computing the logarithms of the faid nine ratios, the numerator of the fraction that is equal to q is always 1, and the denominator of it is a whole number, I think it will be moft convenient to fubftitute $\frac{1}{m}$ inftead of q in the refidual quantity $1 - q$ and the infinite feries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ &c, whereby the faid refidual quantity will be changed into $1 - \frac{1}{m}$, and the faid feries will be changed into the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ &c, or the logarithm of the ratio of 1 to $1 - \frac{1}{m}$ will be the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ &c *ad infinitum*.

71. If we fhould have occafion to exprefs this feries in fuch a manner as to point out the generating fractions by the fucceffive multiplication of which its fecond and other following terms are derived from the firft term $\frac{1}{m}$ and from each other,

it

it may be done by putting the capital letter A for the whole firſt term $\frac{1}{m}$, and B for the whole ſecond term $\frac{1}{2m^2}$, and C for the whole third term $\frac{1}{3m^3}$, and D for the whole fourth term $\frac{1}{4m^4}$, and E, F, G, H, I, K, L, M, and the following capital letters of the alphabet, for the whole fifth, ſixth, ſeventh, eighth, ninth, tenth, eleventh, twelfth, and other following terms of the ſeries; by which ſubſtitution the ſaid Series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \frac{1}{9m^9} + \frac{1}{10m^{10}} +$ &c will be converted into the ſeries $\frac{1}{m} + \frac{A}{2m} + \frac{2B}{3m} + \frac{3C}{4m} + \frac{4D}{5m} + \frac{5E}{6m} + \frac{6F}{7m} + \frac{7G}{8m} + \frac{8H}{9m} + \frac{9I}{10m} +$ &c. But the terms of the ſeries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \frac{1}{9m^9} + \frac{1}{10m^{10}} +$ &c are ſo extremely ſimple and conciſe that there ſeems to be no occaſion to change its form.

EXAMPLES

OF THE

COMPUTATION OF THE LOGARITHMS

OF THE

RATIOS OF 1 TO SEVERAL NUMBERS

Denoted by the residual quantity $1 - \frac{1}{m}$, by means of the infinite series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \frac{1}{9m^9} + \frac{1}{10m^{10}} + \frac{1}{11m^{11}} + \frac{1}{12m^{12}} + \frac{1}{13m^{13}} + \frac{1}{14m^{14}} + \frac{1}{15m^{15}} + \frac{1}{16m^{16}} + \frac{1}{17m^{17}} + \frac{1}{18m^{18}} + \&c$,

INVENTED BY DR. WALLIS.

EXAMPLE I.

72. LET it be required to find by means of the aforesaid series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ &c *ad infinitum*, the logarithm of the ratio of 10 to 9 (or of 10 to $10 - 1$, or of $\frac{10}{10}$ to $\frac{10-1}{10}$), or of 1 to $1 - \frac{1}{10}$.

Here m is $= 10$, and $\frac{1}{m}$ is $= \frac{1}{10} = 0.100,000,000,000,000,000$.

We

We fhall therefore have,

$$\frac{1}{m^2}\left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{10} = \frac{0.100,000,000,000,000,000}{10}\right) = 0.010,000,000,000,000,000$$

And $\frac{1}{m^3}\left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{10} = \frac{0.010,000,000,000,000,000}{10}\right) = 0.001,000,000,000,000,000$

And $\frac{1}{m^4}\left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{10} = \frac{0.001,000,000,000,000,000}{10}\right) = 0.000,100,000,000,000,000$

And $\frac{1}{m^5}\left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{10} = \frac{0.000,100,000,000,000,000}{10}\right) = 0.000,010,000,000,000,000$

And $\frac{1}{m^6}\left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{10} = \frac{0.000,010,000,000,000,000}{10}\right) = 0.000,001,000,000,000,000$

And $\frac{1}{m^7}\left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{10} = \frac{0.000,001,000,000,000,000}{10}\right) = 0.000,000,100,000,000,000$

And $\frac{1}{m^8}\left(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{10} = \frac{0.000,000,100,000,000,000}{10}\right) = 0.000,000,010,000,000,000$

And $\frac{1}{m^9}\left(= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{10} = \frac{0.000,000,010,000,000,000}{10}\right) = 0.000,000,001,000,000,000$

And $\frac{1}{m^{10}}\left(= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{10} = \frac{0.000,000,001,000,000,000}{10}\right) = 0.000,000,000,100,000,000$

And $\frac{1}{m^{11}}\left(= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,100,000,000}{10}\right) = 0.000,000,000,010,000,000$

And $\frac{1}{m^{12}}\left(= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{10} = \frac{0.000,000,000,010,000,000}{10}\right) = 0.000,000,000,001,000,000$

And $\frac{1}{m^{13}}\left(= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{10} = \frac{0.000,000,000,001,000,000}{10}\right) = 0.000,000,000,000,100,000$

And $\frac{1}{m^{14}}\left(= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{10} = \frac{0.000,000,000,000,100,000}{10}\right) = 0.000,000,000,000,010,000$

And $\frac{1}{m^{15}}\left(= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{10} = \frac{0.000,000,000,000,010,000}{10}\right) = 0.000,000,000,000,001,000;$

And $\frac{1}{m^{16}}\left(= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{10} = \frac{0.000,000,000,000,001,000}{10}\right) = 0.000,000,000,000,000,100;$

And $\frac{1}{m^{17}}\left(= \frac{1}{m^{16}} \times \frac{1}{m} = \frac{1}{m^{16}} \times \frac{1}{10} = \frac{0.000,000,000,000,000,100}{10}\right) = 0.000,000,000,000,000,010;$

And confequently

$$\frac{1}{2m^2}\left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.010,000,000,000,000,000}{2}\right) = 0.005,000,000,000,000,000;$$

And $\frac{1}{3m^3}\left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.001,000,000,000,000,000}{3}\right) = 0.000,333,333,333,333,333;$

And $\frac{1}{4m^4}\left(=\frac{1}{m^4}\times\frac{1}{4}=\frac{0.000,100,000,000,000,000}{4}\right)=0.000,025,000,000,000,000$;

And $\frac{1}{5m^5}\left(=\frac{1}{m^5}\times\frac{1}{5}=\frac{0.000,010,000,000,000,000}{5}\right)=0.000,002,000,000,000,000$;

And $\frac{1}{6m^6}\left(=\frac{1}{m^6}\times\frac{1}{6}=\frac{0.000,001,000,000,000,000}{6}\right)=0.000,000,166,666,666,666$;

And $\frac{1}{7m^7}\left(=\frac{1}{m^7}\times\frac{1}{7}=\frac{0.000,000,100,000,000,000}{7}\right)=0.000,000,014,285,714,285$;

And $\frac{1}{8m^8}\left(=\frac{1}{m^8}\times\frac{1}{8}=\frac{0.000,000,010,000,000,000}{8}\right)=0.000,000,001,250,000,000$;

And $\frac{1}{9m^9}\left(=\frac{1}{m^9}\times\frac{1}{9}=\frac{0.000,000,001,000,000,000}{9}\right)=0.000,000,000,111,111,111$;

And $\frac{1}{10m^{10}}\left(=\frac{1}{m^{10}}\times\frac{1}{10}=\frac{0.000,000,000,100,000,000}{10}\right)=0.000,000,000,010,000,000$;

And $\frac{1}{11m^{11}}\left(=\frac{1}{m^{11}}\times\frac{1}{11}=\frac{0.000,000,000,010,000,000}{11}\right)=0.000,000,000,000,909,090$;

And $\frac{1}{12m^{12}}\left(=\frac{1}{m^{12}}\times\frac{1}{12}=\frac{0.000,000,000,001,000,000}{12}\right)=0.000,000,000,000,083,333$;

And $\frac{1}{13m^{13}}\left(=\frac{1}{m^{13}}\times\frac{1}{13}=\frac{0.000,000,000,000,100,000}{13}\right)=0.000,000,000,000,007,692$;

And $\frac{1}{14m^{14}}\left(=\frac{1}{m^{14}}\times\frac{1}{14}=\frac{0.000,000,000,000,010,000}{14}\right)=0.000,000,000,000,000,714$;

And $\frac{1}{15m^{15}}\left(=\frac{1}{m^{15}}\times\frac{1}{15}=\frac{0.000,000,000,000,001,000}{15}\right)=0.000,000,000,000,000,066$;

And $\frac{1}{16m^{16}}\left(=\frac{1}{m^{16}}\times\frac{1}{16}=\frac{0.000,000,000,000,000,100}{16}\right)=0.000,000,000,000,000,006$;

And $\frac{1}{17m^{17}}\left(=\frac{1}{m^{17}}\times\frac{1}{17}=\frac{0.000,000,000,000,000,010}{17}\right)=0.000,000,000,000,000,000$.

Therefore the feries $\frac{1}{m}+\frac{1}{2m^2}+\frac{1}{3m^3}+\frac{1}{4m^4}+\frac{1}{5m^5}+\frac{1}{6m^6}+\frac{1}{7m^7}+\frac{1}{8m^8}+\frac{1}{9m^9}+\frac{1}{10m^{10}}+\frac{1}{11m^{11}}+\frac{1}{12m^{12}}+\frac{1}{13m^{13}}+\frac{1}{14m^{14}}+\frac{1}{15m^{15}}+\frac{1}{16m^{16}}+$ &c is =

$$\begin{array}{r}
0;100,000,000,000,000,000, \\
+\ ;..\,5,000,000,000,000,000, \\
+\ ;...,333,333,333,333,333, \\
+\ ;...,.\,25,000,000,000,000, \\
+\ ;...,..\,2,000,000,000,000, \\
+\ ;...,...,166,666,666,666, \\
+\ ;...,...,.\,14,285,714,285, \\
+\ ;...,...,..\,1,250,000,000, \\
+\ ;...,...,...,111,111,111, \\
+\ ;...,...,...,.\,10,000,000, \\
+\ ;...,...,...,...,909,090, \\
+\ ;...,...,...,...,.\,83,333, \\
+\ ;...,...,...,...,..\,7,692, \\
+\ ;...,...,...,...,...,714, \\
+\ ;...,...,...,...,...,.\,66, \\
+\ ;...,...,...,...,...,..\,6, \\
\hline
=\ 0.105,360,515,657,826,296.
\end{array}$$

Therefore this number 0.105,360,515,657,826,296 is the logarithm of the ratio of 1 to $1 - \frac{1}{10}$ or of 10 to 9. Q. E. I.

N. B. This value of the logarithm of the ratio of 10 to 9 is leſs than its true value by about 5 in the 18th place of decimal fractions, or 0.000,000,000,000, 000,005. For, if we add 0.000,000,000,000,000,005 to 0.105,360,515,657, 826,296, their ſum will be 0.105,360,515,657,826,301, which agrees with the true logarithm of this ratio in all its figures, the more accurate value of this logarithm being 0.105,360,515,657,826,301,227.

EXAMPLE II.

73. Let it be required to find by means of the ſame infinite ſeries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ &c the logarithm of the ratio of 11 to 10 (or of 11 to 11 − 1, or of $\frac{11}{11}$ to $\frac{11-1}{11}$,) or of 1 to $1 - \frac{1}{11}$.

Here m is $= 11$, and $\frac{1}{m}$ is $= \frac{1}{11} = 0.090,909,090,909,090,909$, &c.

We ſhall therefore have

$$\frac{1}{m^2} \left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{11} = \frac{0.090,909,090,909,090,909}{11}\right) = 0.008,264,462,809,917,355;$$

And $$\frac{1}{m^3} \left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{11} = \frac{0.008,264,462,809,917,355}{11}\right) = 0.000,751,314,800,901,577;$$

And

And $\frac{1}{m^4}$ $(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{11} = \frac{0.000,751,314,800,901,577}{11}) = 0.000,068,301,345,536,507$;

And $\frac{1}{m^5}$ $(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{11} = \frac{0.000,068,301,345,536,507}{11}) = 0.000,006,209,213,230,591$;

And $\frac{1}{m^6}$ $(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{11} = \frac{0.000,006,209,213,230,591}{11}) = 0.000,000,564,473,930,053$;

And $\frac{1}{m^7}$ $(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{11} = \frac{0.000,000,564,473,930,053}{11}) = 0.000,000,051,315,811,823$;

And $\frac{1}{m^8}$ $(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{11} = \frac{0.000,000,051,315,811,823}{11}) = 0.000,000,004,665,073,802$;

And $\frac{1}{m^9}$ $(= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{11} = \frac{0.000,000,004,665,073,802}{11}) = 0.000,000,000,424,097,618$;

And $\frac{1}{m^{10}}$ $(= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{11} = \frac{0.000,000,000,424,097,618}{11}) = 0.000,000,000,038,554,328$;

And $\frac{1}{m^{11}}$ $(= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{11} = \frac{0.000,000,000,038,554,328}{11}) = 0.000,000,000,003,504,938$;

And $\frac{1}{m^{12}}$ $(= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,003,504,938}{11}) = 0.000,000,000,000,318,630$;

And $\frac{1}{m^{13}}$ $(= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{11} = \frac{0.000,000,000,000,318,630}{11}) = 0.000,000,000,000,028,966$;

And $\frac{1}{m^{14}}$ $(= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{11} = \frac{0.000,000,000,000,028,966}{11}) = 0.000,000,000,000,002,633$;

And $\frac{1}{m^{15}}$ $(= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{11} = \frac{0.000,000,000,000,002,633}{11}) = 0.000,000,000,000,000,239$;

And $\frac{1}{m^{16}}$ $(= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{11} = \frac{0.000,000,000,000,000,239}{11}) = 0.000,000,000,000,000,021$;

And $\frac{1}{m^{17}}$ $(= \frac{1}{m^{16}} \times \frac{1}{m} = \frac{1}{m^{16}} \times \frac{1}{11} = \frac{0.000,000,000,000,000,021}{11}) = 0.000,000,000,000,000,001$;

And confequently

$\frac{1}{2m^2}$ $(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.008,264,462,809,917,355}{2}) = 0.004,132,231,404,958,677$;

And $\frac{1}{3m^3}$ $(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,751,314,800,901,577}{3}) = 0.000,250,438,266,967,192$;

And $\frac{1}{4m^4}$ $(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,068,301,345,536,507}{4}) = 0.000,017,075,336,384,126$;

And $\frac{1}{5m^5}$ $(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,006,209,213,230,591}{5}) = 0.000,001,241,842,646,118$;

And

And $\frac{1}{6m^6}$ $\left(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,564,473,930,053}{6}\right) = 0.000,000,094,078,988,342$;

And $\frac{1}{7m^7}$ $\left(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,051,315,811,823}{7}\right) = 0.000,000,007,330,830,260$;

And $\frac{1}{8m^8}$ $\left(= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,004,665,073,802}{8}\right) = 0.000,000,000,583,134,225$;

And $\frac{1}{9m^9}$ $\left(= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,424,097,618}{9}\right) = 0.000,000,000,047,121,957$;

And $\frac{1}{10m^{10}}$ $\left(= \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,038,554,328}{10}\right) = 0.000,000,000,003,855,432$;

And $\frac{1}{11m^{11}}$ $\left(= \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,003,504,938}{11}\right) = 0.000,000,000,000,318,630$;

And $\frac{1}{12m^{12}}$ $\left(= \frac{1}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,000,318,630}{12}\right) = 0.000,000,000,000,026,552$;

And $\frac{1}{13m^{13}}$ $\left(= \frac{1}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,028,966}{13}\right) = 0.000,000,000,000,002,228$;

And $\frac{1}{14m^{14}}$ $\left(= \frac{1}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000.002,633}{14}\right) = 0.000,000,000,000,000,188$;

And $\frac{1}{15m^{15}}$ $\left(= \frac{1}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,000,239}{15}\right) = 0.000,000,000,000,000,015$;

And $\frac{1}{16m^{16}}$ $\left(= \frac{1}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,021}{16}\right) = 0.000,000,000,000,000,001$;

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \frac{1}{9m^9} + \frac{1}{10m^{10}} + \frac{1}{11m^{11}} + \frac{1}{12m^{12}} + \frac{1}{13m^{13}} + \frac{1}{14m^{14}} + \frac{1}{15m^{15}} + \frac{1}{16m^{16}} +$ &c

is = 0;090,909,090,909,090,909,
+ ;. . 4,132,231,404,958,677,
+ ;. . . ,250,438,266,967,192,
+ ;. . . ,. 17,075,336,384,126,
+ ;. . . ,. . 1,241,842,646,118,
+ ;. . . ,. . . ,. 94,078,988,342,
+ ;. . . ,. . . ,. . 7,330,830,260,
+ ;. . . ,. . . ,. . . ,583,134,225,
+ ;. . . ,. . . ,. . . ,. 47,121,957,
+ ;. . . ,. . . ,. . . ,. . 3,855,432,
+ ;. . . ,. . . ,. . . ,. . . ,318,630,
+ ;. . . ,. . . ,. . . ,. . . ,. 26,552,
+ ;. . . ,. . . ,. . . ,. . . ,. . 2,228,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,188,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 15,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. . 1,

= 0.095,310,179,804,324,852.

Therefore this number 0.095,310,179,804,324,852, is the logarithm of the ratio of 1 to $1 - \frac{1}{11}$, or of 11 to 10. Q. E. I.

This number agrees with the value of the fame logarithm found above in article 25 by Mercator's feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6}$ + &c. in all the figures but the laft, that value being 0.095,310,179,804, 324,858.

EXAMPLE III.

74. Let it be required to find by means of the fame feries the logarithm of the ratio of 81 to 80, or of 81 to 81 — 1 (or of $\frac{81}{81}$ to $\frac{81-1}{81}$), or of 1 to $1 - \frac{1}{81}$.

Here m is $= 81$, and $\frac{1}{m}$ is $= \frac{1}{81} = 0.012,345,679,012,345,679$.

We fhall therefore have

$$\frac{1}{m^2} \left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{81} = \frac{0.012,345,679,012,345,679}{81}\right) = 0.000,152,415,790,275,872;$$

And $$\frac{1}{m^3} \left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{81} = \frac{0.000,152,415,790,275,872}{81}\right) = 0.000,001,881,676,423,158;$$

And $$\frac{1}{m^4} \left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{81} = \frac{0.000,001,881,676,423,158}{81}\right) = 0.000,000,023,230,573,125;$$

And $$\frac{1}{m^5} \left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{81} = \frac{0.000,000,023,230,573,125}{81}\right) = 0.000,000,000,286,797,199;$$

And $$\frac{1}{m^6} \left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{81} = \frac{0.000,000,000,286,797,199}{81}\right) = 0.000,000,000,003,540,706;$$

And $$\frac{1}{m^7} \left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{81} = \frac{0.000,000,000,003,540,706}{81}\right) = 0.000,000,000,000,043,712;$$

And $$\frac{1}{m^8} \left(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{81} = \frac{0.000,000,000,000,043,712}{81}\right) = 0.000,000,000,000,000,539;$$

And $$\frac{1}{m^9} \left(= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{81} = \frac{0.000,000,000,000,000,539}{81}\right) = 0.000,000,000,000,000,006;$$

And confequently

$$\frac{1}{2m^2} \left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,152,415,790,275,872}{2}\right) = 0.000,076,207,895,137,936;$$

And $$\frac{1}{3m^3} \left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,001,881,676,423,158}{3}\right) = 0.000,000,627,225,474,386;$$

And

And $\frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,023,230,573,125}{4}) = 0.000,000,005,807,643,281$;

And $\frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,286,797,199}{5}) = 0.000,000,000,057,359,439$;

And $\frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,003,540,706}{6}) = 0.000,000,000,000,590,117$;

And $\frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,043,712}{7}) = 0.000,000,000,000,006,244$;

And $\frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,539}{8}) = 0.000,000,000,000,000,067$;

And $\frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,000,000,006}{9}) = 0.000,000,000,000,000,000$.

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} +$ &c is =

0,012,345,679,012,345,679,
\+ ;. . . ,076,207,895,137,936,
\+ ;. . . ,. . . ,627,225,474,386,
\+ ;. . . ,. . . ,. . 5,807,643,281,
\+ ;. . . ,. . . ,. . . ,. 57,359,439,
\+ ;. . . ,. . . ,. . . ,. . . ,590,117,
\+ ;. . . ,. . . ,. . . ,. . . ,. . 6,244,
\+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 67,
\+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. . 0,
= 0.012,422,519,998,557,149.

Therefore this number 0.012,422,519,998,557,149 is the logarithm of the ratio of 1 to $1 - \frac{1}{81}$, or of 81 to 80. Q. E. I.

This number agrees with the value of the same logarithm found above in article 26, by Mercator's series, in all but the two last figures, that value being 0.012,422,519,998,557,152.

EXAMPLE IV.

75. Let it be required to find by means of the same series the logarithm of the ratio of 121 to 120, (or of 121 to 121 − 1, or of $\frac{121}{121}$ to $\frac{121-1}{121}$) or of 1 to $1 - \frac{1}{121}$.

Here m is = 121, and $\frac{1}{m}$ is = $\frac{1}{121}$ = 0.008,264,462,809,917,355.

We ſhall therefore have,

$\frac{1}{m^2}$ $(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{121} = \frac{0.008,264,462,809,917,355}{121}) = 0.000,068,301,345,536,507$;

And $\frac{1}{m^3}$ $(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{121} = \frac{0.000,068,301,345,536,507}{121}) = 0.000,000,564,473,930,053$;

And $\frac{1}{m^4}$ $(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{121} = \frac{0.000,000,564,473,930,053}{121}) = 0.000,000,004,665,073,802$;

And $\frac{1}{m^5}$ $(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{121} = \frac{0.000,000,004,665,073,802}{121}) = 0.000,000,000,038,554,328$;

And $\frac{1}{m^6}$ $(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{121} = \frac{0.000,000,000,038,554,328}{121}) = 0.000,000,000,000,318,630$;

And $\frac{1}{m^7}$ $(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{121} = \frac{0.000,000,000,000,318,630}{121}) = 0.000,000,000,000,002,633$;

And $\frac{1}{m^8}$ $(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{121} = \frac{0.000,000,000,000,002,633}{121}) = 0.000,000,000,000,000,021$;

And $\frac{1}{m^9}$ $(= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{121} = \frac{0.000,000,000,000,000,021}{121}) = 0.000,000,000,000,000,000$;

And conſequently

$\frac{1}{2m^2}$ $(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,068,301,345,536,507}{2}) = 0.000,034,150,672,768,253$;

And $\frac{1}{3m^3}$ $(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,564,473,930,053}{3}) = 0.000,000,188,157,976,684$;

And $\frac{1}{4m^4}$ $(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,004,665,073,802}{4}) = 0.000,000,001,166,268,450$;

And $\frac{1}{5m^5}$ $(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,038,554,328}{5}) = 0.000,000,000,007,710,865$;

And $\frac{1}{6m^6}$ $(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,318,630}{6}) = 0.000,000,000,000,053,105$;

And $\frac{1}{7m^7}$ $(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,002,633}{7}) = 0.000,000,000,000,000,376$;

And $\frac{1}{8m^8}$ $(= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,021}{8}) = 0.000,000,000,000,000,002$.

Therefore the ſeries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \&c$ is =

$$\begin{array}{r} 0;008,264,462,809,917,355, \\ +\ ;\ldots,\ldots 34,150,672,768,253, \\ +\ ;\ldots,\ldots,188,157,976,684, \\ +\ ;\ldots,\ldots,\ldots 1,166,268,450, \\ +\ ;\ldots,\ldots,\ldots,\ldots 7,710,865, \\ +\ ;\ldots,\ldots,\ldots,\ldots,\ldots 53,105, \\ +\ ;\ldots,\ldots,\ldots,\ldots,\ldots,376, \\ +\ ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots 2, \\ \hline = 0.008,298,802,814,695,090. \end{array}$$

Therefore this number 0.008,298,802,814,695,090, is the logarithm of the ratio of 1 to $1 - \frac{1}{121}$, or of 121 to 120. Q. E. I.

This number agrees with the value of the same logarithm found above in article 27, by Mercator's series, in all but the last figure, that value being 0.008, 298,802,814,695,093.

EXAMPLE V.

76. Let it be required to find by means of the same series the logarithm of the ratio of 2401 to 2400, (or of 2401 to 2401 — 1, or of $\frac{2401}{2401}$ to $\frac{2401-1}{2401}$,) or of 1 to $1 - \frac{1}{2401}$.

Here m is $=$ 2401, and $\frac{1}{m}$ is $= \frac{1}{2401} =$ 0.000,416,493,127,863,390, &c.

We shall therefore have

$$\frac{1}{m^2}\ (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{2401} = \frac{0.000,416,493,127,863,390}{2401}) = 0.000,000,173,466,525,557;$$

And $$\frac{1}{m^3}\ (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{2401} = \frac{0.000,000,173,466,525,557}{2401}) = 0.000,000,000,072,247,615;$$

And $$\frac{1}{m^4}\ (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{2401} = \frac{0.000,000,000,072,247,615}{2401}) = 0.000,000,000,000,030,090;$$

And $$\frac{1}{m^5}\ (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{2401} = \frac{0.000,000,000,000,030,090}{2401}) = 0.000,000,000,000,000,012;$$

And $$\frac{1}{m^6}\ (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{2401} = \frac{0.000,000,000,000,000,012}{2401}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{1}{2m^2}\ (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,000,173,466,525,557}{2}) = 0.000,000,086,733,262,778;$$

And $\frac{1}{3m^3}$ $(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,000,072,247,615}{3}) = 0.000,000,000,024,082,538$;

And $\frac{1}{4m^4}$ $(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,000,030,090}{4}) = 0.000,000,000,000,007,522$;

And $\frac{1}{5m^5}$ $(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,000,012}{5}) = 0.000,000,000,000,000,002$.

Therefore the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} +$ &c is in this cafe $=$

$$\begin{array}{rl} & 0.000,416,493,127,863,390, \\ + & ;\ldots,\ldots,.\ 86,733,262,778, \\ + & ;\ldots,\ldots,\ldots,.\ 24,082,538, \\ + & ;\ldots,\ldots,\ldots,\ldots,\ldots 7,522, \\ + & ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots 2, \\ \hline = & 0.000,416,579,885,216,230. \end{array}$$

Therefore this number 0.000,416,579,885,216,230 is the logarithm of the ratio of 1 to $1 - \frac{1}{2401}$, or of 2401 to 2400. Q. E. I.

This number agrees with the value of the fame logarithm found above in article 28, by Mercator's feries, in all the figures except the laft, that value of it being 0.000,416,579,885,216,232.

EXAMPLE VI.

77. Let it be required to find by the fame feries the logarithm of the ratio of 169 (which is the fquare of 13) to 168, (or of the ratio of 169 to $169 - 1$, or of $\frac{169}{169}$ to $\frac{169-1}{169}$,) or of 1 to $1 - \frac{1}{169}$.

Here m is $= 169$, and $\frac{1}{m}$ is $= \frac{1}{169} = 0.005,917,159,763,313,609$, &c.

We fhall therefore have

$\frac{1}{m^2}$ $(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{169} = \frac{0.005,917,159,763,313,609}{169}) = 0.000,035,012,779,664,577$;

And $\frac{1}{m^3}$ $(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{169} = \frac{0.000,035,012,779,664,577}{169}) = 0.000,000,207,176,211,033$;

And $\frac{1}{m^4}$ $(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{169} = \frac{0.000,000,207,176,211,033}{169}) = 0.000,000,001,225,894,739$;

And $\frac{1}{m^5}$ $(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{169} = \frac{0.000,000,001,225,894,739}{169}) = 0.000,000,000,007,253,815$;

And

And $\frac{1}{m^6}$ $\left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{169} = \frac{0.000,000,000,007,253,815}{169}\right) = 0.000,000,000,000,042,921$;

And $\frac{1}{m^7}$ $\left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{169} = \frac{0.000,000,000,000,042,921}{169}\right) = 0.000,000,000,000,000,253$;

And $\frac{1}{m^8}$ $\left(= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{169} = \frac{0.000,000,000,000,000,253}{169}\right) = 0.000,000,000,000,000,001$;

And confequently

$\frac{1}{2m^2}$ $\left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,035,012,779,664,577}{2}\right) = 0.000,017,506,389,832,288$;

And $\frac{1}{3m^3}$ $\left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,207,176,211,033}{3}\right) = 0.000,000,069,058,737,011$;

And $\frac{1}{4m^4}$ $\left(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,001,225,894,739}{4}\right) = 0.000,000,000,306,473,684$;

And $\frac{1}{5m^5}$ $\left(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,007,253,815}{5}\right) = 0.000,000,000,001,450,763$;

And $\frac{1}{6m^6}$ $\left(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,042,921}{6}\right) = 0.000,000,000,000,007,153$;

And $\frac{1}{7m^7}$ $\left(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,253}{7}\right) = 0.000,000,000,000,000,036$;

And $\frac{1}{8m^8}$ $\left(= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,001}{8}\right) = 0.000,000,000,000,000,000.$

Therefore the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} +$ &c is =

0;005,917,159,763,313,609,
+ ;. . . ,. 17,506,389,832,288,
+ ;. . . ,. . . ,. 69,058,737,011,
+ ;. . . ,. . . ,. . . ,306,473,684,
+ ;. . . ,. . . ,. . . ,. . 1,450,763,
+ ;. . . ,. . . ,. . . ,. . . ,. . 7,153,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 36,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. . . ,
= 0.005,934,735,519,814,544.

Therefore this number 0.005,934,735,519,814,544, is the logarithm of the ratio of 1 to $1 - \frac{1}{169}$, or of 169 to 168. Q. E. I.

This number agrees with the value of the fame logarithm found above in article 29, by Mercator's feries, in all the figures except the laft, that value of it being 0.005,934,735,519,814,547.

EXAMPLE

EXAMPLE VII.

78. Let it be required to find by means of the fame feries the logarithm of the ratio of 289 (which is the fquare of 17) to 288, (or of the ratio of 289 to 289 − 1, or of $\frac{289}{289}$ to $\frac{289-1}{289}$) or of 1 to $1 - \frac{1}{289}$.

Here m is $= 289$, and $\frac{1}{m}$ is $= \frac{1}{289} = 0.003,460,207,612,456,747$, *&c.*

We fhall therefore have

$$\frac{1}{m^2}\ (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{289} = \frac{0.003,460,207,612,456,747}{289}) = 0.000,011,973,036,721,303;$$

And $\frac{1}{m^3}\ (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{289} = \frac{0.000,011,973,036,721,303}{289}) = 0.000,000,041,429,192,807;$

And $\frac{1}{m^4}\ (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{289} = \frac{0.000,000,041,429,192,807}{289}) = 0.000,000,000,143,353,608;$

And $\frac{1}{m^5}\ (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{289} = \frac{0.000,000,000,143,253,608}{289}) = 0.000,000,000,000,496,033;$

And $\frac{1}{m^6}\ (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{289} = \frac{0.000,000,000,000,496,033}{289}) = 0.000,000,000,000,001,716;$

And $\frac{1}{m^7}\ (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{289} = \frac{0.000,000,000,000,001,716}{289}) = 0.000,000,000,000,000,005;$

And confequently

$$\frac{1}{2m^2}\ (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,011,973,036,721,303}{2}) = 0.000,005,986,518,360,651;$$

And $\frac{1}{3m^3}\ (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,041,429,192,807}{3}) = 0.000,000,013,809,730,935;$

And $\frac{1}{4m^4}\ (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,143,353,608}{4}) = 0.000,000,000,035,838,402;$

And $\frac{1}{5m^5}\ (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,496,033}{5}) = 0.000,000,000,000,099,206;$

And $\frac{1}{6m^6}\ (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,001,716}{6}) = 0.000,000,000,000,000,286;$

And $\frac{1}{7m^7}\ (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,005}{7}) = 0.000,000,000,000,000,000.$

Therefore the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} +$ *&c* is in this cafe =

0;003,460,207,612,456,747,
\+ ;. . . ,. . 5,986,518,360,651,
\+ ;. . . ,. . . ,. 13,809,730,935,
\+ ;. . . ,. . . ,. . . ,. 35,838,402,
\+ ;. . . ,. . . ,. . . ,. . . ,. 99,206,
\+ ;. . . ,. . . ,. . . ,. . . ,. . . ,286,
\+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. . . ,
= 0;003,466,207,976,486,227.

Therefore this number 0.003,466,207,976,486,227, is the logarithm of the ratio of 1 to $1 - \frac{1}{289}$, or of 289 to 288. Q. E. I.

This number agrees with the value of the same logarithm found above in article 30, by Mercator's series, in all the figures except the last, that value being 0.003, 466,207,976,486,229.

EXAMPLE VIII.

79. Let it be required to find by means of the same series the logarithm of the ratio of 361 (which is the square of 19) to 360, (or of 361 to 361 − 1, or of $\frac{361}{361}$ to $\frac{361-1}{361}$) or of 1 to $1 - \frac{1}{361}$.

Here m is $= 361$, and $\frac{1}{m}$ is $= \frac{1}{361} = 0.002,770,083,102,493,074$.

We shall therefore have

$$\frac{1}{m^2} \left(= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{361} = \frac{0.002,770,083,102,493,074}{361}\right) = 0.000,007,673,360,394,717;$$

And $$\frac{1}{m^3} \left(= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{361} = \frac{0.000,007,673,360,394,717}{361}\right) = 0.000,000,021,255,845,968;$$

And $$\frac{1}{m^4} \left(= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{361} = \frac{0.000,000,021,255,845,968}{361}\right) = 0.000,000,000,058,880,459;$$

And $$\frac{1}{m^5} \left(= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{361} = \frac{0.000,000,000,058,880,459}{361}\right) = 0.000,000,000,000,163,103;$$

And $$\frac{1}{m^6} \left(= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{361} = \frac{0.000,000,000,000,163,103}{361}\right) = 0.000,000,000,000,000,451;$$

And $$\frac{1}{m^7} \left(= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{361} = \frac{0.000,000,000,000,000,451}{361}\right) = 0.000,000,000,000,000,001;$$

And

And confequently

$$\frac{1}{2m^2} \left(= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,007,673,360,394,717}{2}\right) = 0.000,003,836,680,197,358;$$

And $$\frac{1}{3m^3} \left(= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,021,255,845,968}{3}\right) = 0.000,000,007,085,281,989;$$

And $$\frac{1}{4m^4} \left(= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,058,880,459}{4}\right) = 0.000,000,000,014,720,114;$$

And $$\frac{1}{5m^5} \left(= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,163,103}{5}\right) = 0.000,000,000,000,032,620;$$

And $$\frac{1}{6m^6} \left(= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,451}{6}\right) = 0.000,000,000,000,000,075;$$

And $$\frac{1}{7m^7} \left(= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,001}{7}\right) = 0.000,000,000,000,000,000.$$

Therefore the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ &c is =

$$\begin{array}{r} 0;002,770,083,102,493,074, \\ + \; ;\ldots,\ldots 3,836,680,197,358, \\ + \; ;\ldots,\ldots,\ldots 7,085,281,989, \\ + \; ;\ldots,\ldots,\ldots,\ldots 14,720,114, \\ + \; ;\ldots,\ldots,\ldots,\ldots,\ldots 32,620, \\ + \; ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots 75, \\ \hline = 0.002,773,926,882,725,230. \end{array}$$

Therefore this number 0.002,773,926,882,725,230, is the logarithm of the ratio of 1 to $1 - \frac{1}{361}$, or of 361 to 360. Q. E. I.

This number, 0.002,773,926,882,725,230, agrees with the value of the logarithm of the fame ratio found above in article 31, by Mercator's feries, in all the figures except the laft, that value being 0.002,773,926,882,725,234.

EXAMPLE IX.

80. Laftly, let it be required to find by means of the fame feries the logarithm of the ratio of 529 (which is the fquare of 23) to 528, (or of 529 to 529 — 1, or of $\frac{529}{529}$ to $\frac{529-1}{529}$), or of 1 to $1 - \frac{1}{529}$.

Here m is = 529, and $\frac{1}{m}$ is = $\frac{1}{529}$ = 0.001,890,359,168,241,965.

We

We shall therefore have

$$\frac{1}{m^2}\ (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{529} = \frac{0.001,890,359,168,241,965}{529}) = 0.000,003,573,457,784,956\ ;$$

And $$\frac{1}{m^3}\ (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{529} = \frac{0.000,003,573,457,784,956}{529}) = 0.000,000,006,755,118,686\ ;$$

And $$\frac{1}{m^4}\ (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{529} = \frac{0.000,000,006,755,118,686}{529}) = 0.000,000,000,012,769,600\ ;$$

And $$\frac{1}{m^5}\ (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{529} = \frac{0.000,000,000,012,769,600}{529}) = 0.000,000,000,000,024,139\ ;$$

And $$\frac{1}{m^6}\ (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{529} = \frac{0.000,000,000,000,024,139}{529}) = 0.000,000,000,000,000,045\ ;$$

And $$\frac{1}{m^7}\ (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{529} = \frac{0.000,000,000,000,000,045}{529}) = 0.000,000,000,000,000,000\ ;$$

And consequently

$$\frac{1}{2m^2}\ (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,003,573,457,784,956}{2}) = 0.000,001,786,728,892,478\ ;$$

And $$\frac{1}{3m^3}\ (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,006,755,118,686}{3}) = 0.000,000,002,251,706,228\ ;$$

And $$\frac{1}{4m^4}\ (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,012,769,600}{4}) = 0.000,000,000,003,192,400\ ;$$

And $$\frac{1}{5m^5}\ (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,024,139}{5}) = 0.000,000,000,000,004,827\ ;$$

And $$\frac{1}{6m^6}\ (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,045}{6}) = 0.000,000,000,000,000,007.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ &c is, in this case, $=$

$$\begin{array}{rl} & 0;001,890,359,168,241,965, \\ + & ;\ldots,\ldots 1,786,728,892,478, \\ + & ;\ldots,\ldots,\ldots 2,251,706,228, \\ + & ;\ldots,\ldots,\ldots,\ldots 3,192,400, \\ + & ;\ldots,\ldots,\ldots,\ldots,\ldots 4,827, \\ + & ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots 7, \\ \hline = & 0.001,892,148,152,037,905. \end{array}$$

Therefore this number 0.001,892,148,152,037,905, is the logarithm of the ratio of 1 to $1 - \frac{1}{529}$ or of 529 to 528. Q. E. I.

This value of the logarithm of the ratio of 529 to 528, agrees with the value of it found above in article 32, by Mercator's series, in all the figures except the last, that value being 0.001,892,148,152,037,909.

81. As

81. As the value of thefe nine logarithms obtained in the preceding examples, by means of the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ *&c*, invented by Dr. Wallis, agree fo nearly with the values of them obtained above, by means of the feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ *&c*, invented by Mercator, there is no reafon to apprehend that any miftakes have been made in computing them; but we may juftly be confident that they are exact in all the places of figures to which they agree with each other. And thus thefe two feriefes ferve as checks upon each other, or as means of difcovering any miftakes that may have been made in computing logarithms by either of them, and of confirming, or proving the truth of, the calculations by which they have been obtained, when thofe calculations have been made without an error. And this is, indeed, the principal benefit that can be derived from the poffeffion of both thefe feriefes, rather than of only one of them, fince each of them feems juft as fit as the other for the purpofe of computing logarithms.

Of the Identity of the Logarithms of any given Ratios obtained by the Series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c ad infinitum, *which was invented by Mr. Mercator, with the Logarithms of the fame Ratios obtained by the Series* $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c ad infinitum, *which was invented by Dr. Wallis.*

82. We have feen in the foregoing examples of the computation of logarithms, by the two foregoing feriefes, that the logarithm of the fame ratio is always the fame number, whether it be computed by Mr. Mercator's feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c*, or by Dr. Wallis's feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c*. Thus, for example, the logarithm of the ratio of 10 to 9, obtained in article 24, by means of the feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c*, or $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ *&c*, is $= 0.105,360,515,657,826,302$; and the logarithm of the fame ratio of 10 to 9, obtained in article 72, by means of the feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c*, or $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ *&c*, is $= 0.105,360,515,657,826,296$; which agrees with the former logarithm of it, obtained in article 24, in all but the three laft figures. The logarithms, therefore, of two different ratios obtained by means of Dr. Wallis's feries, are not only proportional to the logarithms of the fame two ratios obtained by means of Mr. Mercator's feries, (as they neceffarily muft be, becaufe both pairs of logarithms are proportional to, or meafures of, the fame two ratios,) but they are equal

 to

to them; ſo that the two ſerieſes $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c*, and $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c*, exhibit the logarithms of the ratios of 1 to $1 - k$, and of $1 + k$ to 1, in one and the ſame ſyſtem, or, in other words, the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c, ad infinitum*, is to the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c, ad infinitum*, in the ſame proportion as the ratio of 1 to $1 - k$ is to the ratio of $1 + k$ to 1. Now "that this ſhould be ſo," is by no means ſelf-evident; but it may be demonſtrated in the manner following.

THEOREM III.

83. If k be any quantity leſs than 1, the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6}$ + *&c ad infinitum*, will be the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum*, in the ſame proportion as the ratio of 1 to $1 - k$ to the ratio of $1 + k$ to 1.

DEMONSTRATION.

Let q be any quantity leſs than k, and conſequently, *à fortiori*, leſs than 1.

Then will the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* be to the ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ *&c ad infinitum*, in the ſame proportion as the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$. This is demonſtrated in theorem 2, article 66.

Let q be extremely ſmall, as, for example, equal to $\frac{1}{\overline{1000,000}|^9}$, or the quotient of the diviſion of 1 by a nonillion, or by the ninth power of a million, that is, to 0.000000,000000,000000,000000,000000,000000,000000,000000,000001, or an unit in the 54th place of decimal fractions, then will the whole ſeries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ *&c ad infinitum* be very nearly equal to its firſt term q. Therefore in this caſe the ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6}$ *&c* will be to the ſingle term q very nearly in the ſame proportion as the ratio of 1 to $1 - k$ to the ratio of 1 to $1 - q$.

Further, by theorem 1, article 18, the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6}$ + *&c ad infinitum* is to the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad*

infinitum,

infinitum, in the fame proportion as the ratio of $1 + q$ to 1 to the ratio of $1 + k$ to 1. And when q is exrremely fmall, as, for example, equal to $\frac{1}{1000,000)^9}$, or 0.000000,000000,000000,000000,000000,000000,000000,000000,000001, or an unit in the 54th place of decimal fractions, the whole feries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c ad infinitum* is very nearly equal to its firft term q. Therefore in this cafe the fingle term q will be to the feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* very nearly in the fame proportion as the ratio of $1 + q$ to 1 to the ratio of $1 + k$ to 1.

Laftly, it follows, from lemma 1, article 11, that when q is extremely fmall, the ratio of $1 + q$ to 1 is very nearly equal to the ratio of 1 to $1 - q$.

Therefore, when q is extremely fmall, the fingle term q will be to the feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* very nearly in the fame proportion as the ratio of 1 to $1 - q$ to the ratio of $1 + k$ to 1.

But it has been before fhewn, that when q is extremely fmall, the feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* will be to the fingle term q very nearly in the fame proportion as the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$.

It follows therefore, *ex æquo*, that, when q is extremely fmall, the feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* will be to the feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* very nearly in the fame proportion as the ratio of 1 to $1 - k$ is to the ratio of $1 + k$ to 1.

And by taking q fufficiently fmall, we may prove that the proportion of the faid two feriefes to each other comes as near as we pleafe to an equality with the proportion of the faid two ratios to each other. Therefore the proportion of the faid two feriefes to each other muft be accurately equal to the proportion of the faid two ratios to each other; or the feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* muft be to the feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* in exactly the fame proportion as the ratio of 1 to $1 - k$ is to the ratio of $1 + k$ to 1. Q. E. D.

Of the Difference between Napier's Logarithms and Briggs's Logarithms.

84. The logarithms found by computing either of thefe feriefes are thofe which Lord Napier (the learned Scotch nobleman who invented logarithms) firft made ufe

ufe of; and, therefore, are often called *Napier's logarithms.* They are alfo often called *natural* logarithms; becaufe in moft of the methods of computing logarithms they occur more readily, as meafures of ratios, than any other numbers. They are alfo often called *hyperbolick* logarithms; becaufe they exprefs the magnitudes of the afymptotick areas of an hyperbola, when the fquare, or parallelogram, of the hyperbola is called 1; as it ufually is when mathematicians apply numbers to the menfuration of its areas. But if any other area, either greater or lefs than the fquare, or parallelogram of the hyperbola, fhould be denoted by 1, (which is entirely a matter of choice) thefe logarithms would no longer exprefs the magnitudes of the faid afymptotick areas: and therefore the name of *hyperbolick* feems to have been given them without fufficient reafon. If the afymptotick area of an hyperbola that correfponds to the ratio of 10 to 1, or that is bounded by two ordinates, of which the greater is equal to ten times the leffer, fhould be called 1, the numeral expreffions of the magnitudes of the afymptotick areas of fuch hyperbola would be the logarithms of Briggs's fyftem, or thofe which are now in common ufe. But it is evident, that the logarithms of the fame ratios in all fyftems muft be proportional to each other, becaufe they are meafures of, or proportional to, the fame quantities, to wit, the fame ratios. Thus, for example, the logarithm of the ratio of 10 to 1 in any fyftem of logarithms whatfoever, muft be to the logarithm of the ratio of 2 to 1 in the fame fyftem in the fame proportion as the logarithm of the former ratio found above in article 41 to the logarithm of the latter ratio found above in article 35, that is, as 2.302,585,092,994,045,668 to 0.693,147,180,559,945,304; becaufe in both fyftems the logarithms of the faid ratios are proportional to the ratios themfelves. And, *permutando*, the logarithm of any one ratio in one fyftem of logarithms, muft be to the logarithm of the fame ratio in any other fyftem in the fame proportion as the logarithm of any other ratio in the former fyftem to the logarithm of the faid other ratio in the latter fyftem; and confequently the logarithms of any one fyftem may be derived from thofe of any other fyftem by increafing or diminifhing them all in the fame proportion. In Briggs's fyftem of logarithms the logarithm of the ratio of 10 to 1 is an unit, or 1; and confequently that of the ratio of 100 to 1 (which is double of the ratio of 10 to 1) is 2, and that of the ratio of 1000 to 1 (which is triple of the ratio of 10 to 1) is 3, and fo on of the other powers of 10. The logarithms of this fyftem will therefore be lefs than the logarithms of the fame ratios in Napier's fyftem in the proportion of 1 (the logarithm of the ratio of 10 to 1, in Briggs's fyftem,) to 2.302,585,092,994,045,668, which is the logarithm of the ratio of 10 to 1 in Napier's fyftem. Therefore, in order to derive Briggs's logarithms from Napier's, we muft diminifh the latter in the proportion of 1 to 2.302,585,092,994,045,668, or divide them by 2.302,585,092,994,045,668, or (which comes to the fame thing) multiply them by the fraction $\frac{1}{2.302,585,092,994.045,668}$, or its equal, the decimal fraction 0.434,294,481,903,251,830.

85. The

85. The divifion of 1 by the long number 2.302,585,092,994,045,668, in order to obtain the decimal fraction 0.434,294,481,903,251,830, is a troublefome and tedious operation, but a very important one in the bufinefs of computing logarithms: I fhall therefore, for the reader's more complete fatisfaction, fet it down at full length; and, in order to avoid miftakes in the calculation in the multiplications of the long divifor 2.302,585,092,994,045,668 by the feveral figures in the quotient, I fhall raife the feveral products of the multiplication of this divifor into the numbers 2, 3, 4, 5, 6, 7, 8, and 9, (which are all the figures that can occur in the quotient) by addition inftead of multiplication, before I begin to perform the operation of divifion; and then, when I come to the operation of divifion, I fhall make ufe of thefe products that have been thus previoufly found by the fimple operations of adding one number to another, in which it is not likely we fhould make any miftake. Thefe additions are as follows:

$$
\begin{array}{r}
2.302,585,092,994,045,668, \\
\underline{2.302,585,092,994,045,668,} \\
4.605,170,185,988,091,336, \\
\underline{2.302,585,092,994,045,668,} \\
6.907,755,278,982,137,004, \\
\underline{2.302,585,092,994,045,668,} \\
9.210,340,371,976,182,672, \\
\underline{2.302,585,092,994,045,668,} \\
11.512,925,464,970,228,340, \\
\underline{2.302,585,092,994,045,668,} \\
13.815,510,557,964,274,008, \\
\underline{2.302,585,092,994,045,668,} \\
16.118,095,650,958,319,676, \\
\underline{2.302,585,092,994,045,668,} \\
18.420,680,743,952,365,344, \\
\underline{2.302,585,092,994,045,668,} \\
20.723,265,836,946,411,012.
\end{array}
$$

It appears therefore that

$2 \times 2.302,585,092,994,045,668$ is $=$ 4.605,170,185,988,091,336;
And that $3 \times 2,302,585,092,994,045,668$ is $=$ 6.907,755,278,982,137,004;
And that $4 \times 2.302,585,092,994,045,668$ is $=$ 9.210,340,371,976,182,672;
And that $5 \times 2.302,585,092,994,045,668$ is $=$ 11.512,925,464,970,228,340;
And that $6 \times 2.302,585,092,994,045,668$ is $=$ 13.815,510,557,964,274,008;
And that $7 \times 2.302,585,092,994,045,668$ is $=$ 16.118,095,650,958,319,676;
And that $8 \times 2.302,585,092,994,045,668$ is $=$ 18.420,680,743,952,365,344;
And that $9 \times 2.302,585,092,994,045,668$ is $=$ 20.723,265,836,946,411,012.

Having thus found the products of the multiplication of the long number 2.302,585,092,994,045,668 into the feveral numbers 2, 3, 4, 5, 6, 7, 8, and

and 9, in a manner that is hardly liable to error, we may venture to proceed to the grand operation of dividing 1 by the ſaid long number; which is as follows:

2.302,585,092,994,045,668.) (0.434,294,481,903,251,830.

1.000,000,000,000,000,000,000,000,000,000,000,000,
9210340371976182672
.7896596280238173280
6907755278982137004
.9888410012560362760
9210340371976182672
.6780696405841800880
4605170185988091336
21755262198537095440
2072326583694641101 2
.1031996361590684428 0
9210340371976182672
.1109623243930661608 0
9210340371976182672
.1885892067330433408 0
1842068074395236534 4
..4382399293519687360
2302585092994045668
2079814200525641692 0
2072326583694641101 2
...7487616831000590800
6907755278982137004
.5798615520184537960
4605170185988091336
1193445334196446624 0
1151292546497022834 0
..4215278769942379000
2302585092994045668
1912693676948333332 0
1842068074395236534 4
..7062560255309679760
6907755278982137004
.1548049763275427560

We may conclude therefore that 0.434,294,481,903,251,830 is the decimal fraction by which Napier's logarithm of any ratio muſt be multiplied, in order to produce Briggs's logarithm of the ſame ratio.

N. B.

N. B. This number is fomewhat greater than the truth; the more accurate value of the decimal fraction that is equal to $\frac{1}{2.302,585,092,994,045,668}$, *&c*, being 0.434,294,481,903,251,827,651,128.

Of the Computation of Briggs's Logarithms, by Means of the Two foregoing Infinite Seriefes of Mr. Mercator and Dr. Wallis.

86. Whenever it is required to derive the logarithm of any given ratio in Briggs's fyftem from the logarithm of the fame ratio in Napier's fyftem, the moft direct way of obtaining it will be, either to divide the latter logarithm (which is fuppofed to be already known) by the long number 2.302,585,092, 994,045,668, or to multiply it into the decimal fraction 0.434,294,481,903, 251,830; and of thefe operations I conceive the latter, to wit, the multiplication by the decimal fraction 0.434,294,481,903,251,830, to be rather the lefs laborious. But if we are defirous of computing a whole table of Briggs's logarithms, or a very great number of them, the labour of performing either of thefe operations for every different logarithm we wanted to compute would be found almoft intolerable. In fuch a cafe, therefore, it would be adviseable to perform one of thefe operations only with refpect to the few original logarithms of fmall ratios, which are computed by means of either Mercator's feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ *&c ad infinitum*, or Dr. Wallis's feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ *&c ad infinitum*, and from which the logarithms of greater ratios are afterwards derived by the eafy operations of addition and fubtraction, or by multiplication or divifion by the fmall numbers 2, 3, or 4. Thus, for example, if we wanted to compute the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, according to Briggs's fyftem, it would be moft convenient to perform this operation of multiplication by the decimal fraction 0.434,294,481,903,251,830, only with refpect to the logarithms of the fmall original ratios of 10 to 9, of 11 to 10, of 81 to 80, of 121 to 120, of 2401 to 2400, of 169 to 168, of 289 to 288, of 361 to 360, and of 529 to 528; after which we might derive the logarithms of the greater ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, from the faid leffer logarithms by addition or fubtraction, or by multiplication and divifion of them by the fmall numbers 2, 3, and 4, in the manner above fet forth. And even in multiplying thofe fmall original logarithms into the faid long decimal fraction, it will be of importance to feek out the eafieft way of proceeding, in order to leffen the labour of fo tedious an operation. Now there are two ways of multiplying

tiplying one of thefe original logarithms reprefented by Mercator's feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6}$ + *&c*, or Dr. Wallis's feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6}$ + *&c*, into the decimal fraction 0.434,294,481,903,251,830. For we may either compute the value of the faid feries to the intended degree of exactnefs, and then multiply the value fo found into the faid decimal fraction, or we may multiply each of the terms of the faid feries into the faid decimal fraction at the time we are computing them. By proceeding in the former method, the logarithm fought will be = 0.434,294,481,903,251,830 × the feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6}$ + *&c*, or 0.434,294,481,903,251,830 × the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6}$ + *&c*; or if, for the fake of brevity, we denote the faid long multiplier 0.434,294,481,903,251,830 by the capital letter M, the logarithm fought will be = M × the feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6}$ + *&c*, or M × the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6}$ + *&c*, according as Mercator's or Doctor Wallis's feries is made ufe of; and by proceeding in the latter method, the logarithm fought will be = to the feries $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6}$ + *&c*, or to the feries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6}$ + *&c*; of which feriefes it is evident that the former, to wit, $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6}$ + *&c*, is = M × the feries $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6}$ + *&c*, and the latter, to wit, $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6}$ + *&c*, is = M × the feries $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6}$ + *&c*. But the latter method of proceeding (by which the logarithm fought is derived from the feries $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6}$ + *&c*, or the feries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6}$ + *&c*) is much the more convenient of the two for the purpofes of calculation. In order, therefore, to find Briggs's logarithms of the ratios of the feveral numbers 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, to 1, it would be advifeable to compute the logarithms of the feveral leffer ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, 2401 to 2400, 169 to 168, 289 to 288, 361 to 360, and 529 to 528, by means of one of the two feriefes $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6}$ + *&c*, and $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6}$ + *&c*. This I fhall now proceed to do by means of the latter of thefe two feriefes, to wit, the feries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6}$ + *&c ad infinitum.*

EXAMPLES

OF THE

COMPUTATION OF BRIGGS's LOGARITHMS

OF THE

RATIOS OF 1 TO SEVERAL FRACTIONAL NUMBERS,

OR NUMBERS LESS THAN 1,

Denoted by the residual quantity $1 - \frac{1}{m}$,

by means of the infinite series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \&c$, which is derived from the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \&c$, (invented by Dr. Wallis,)

by multiplying the several terms into the decimal fraction M, or 0.434,294, 481,903,251,830.

EXAMPLE I.

87. LET it be required to find by means of the said series the logarithm of the ratio of 10 to 9, (or of 10 to 10 — 1, or of $\frac{10}{10}$ to $\frac{10-1}{10}$,) or of 1 to $1 - \frac{1}{10}$.

Here m is = 10, and consequently $\frac{M}{m}$ is $= \frac{M}{10} = \frac{0.434,294,481,903,251,830}{10}$ = 0.043,429,448,190,325,183.

We

We fhall therefore have

$$\frac{M}{m^2} \left(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{10} = \frac{0.043,429,448,190,325,183}{10}\right) = 0.004,342,944,819,032,518;$$

And $$\frac{M}{m^3} \left(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{10} = \frac{0.004,342,944,819,032,518}{10}\right) = 0.000,434,294,481,903,251;$$

And $$\frac{M}{m^4} \left(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{10} = \frac{0.000,434,294,481,903,251}{10}\right) = 0.000,043,429,448,190,325;$$

And $$\frac{M}{m^5} \left(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{10} = \frac{0.000,043,429,448,190,325}{10}\right) = 0.000,004,342,944,819,032;$$

And $$\frac{M}{m^6} \left(= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{10} = \frac{0.000,004,342,944,819,032}{10}\right) = 0.000,000,434,294,481,903;$$

And $$\frac{M}{m^7} \left(= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{10} = \frac{0.000,000,434,294,481,903}{10}\right) = 0.000,000,043,429,448,190;$$

And $$\frac{M}{m^8} \left(= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{10} = \frac{0.000,000,043,429,448,190}{10}\right) = 0.000,000,004,342,944,819;$$

And $$\frac{M}{m^9} \left(= \frac{M}{m^8} \times \frac{1}{m} = \frac{M}{m^8} \times \frac{1}{10} = \frac{0.000,000,004,342,944,819}{10}\right) = 0.000,000,000,434,294,481;$$

And $$\frac{M}{m^{10}} \left(= \frac{M}{m^9} \times \frac{1}{m} = \frac{M}{m^9} \times \frac{1}{10} = \frac{0.000,000,000,434,294,481}{10}\right) = 0.000,000,000,043,429,448;$$

And $$\frac{M}{m^{11}} \left(= \frac{M}{m^{10}} \times \frac{1}{m} = \frac{M}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,043,429,448}{10}\right) = 0.000,000,000,004,342,944;$$

And $$\frac{M}{m^{12}} \left(= \frac{M}{m^{11}} \times \frac{1}{m} = \frac{M}{m^{11}} \times \frac{1}{10} = \frac{0.000,000,000,004,342,944}{10}\right) = 0.000,000,000,000,434,294;$$

And $$\frac{M}{m^{13}} \left(= \frac{M}{m^{12}} \times \frac{1}{m} = \frac{M}{m^{12}} \times \frac{1}{10} = \frac{0.000,000,000,000,434,294}{10}\right) = 0.000,000,000,000,043,429;$$

And $$\frac{M}{m^{14}} \left(= \frac{M}{m^{13}} \times \frac{1}{m} = \frac{M}{m^{13}} \times \frac{1}{10} = \frac{0.000,000,000,000,043,429}{10}\right) = 0.000,000,000,000,004,342;$$

And $$\frac{M}{m^{15}} \left(= \frac{M}{m^{14}} \times \frac{1}{m} = \frac{M}{m^{14}} \times \frac{1}{10} = \frac{0.000,000,000,000,004,342}{10}\right) = 0.000,000,000,000,000,434;$$

And $$\frac{M}{m^{16}} \left(= \frac{M}{m^{15}} \times \frac{1}{m} = \frac{M}{m^{15}} \times \frac{1}{10} = \frac{0.000,000,000,000,000,434}{10}\right) = 0.000,000,000,000,000,043;$$

And $$\frac{M}{m^{17}} \left(= \frac{M}{m^{16}} \times \frac{1}{m} = \frac{M}{m^{16}} \times \frac{1}{10} = \frac{0.000,000,000,000,000,043}{10}\right) = 0.000,000,000,000,000,004;$$

And confequently

$$\frac{M}{2m^2} \left(= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.004,342,944,819,032,518}{2}\right) = 0.002,171,472,409,516,259;$$

And $$\frac{M}{3m^3} \left(= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,434,294,481,903,251}{3}\right) = 0.000,144,764,827,301,083;$$

And $$\frac{M}{4m^4} \left(= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,043,429,448,190,325}{4}\right) = 0.000,010,857,362,047,581;$$

 And

And $\frac{M}{5m^5}$ $\left(= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,004,342,944,819,032}{5}\right) = 0.000,000,868,588,963,806$;

And $\frac{M}{6m^6}$ $\left(= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,434,294,481,903}{6}\right) = 0.000,000,072,382,413,650$;

And $\frac{M}{7m^7}$ $\left(= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,043,429,448,190}{7}\right) = 0.000,000,006,204,206,884$;

And $\frac{M}{8m^8}$ $\left(= \frac{M}{m^8} \times \frac{1}{8} = \frac{0.000,000,004,342,944,819}{8}\right) = 0.000,000,000,542,868,102$;

And $\frac{M}{9m^9}$ $\left(= \frac{M}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,434,294,481}{9}\right) = 0.000,000,000,048,254,942$;

And $\frac{M}{10m^{10}}$ $\left(= \frac{M}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,043,429,448}{10}\right) = 0.000,000,000,004,342,944$;

And $\frac{M}{11m^{11}}$ $\left(= \frac{M}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,004,342,944}{11}\right) = 0.000,000,000,000,394,813$;

And $\frac{M}{12m^{12}}$ $\left(= \frac{M}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,000,434,294}{12}\right) = 0.000,000,000,000,036,191$;

And $\frac{M}{13m^{13}}$ $\left(= \frac{M}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,043,429}{13}\right) = 0.000,000,000,000,003,340$;

And $\frac{M}{14m^{14}}$ $\left(= \frac{M}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,004,342}{14}\right) = 0.000,000,000,000,000,310$

And $\frac{M}{15m^{15}}$ $\left(= \frac{M}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,000,434}{15}\right) = 0.000,000,000,000,000,028$;

And $\frac{M}{16m^{16}}$ $\left(= \frac{M}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,043}{16}\right) = 0.000,000,000,000,000,002$;

And $\frac{M}{17m^{17}}$ $\left(= \frac{M}{m^{17}} \times \frac{1}{17} = \frac{0.000,000,000,000,000,004}{17}\right) = 0.000,000,000,000,000,000.$

Therefore the ſeries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \frac{M}{8m^8} + \frac{M}{9m^9} + \frac{M}{10m^{10}} + \frac{M}{11m^{11}} + \frac{M}{12m^{12}} + \frac{M}{13m^{13}} + \frac{M}{14m^{14}} + \frac{M}{15m^{15}} + \frac{M}{16m^{16}} + \frac{M}{17m^{17}} +$ &c is =

0;043,429,448,190,325,183,
+ ;. . 2,171,472,409,516,259,
+ ;. . . ,144,764,827,301,083,
+ ;. . . ,. 10,857,362,047,581,
+ ;. . . ,. . . ,868,588,963,806,
+ ;. . . ,. . . ,. 72,382,413,650,
+ ;. . . ,. . . ,. . 6,204,206,884,
+ ;. . . ,. . . ,. . . ,542,868,102,
+ ;. . . ,. . . ,. . . ,. 48,254,942,
+ ;. . . ,. . . ,. . . ,. . 4,342,944,
+ ;. . . ,. . . ,. . . ,. . . ,394,813,
+ ;. . . ,. . . ,. . . ,. . . ,. 36,191,
+ ;. . . ,. . . ,. . . ,. . . ,. . 3,340,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,310,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 28,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. . 2,

= 0.045,757,490,560,675,118.

Therefore

Therefore this number 0.045,757,490,560,675,118 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{10}$, or of 10 to 9. Q. E. I.

This value of the logarithm of the ratio of 10 to 9 is exact in all the figures, except the two last, which ought to be 25 instead of 18; the more exact value of this logarithm being 0.045,757,490,560,675,125,410, as will appear by subtracting Briggs's logarithm of 9, as computed by Mr. Abraham Sharp, to wit, 0.954,242, 509,439,324,874,590, *&c*, from Briggs's logarithm of 10, to wit, 1.000,000,000, 000,000,000,000.

EXAMPLE II.

88. Let it be required to find by means of the same series Briggs's logarithm of the ratio of 11 to 10, (or of 11 to 11 — 1, or of $\frac{11}{11}$ to $\frac{11-1}{11}$,) or of 1 to $1 - \frac{1}{11}$.

Here m is $= 11$, and $\frac{1}{m}$ is $= \frac{1}{11}$, and consequently $\frac{M}{m}$ is $(= M \times \frac{1}{11} = \frac{M}{11} = \frac{0.434,294,481,903,251,830}{11}) = 0.039,481,316,536,659,257$.

We shall therefore have

$$\frac{M}{m^2} \left(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{11} = \frac{0.039,481,316.536,659,257}{11}\right) = 0.003,589,210,594,241,750;$$

And $$\frac{M}{m^3} \left(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{11} = \frac{0.003,589,210,594,241,750}{11}\right) = 0.000,326,291,872,203,795;$$

And $$\frac{M}{m^4} \left(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{11} = \frac{0.000,326,291,872,203,795}{11}\right) = 0.000,029,662,897,473,072;$$

And $$\frac{M}{m^5} \left(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{11} = \frac{0.000,029,662,897,473,072}{11}\right) = 0.000,002,696,627,043,006;$$

And $$\frac{M}{m^6} \left(= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{11} = \frac{0.000,002,696,627,043,006}{11}\right) = 0.000,000,245,147,913,000;$$

And $$\frac{M}{m^7} \left(= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{11} = \frac{0.000,000,245,147,913,000}{11}\right) = 0.000,000,022,286,173,909;$$

And $$\frac{M}{m^8} \left(= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{11} = \frac{0.000,000,022,286,173,909}{11}\right) = 0.000,000,002,026,015,809;$$

And $$\frac{M}{m^9} \left(= \frac{M}{m^8} \times \frac{1}{m} = \frac{M}{m^8} \times \frac{1}{11} = \frac{0.000,000,002,026,015,809}{11}\right) = 0.000,000,000,184,183,255;$$

And $$\frac{M}{m^{10}} \left(= \frac{M}{m^9} \times \frac{1}{m} = \frac{M}{m^9} \times \frac{1}{11} = \frac{0.000,000,000,184,183,255}{11}\right) = 0.000,000,000,016,743,932;$$

And $$\frac{M}{m^{11}} \left(= \frac{M}{m^{10}} \times \frac{1}{m} = \frac{M}{m^{10}} \times \frac{1}{11} = \frac{0.000,000,000,016,743,932}{11}\right) = 0.000,000,000,001,522,175;$$

And $$\frac{M}{m^{12}} \left(= \frac{M}{m^{11}} \times \frac{1}{m} = \frac{M}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,001,522,175}{11}\right) = 0.000,000,000,000,138,379;$$

And

And $\frac{M}{m^{13}}(=\frac{M}{m^{12}}\times\frac{1}{m}=\frac{M}{m^{12}}\times\frac{1}{11}=\frac{0.000,000,000,000,138,379}{11})=0.000,000,000,000,012,579$;

And $\frac{M}{m^{14}}(=\frac{M}{m^{13}}\times\frac{1}{m}=\frac{M}{m^{13}}\times\frac{1}{11}=\frac{0.000,000,000,000,012,579}{11})=0.000,000,000,000,001,143$;

And $\frac{M}{m^{15}}(=\frac{M}{m^{14}}\times\frac{1}{m}=\frac{M}{m^{14}}\times\frac{1}{11}=\frac{0.000,000,000,000,001,143}{11})=0.000,000,000,000,000,103$;

And $\frac{M}{m^{16}}(=\frac{M}{m^{15}}\times\frac{1}{m}=\frac{M}{m^{15}}\times\frac{1}{11}=\frac{0.000,000,000,000,000,103}{11})=0.000,000,000,000,000,009$;

And confequently

$\frac{M}{2m^2}(=\frac{M}{m^2}\times\frac{1}{2}=\frac{0.003,589,210,594,241,750}{2})=0.001,794,605,297,120,875$;

And $\frac{M}{3m^3}(=\frac{M}{m^3}\times\frac{1}{3}=\frac{0.000,326,291,872,203,795}{3})=0.000,108,763,957,401,265$;

And $\frac{M}{4m^4}(=\frac{M}{m^4}\times\frac{1}{4}=\frac{0.000,029,662,897,473,072}{4})=0.000,007,415,724,368,268$;

And $\frac{M}{5m^5}(=\frac{M}{m^5}\times\frac{1}{5}=\frac{0.000,002,696,627,043,006}{5})=0.000,000,539,325,408,601$;

And $\frac{M}{6m^6}(=\frac{M}{m^6}\times\frac{1}{6}=\frac{0.000,000,245,147,913,000}{6})=0.000,000,040,857,985,500$;

And $\frac{M}{7m^7}(=\frac{M}{m^7}\times\frac{1}{7}=\frac{0.000,000,022,286,173,909}{7})=0.000,000,003,183,739,129$;

And $\frac{M}{8m^8}(=\frac{M}{m^8}\times\frac{1}{8}=\frac{0.000,000,002,026,015,809}{8})=0.000,000,000,253,251,976$;

And $\frac{M}{9m^9}(=\frac{M}{m^9}\times\frac{1}{9}=\frac{0.000,000,000,184,183,255}{9})=0.000,000,000,020,464,806$;

And $\frac{M}{10m^{10}}(=\frac{M}{m^{10}}\times\frac{1}{10}=\frac{0.000,000,000,016,743,932}{10})=0.000,000,000,001,674,393$;

And $\frac{M}{11m^{11}}(=\frac{M}{m^{11}}\times\frac{1}{11}=\frac{0.000,000,000,001,522,175}{11})=0.000,000,000,000,138,379$;

And $\frac{M}{12m^{12}}(=\frac{M}{m^{12}}\times\frac{1}{12}=\frac{0.000,000,000,000,138,379}{12})=0.000,000,000,000,011,531$;

And $\frac{M}{13m^{13}}(=\frac{M}{m^{13}}\times\frac{1}{13}=\frac{0.000,000,000,000,012,579}{13})=0.000,000,000,000,000,967$;

And $\frac{M}{14m^{14}}(=\frac{M}{m^{14}}\times\frac{1}{14}=\frac{0.000,000,000,000,001,143}{14})=0.000,000,000,000,000,081$;

And $\frac{M}{15m^{15}}(=\frac{M}{m^{15}}\times\frac{1}{15}=\frac{0.000,000,000,000,000,103}{15})=0.000,000,000,000,000,006$.

Therefore the feries $\frac{M}{m}+\frac{M}{2m^2}+\frac{M}{3m^3}+\frac{M}{4m^4}+\frac{M}{5m^5}+\frac{M}{6m^6}+\frac{M}{7m^7}+\frac{M}{8m^8}+\frac{M}{9m^9}+\frac{M}{10m^{10}}+\frac{M}{11m^{11}}+\frac{M}{12m^{12}}+\frac{M}{13m^{13}}+\frac{M}{14m^{14}}+\frac{M}{15m^{15}}+$ &c is =

0;039,481,316,536,659,257,
+ ;. . 1,794,605,297,120,875,
+ ;. . . ,108,763,957,401,265,
+ ;. . . ,. . 7,415,724,368,268,
+ ;. . . ,. . . ,539,325,408,601,
+ ;. . . ,. . . ,. 40,857,985,500,
+ ;. . . ,. . . ,. . 3,183,739,129,
+ ;. . . ,. . . ,. . . ,253,251,976,
+ ;. . . ,. . . ,. . . ,. 20,464,806,
+ ;. . . ,. . . ,. . . ,. . 1,674,393,
+ ;. . . ,. . . ,. . . ,. . . ,138,379,
+ ;. . . ,. . . ,. . . ,. . . ,. 11,531,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,967,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 81,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. . 6,
= 0.041,392,685,158,225,034.

Therefore this number 0.041,392,685,158,225,034, is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{11}$, or of 11 to 10. Q. E. I.

This value of the logarithm of the ratio of 11 to 10 is exact in all the figures except the two last, its more exact value being 0.041,392,685,158,225, 040,750.

EXAMPLE III.

89. Let it be required to find by means of the same series Briggs's logarithm of the ratio of 81 to 80, (or of 81 to 81 − 1, or of $\frac{81}{81}$ to $\frac{81-1}{81}$,) or of 1 to $1 - \frac{1}{81}$.

Here m is $= 81$, and $\frac{1}{m}$ is $= \frac{1}{81}$, and consequently $\frac{M}{m}$ is $(= M \times \frac{1}{m} = M \times \frac{1}{81} = \frac{M}{81} = \frac{0.434,294,481,903,251,830}{81}) = 0.005,361,660,270,410,516$.

We shall therefore have,

$\frac{M}{m^2}$ $(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{81} = \frac{0.005,361,660,270,410,516}{81}) = 0.000,066,193,336,671,734$;

And $\frac{M}{m^3}$ $(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{81} = \frac{0.000,066,193,336,671,734}{81}) = 0.000,000,817,201,687,305$;

And $\frac{M}{m^4}$ $(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{81} = \frac{0.000,000,817,201,687,305}{81}) = 0.000,000,010,088,909,719$;

And $\frac{M}{m^5}$ $(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{81} = \frac{0.000,000,010,088,909,719}{81}) = 0.000,000,000,124,554,440$;

And $\frac{M}{m^6}$ $(= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{81} = \frac{0.000,000,000,124,554,440}{81}) = 0.000,000,000,001,537,709$;

And $\frac{M}{m^7}$ $\left(= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{81} = \frac{0.000,000,000,001,537,709}{81}\right) = 0.000,000,000,000,018,984$;

And $\frac{M}{m^8}$ $\left(= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{81} = \frac{0.000,000,000,000,018,984}{81}\right) = 0.000,000,000,000,000,234$;

And $\frac{M}{m^9}$ $\left(= \frac{M}{m^8} \times \frac{1}{m} = \frac{M}{m^8} \times \frac{1}{81} = \frac{0.000,000,000,000,000,234}{81}\right) = 0.000,000,000,000,000,002$;

And confequently

$$\frac{M}{2m^2} \left(= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,066,193,336,671,734}{2}\right) = 0.000,033,096,668,335,867;$$

And $\frac{M}{3m^3}$ $\left(= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,817,201,687,305}{3}\right) = 0.000,000,272,400,562,435$;

And $\frac{M}{4m^4}$ $\left(= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,010,088,909,719}{4}\right) = 0.000,000,002,522,227,429$;

And $\frac{M}{5m^5}$ $\left(= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,124,554,440}{5}\right) = 0.000,000,000,024,910,888$;

And $\frac{M}{6m^6}$ $\left(= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,001,537,709}{6}\right) = 0.000,000,000,000,256,284$;

And $\frac{M}{7m^7}$ $\left(= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,018,984}{7}\right) = 0.000,000,000,000,002,712$;

And $\frac{M}{8m^8}$ $\left(= \frac{M}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,234}{8}\right) = 0.000,000,000,000,000,029$;

And $\frac{M}{9m^9}$ $\left(= \frac{M}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,000,000,002}{9}\right) = 0.000,000,000,000,000,000$.

Therefore the feries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \frac{M}{8m^8} + \&c$ is =

0;005,361,660,270,410,516,
\+ ;. . . ,. 33,096,668,335,867,
\+ ;. . . ,. . . ,272,400,562,435,
\+ ;. . . ,. . . ,. . 2,522,227,429,
\+ ;. . . ,. . . ,. . . ,. 24,910,888,
\+ ;. . . ,. . . ,. . . ,. . . ,256,284,
\+ ;. . . ,. . . ,. . . ,. . . ,. . 2,712,
\+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 29,
\+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. . 0,
= 0.005,395,031,886,706,160.

Therefore this number 0.005,395,031,886,706,160 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{81}$, or of 81 to 80. Q. E. I.

This value of the logarithm of the ratio of 81 to 80 is exact in all but the laft place of figures, in which there ought to be a 3 inftead of a cypher; the more exact value of this logarithm being the excefs of Briggs's logarithm of the ratio of 80 to 1, above

above Briggs's logarithm of the ratio of 80 to 1, that is, (according to Mr. Abraham Sharp's computation of thofe logarithms,) to the excefs of 1.908, 485,018,878,649,749,180, above 1.903,089,986,991,943,585,641, or to 0.005, 395,031,886,706,163,539.

EXAMPLE IV.

90. Let it be required to find by means of the fame feries Briggs's logarithm of the ratio of 121 to 120, (or of 121 to 121 − 1, or of $\frac{121}{121}$ to $\frac{121-1}{121}$) or of 1 to $1 - \frac{1}{121}$.

Here m is $= 121$, and $\frac{1}{m}$ is $= \frac{1}{121}$, and confequently $\frac{\text{M}}{m}$ is $(= \text{M} \times \frac{1}{m} = \text{M} \times \frac{1}{121} = \frac{\text{M}}{121} = \frac{0.434,294,481,903,251,830}{121}) = 0.003,589,210,594,241,750$.

We fhall therefore have

$$\frac{\text{M}}{m^2} (= \frac{\text{M}}{m} \times \frac{1}{m} = \frac{\text{M}}{m} \times \frac{1}{121} = \frac{0.003,589,210,594,241,750}{121}) = 0.000,029,662,897,473,072;$$

And $$\frac{\text{M}}{m^3} (= \frac{\text{M}}{m^2} \times \frac{1}{m} = \frac{\text{M}}{m^2} \times \frac{1}{121} = \frac{0.000,029,662,897,473,072}{121}) = 0.000,000,245,147,913,000;$$

And $$\frac{\text{M}}{m^4} (= \frac{\text{M}}{m^3} \times \frac{1}{m} = \frac{\text{M}}{m^3} \times \frac{1}{121} = \frac{0.000,000,245,147,913,000}{121}) = 0.000,000,002,026,015,809;$$

And $$\frac{\text{M}}{m^5} (= \frac{\text{M}}{m^4} \times \frac{1}{m} = \frac{\text{M}}{m^4} \times \frac{1}{121} = \frac{0.000,000,002,026,015,809}{121}) = 0.000,000,000,016,743,932;$$

And $$\frac{\text{M}}{m^6} (= \frac{\text{M}}{m^5} \times \frac{1}{m} = \frac{\text{M}}{m^5} \times \frac{1}{121} = \frac{0.000,000,000,016,743,932}{121}) = 0.000,000,000,000,138,379;$$

And $$\frac{\text{M}}{m^7} (= \frac{\text{M}}{m^6} \times \frac{1}{m} = \frac{\text{M}}{m^6} \times \frac{1}{121} = \frac{0.000,000,000,000,138,379}{121}) = 0.000,000,000,000,001,143;$$

And $$\frac{\text{M}}{m^8} (= \frac{\text{M}}{m^7} \times \frac{1}{m} = \frac{\text{M}}{m^7} \times \frac{1}{121} = \frac{0.000,000,000,000,001,143}{121}) = 0.000,000,000,000,000,009;$$

And confequently

$$\frac{\text{M}}{2m^2} (= \frac{\text{M}}{m^2} \times \frac{1}{2} = \frac{0.000,029,662,897,473,072}{2}) = 0.000,014,831,448,736,536;$$

And $$\frac{\text{M}}{3m^3} (= \frac{\text{M}}{m^3} \times \frac{1}{3} = \frac{0.000,000,245,147,913,000}{3}) = 0.000,000,081,715,971,000;$$

And $$\frac{\text{M}}{4m^4} (= \frac{\text{M}}{m^4} \times \frac{1}{4} = \frac{0.000,000,002,026,015,809}{4}) = 0.000,000,000,506,503,952;$$

And $$\frac{\text{M}}{5m^5} (= \frac{\text{M}}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,016,743,932}{5}) = 0.000,000,000,003,348,786;$$

And

And $\frac{M}{6m^6}$ $(= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,138,379}{6}) = 0.000,000,000,000,023,063$;

And $\frac{M}{7m^7}$ $(= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,001,143}{7}) = 0.000,000,000,000,000,163$;

And $\frac{M}{8m^8}$ $(= \frac{M}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,009}{8}) = 0.000,000,000,000,000,001$.

Therefore the feries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \frac{M}{8m^8} +$ &c will be =

0;003,589,210,594,241,750,
+ ;...,. 14,831,448,736,536,
+ ;...,...,. 81,715,971,000,
+ ;...,...,...,506,503,952,
+ ;...,...,...,.. 3,348,786,
+ ;...,...,...,...,. 23,063,
+ ;...,...,...,...,...,163,
+ ;...,...,...,...,...,.. 1,
= 0.003,604,124,268,825,251.

Therefore this number 0.003,604,124,268,825,251, is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{121}$, or of 121 to 120. Q. E. I.

This value of the logarithm of the ratio of 121 to 120 is exact in all the figures except the laft, which ought to be a 3 inftead of an unit; for the more accurate value of this logarithm is 0.003,604,124,268,825,253,778, as will appear from fubtracting the logarithm of the ratio of 120 to 1 from the logarithm of the ratio of 121 to 1; which fubtraction may be performed as follows. The logarithm of 11 is (according to Mr. Sharp's computation) 1.041,392,685,158,225,040,750; the double of which, to wit, 2.082,785,370,316,450,081,500, is the logarithm of 121, which is the fquare of 11. The logarithm of 12 is (according to Mr. Sharp's computation) = 1.079,181,246,047,624,827,722; and confequently the logarithm of 10 times 12, or 120, will be = 1 + 1.079,181,246,047,624,827,722, or 2.079, 181,246,047,624,827,722; which being fubtracted from 2.082,785,370,316,450, 081,500, leaves 0.003,604,124,268,825,253,778 for the logarithm of the ratio of 121 to 120.

EXAMPLE V.

91. Let it be required to find by means of the fame feries the logarithm of the ratio of 2401 to 2400, (or of 2401 to 2401 − 1, or of $\frac{2401}{2401}$ to $\frac{2401 - 1}{2401}$,) or of 1 to $1 - \frac{1}{2401}$.

Here

Here m is $= 2401$, and $\frac{1}{m}$ is $= \frac{1}{2401}$, and $\frac{M}{m}$ is $(= \frac{M}{2401} = \frac{0.434,294,481,903,251,830}{2401})$ $= 0.000,180,880,667,181,695$.

We shall therefore have

$$\frac{M}{m^2} \left(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{2401} = \frac{0.000,180,880,667,181,695}{2401}\right) = 0.000,000,075,335,554,844;$$

And $$\frac{M}{m^3} \left(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{2401} = \frac{0.000,000,075,335,554,844}{2401}\right) = 0.000,000,000,031,376,740;$$

And $$\frac{M}{m^4} \left(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{2401} = \frac{0.000,000,000,031,376,740}{2401}\right) = 0.000,000,000,000,013,068;$$

And $$\frac{M}{m^5} \left(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{2401} = \frac{0.000,000,000,000,013,068}{2401}\right) = 0.000,000,000,000,000,005;$$

And consequently

$$\frac{M}{2m^2} \left(= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,000,075,335,554,844}{2}\right) = 0.000,000,037,667,777,422;$$

And $$\frac{M}{3m^3} \left(= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,000,031,376,740}{3}\right) = 0.000,000,000,010,458,913;$$

And $$\frac{M}{4m^4} \left(= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,000,013,068}{4}\right) = 0.000,000,000,000,003,267;$$

And $$\frac{M}{5m^5} \left(= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,000,005}{5}\right) = 0.000,000,000,000,000,001.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \&c$ will be $=$

$$\begin{array}{r} 0.000,180,880,667,181,695, \\ + \;\ldots,\ldots,\ldots,37,667,777,422, \\ + \;\ldots,\ldots,\ldots,\ldots,10,458,913, \\ + \;\ldots,\ldots,\ldots,\ldots,\ldots,3,267, \\ + \;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots,1, \\ \hline = 0.000,180,918,345,421,298. \end{array}$$

Therefore this number 0.000,180,918,345,421,298 is the logarithm of the ratio of 1 to $1 - \frac{1}{2401}$, or of 2401 to 2400. Q. E. I.

This value of the logarithm of 2401 to 2400 is exact in all the figures except the last; its more exact value being 0.000,180,918,345,421,299,912, as will appear by subtracting the logarithm of 2400 from the logarithm of 2401, which may be done as follows. According to Mr. Sharp's computation, the logarithm of 7 is $=$ 0.845, 098,040,014,256,830,712; consequently the logarithm of 2401 (which is the fourth power of 7) will be $= 4 \times 0.845,098,040,014,256,830,712 = 3.380,392,160$, 057,027,322,848. And the logarithm of 24 is $=$ 1.380,211,241,711,606,022,936; and consequently the logarithm of 24×100, or 2400, is $= 2 + 1.380,211,241$,

 711,

711,606,022,936 = 3.380,211,241,711,606,022,936. And, if this laſt logarithm be ſubtracted from 3.380,392,160,057,027,322,848, (which is the logarithm of 2401,) the remainder will be 0.000,180,918,345,421,299,912.

EXAMPLE VI.

92. Let it be required to find by means of the ſame ſeries the logarithm of the ratio of 169 to 168, (or of 169 to 169 − 1, or of $\frac{169}{169}$ to $\frac{169-1}{169}$) or of 1 to 1 $-\frac{1}{169}$.

Here m is = 169, and $\frac{1}{m}$ is = $\frac{1}{169}$, and $\frac{M}{m}$ is $\left(= \frac{M}{169} = \frac{0.434,294,481,903,251,830}{169}\right.$ = 0.002,569,789,833,747,052.

We ſhall therefore have

$$\frac{M}{m^2} \left(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{169} = \frac{0.002,569,789,833,747,052}{169}\right) = 0.000,015,205,857,004,420;$$

And $$\frac{M}{m^3} \left(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{169} = \frac{0.000,015,205,857,004,420}{169}\right) = 0.000,000,089,975,485,233;$$

And $$\frac{M}{m^4} \left(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{169} = \frac{0.000,000,089,975,485,233}{169}\right) = 0.000,000,000,532,399,320;$$

And $$\frac{M}{m^5} \left(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{169} = \frac{0.000,000,000,532,399,320}{169}\right) = 0.000,000,000,003,150,291;$$

And $$\frac{M}{m^6} \left(= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{169} = \frac{0.000,000,000,003,150,291}{169}\right) = 0.000,000,000,000,018,640;$$

And $$\frac{M}{m^7} \left(= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{169} = \frac{0.000,000,000,000,018,640}{169}\right) = 0.000,000,000,000,000,110;$$

And $$\frac{M}{m^8} \left(= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{169} = \frac{0.000,000,000,000,000,110}{169}\right) = 0.000,000,000,000,000,000;$$

And conſequently

$$\frac{M}{2m^2} \left(= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,015,205,857,004,420}{2}\right) = 0.000,007,602,928,502,210;$$

And $$\frac{M}{3m^3} \left(= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,089,975,485,233}{3}\right) = 0.000,000,029,991,828,411;$$

And $$\frac{M}{4m^4} \left(= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,532,399,320}{4}\right) = 0.000,000,000,133,099,830;$$

And $$\frac{M}{5m^5} \left(= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,003,150,291}{5}\right) = 0.000,000,000,000,630,058;$$

And

And $\frac{M}{6m^6}$ $(= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,018,640}{6}) = 0.000,000,000,000,003,106$;

And $\frac{M}{7m^7}$ $(= \frac{M}{m^7} \times \frac{1}{7} \times \frac{0.000,000,000,000,000,110}{7}) = 0.000,000,000,000,000,015$.

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} +$ &c will be =

$$\begin{array}{rl} & 0;002,569,789,833,747,052, \\ + & ;\ldots,\ldots 7,602,928,502,210, \\ + & ;\ldots,\ldots,\ldots 29,991,828,411, \\ + & ;\ldots,\ldots,\ldots,133,099,830, \\ + & ;\ldots,\ldots,\ldots,\ldots,630,058, \\ + & ;\ldots,\ldots,\ldots,\ldots,\ldots 3,106, \\ + & ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots 15, \\ \hline = & 0.002,577,422,887,810,682. \end{array}$$

Therefore this number 0.002,577,422,887,810,682 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{169}$, or of 169 to 168. Q. E. I.

This value of the logarithm of the ratio of 169 to 168 is exact in all the figures except the last, which should be a 4 instead of a 2; the more accurate value of this logarithm being 0.002,577,422,887,810,684,765. This will appear from Mr. Sharp's computations, in the following manner. The logarithm of 13, according to Mr. Sharp, is = 1.113,943,352,306,836,769,206; therefore the logarithm of 169 (which is the square of 13) is = 2 × 1.113,943,352,306,836,769,206, or 2.227,886,704,613,673,538,412. The logarithm of 12 is = 1.079,181,246,047, 624,827,722; and the logarithm of 14 is 1.146,128,035,678,238,025,925. Therefore the logarithm of 168 (which is = 12 × 14) is =

$$\left.\begin{array}{rl} & 1.079,181,246,047,624,827,722 \\ + & 1.146,128,035,678,238,025,925 \end{array}\right\} = 2.225,309,281,725,862,853,647;$$

which being subtracted from 2.227,886,704,136,673,538,412 (which is the logarithm of 169) leaves 0.002,577,422,887,810,684,765 for the logarithm of the ratio of 169 to 168. Q. E. D.

EXAMPLE VII.

93. Let it be required to find by means of the same series the logarithm of the ratio of 289 to 288, (or of 289 to 289 − 1, or of $\frac{289}{289}$ to $\frac{289-1}{289}$), or of 1 to $1 - \frac{1}{289}$.

Here m is = 289, and $\frac{1}{m}$ is $= \frac{1}{289}$, and consequently $\frac{M}{m}$ is $(= \frac{M}{289} = \frac{0.434,294,481,903,251,830}{289}$ $= 0.001,502,749,072,329,591$.

We

We ſhall therefore have

$$\frac{M}{m^2}\ \left(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{289} = \frac{0.001,502,749,072,329,591}{289}\right) = 0.000,005,199,823,779,687;$$

And $$\frac{M}{m^3}\ \left(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{289} = \frac{0.000,005,199,823,779,687}{289}\right) = 0.000,000,017,992,469,825;$$

And $$\frac{M}{m^4}\ \left(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{289} = \frac{0.000,000,017,992,469,825}{289}\right) = 0.000,000,000,062,257,681;$$

And $$\frac{M}{m^5}\ \left(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{289} = \frac{0.000,000,000,062,257,681}{289}\right) = 0.000,000,000,000,215,424;$$

And $$\frac{M}{m^6}\ \left(= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{289} = \frac{0.000,000,000,000,215,424}{289}\right) = 0.000,000,000,000,000,745;$$

And $$\frac{M}{m^7}\ \left(= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{289} = \frac{0.000,000,000,000,000,745}{289}\right) = 0.000,000,000,000,000,002;$$

And conſequently

$$\frac{M}{2m^2}\ \left(= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,005,199,823,779,687}{2}\right) = 0.000,002,599,911,889,843;$$

And $$\frac{M}{3m^3}\ \left(= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,017,992,469,825}{3}\right) = 0.000,000,005,997,489,941;$$

And $$\frac{M}{4m^4}\ \left(= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,062,257,681}{4}\right) = 0.000,000,000,015,564,420;$$

And $$\frac{M}{5m^5}\ \left(= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,215,424}{5}\right) = 0.000,000,000,000,043,084;$$

And $$\frac{M}{6m^6}\ \left(= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,745}{6}\right) = 0.000,000,000,000,000,124;$$

And $$\frac{M}{7m^7}\ \left(= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,002}{7}\right) = 0.000,000,000,000,000,000.$$

Therefore the ſeries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \&c$ is =

$$\begin{array}{rr}
 & 0;001,502,749,072,329,591, \\
+ & ;\ldots,\ldots 2,599,911,889,843, \\
+ & ;\ldots,\ldots,\ldots 5,997,489,941, \\
+ & ;\ldots,\ldots,\ldots,\ldots 15,564,420, \\
+ & ;\ldots,\ldots,\ldots,\ldots,\ldots 43,084, \\
+ & ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots 124, \\
+ & ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots,\ldots, \\
\hline
= & 0.001,505,354,997,317,003.
\end{array}$$

Therefore this number 0.001,505,354,997,317,003 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{289}$, or of 289 to 288. Q. E. I.

This

This value of Briggs's logarithm of the ratio of 289 to 288 is exact in all the figures except the last, which should be a 6 instead of a 3; the more accurate value of this logarithm being 0.001,505,354,997,317,006,423. This will appear from Mr. Sharp's computations, in the following manner. The logarithm of 17, according to Mr. Sharp's computation of it, is 1.230,448,921,378,273,928,540; therefore the logarithm of 289 (which is the square of 17) will be $= 2 \times$ 1.230, 448,921,378,273,928,540, or to 2.460,897,842,756,547,857,080. The logarithm of 16 is $=$ 1.204,119,982,655,924,780,854; and the logarithm of 18 is $=$ 1.255,272,505,103,306,069,803. Therefore the logarithm of 288 (which is $= 16 \times 18$) is $=$ 1.204,119,982,655,924,780,854 $+$ 1.255,272,505,103,306, 069,803 $=$ 2.459,392,487,759,230,850,657; which being subtracted from the logarithm of 289, which has been shewn to be 2.460,897,842,756,547,857,080, will leave 0.001,505,354,997,317,006,423 for the logarithm of the ratio of 289 to 288.

EXAMPLE VIII.

94. Let it be required to find by means of the same series the logarithm of the ratio of 361 to 360, (or of 361 to 361 − 1, or of $\frac{361}{361}$ to $\frac{361-1}{361}$,) or of 1 to $1 - \frac{1}{361}$.

Here m is $= 361$, and $\frac{1}{m}$ is $= \frac{1}{361}$, and $\frac{M}{m}$ is $(= \frac{M}{361} = \frac{0.434,294,481,903,251,830}{361})$ $=$ 0.001,203,031,805,826,182.

We shall therefore have

$$\frac{M}{m^2} \left(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{361} = \frac{0.001,203,031,805,826,182}{361}\right) = 0.000,003,332,498,077,080;$$

And $$\frac{M}{m^3} \left(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{361} = \frac{0.000,003,332,498,077,080}{361}\right) = 0.000,000,009,231,296,612;$$

And $$\frac{M}{m^4} \left(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{361} = \frac{0.000,000,009,231,296,612}{361}\right) = 0.000,000,000,025,571,458;$$

And $$\frac{M}{m^5} \left(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{361} = \frac{0.000,000,000,025,571,458}{361}\right) = 0.000,000,000,000,070,835;$$

And $$\frac{M}{m^6} \left(= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{361} = \frac{0.000,000,000,000,070,835}{361}\right) = 0.000,000,000,000,000,196;$$

And $$\frac{M}{m^7} \left(= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{361} = \frac{0.000,000,000,000,000,196}{361}\right) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{M}{2m^2} \left(= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,003,332,498,077,080}{2}\right) = 0.000,001,666,249,038,540;$$

And $$\frac{M}{3m^3} \left(= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,009,231,296,612}{3}\right) = 0.000,000,003,077,098,870;$$

And

And $\frac{M}{4m^4}$ $(= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,025,571,458}{4}) = 0.000,000,000,006,392,864$;

And $\frac{M}{5m^5}$ $(= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,070,835}{5}) = 0.000,000,000,000,014,167$;

And $\frac{M}{6m^6}$ $(= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,196}{6}) = 0.000,000,000,000,000,032$.

Therefore the feries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} +$ &c is =

0;001,203,031,805,826,182,
+ ;. . . ,. . 1,666,249,038,540,
+ ;. . . ,. . . ,. . 3,077,098,870,
+ ;. . . ,. . . ,. . . ,. . 6,392,864,
+ ;. . . ,. . . ,. . . ,. . . ,. 14,167,
+ ;. . . ,. . . ,. . . ,. . . ,. . . ,. 32,
= 0.001,204,701,138,370,655.

Therefore this number 0.001,204,701,138,370,655 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{361}$, or of 361 to 360. Q. E. I.

This value of the logarithm of the ratio of 361 to 360 is exact in all the figures except the laft, which ought to be an 8 inftead of a 5; the more accurate value of this logarithm being 0.001,204,701,138,370,658,056. For, by Mr. Sharp's table of logarithms to 61 places of figures, it appears that the logarithm of 19 is 1.278, 753,600,952,828,961,536; therefore the logarithm of 361 (which is the fquare of 19) will be = 2 × 1.278,753,600,952,828,961,536 = 2.557,507,201,905,657, 923,072. And the logarithm of 18 is 1.255,272,505,103,306,069,803, and that of 20 is 1.301,029,995,663,981,195,213; and confequently the logarithm of 360 (which is = 18 × 20) will be = 1.255,272,505,103,306,069,803 + 1.301,029 995,663,981,195,213 = 2.556,302,500,767,287,265,016; which if we fubtract from 2.557,507,201,905,657,923,072, the remainder, to wit, 0.001,204,701,138, 370,658,056, will be the logarithm of the ratio of 361 to 360.

EXAMPLE IX.

95. Let it be required to find by means of the fame feries the logarithm of the ratio of 529 to 528, (or of 529 to 529 − 1, or of $\frac{529}{529}$ to $\frac{529-1}{529}$,) or of 1 to $1 - \frac{1}{529}$.

Here m is = 529, and $\frac{1}{m}$ is = $\frac{1}{529}$, and $\frac{M}{m}$ is $(= \frac{M}{529} = \frac{0.434,294,481,903,251,830}{529})$ = 0.000,820,972,555,582,706.

We

We shall therefore have

$$\frac{M}{m^2} \left(= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{529} = \frac{0.000,820,972,555,582,706}{529}\right) = 0.000,001,551,932,997,320;$$

And $\frac{M}{m^3} \left(= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{529} = \frac{0.000,001,551,932,997,320}{529}\right) = 0.000,000,002,933,710,769;$

And $\frac{M}{m^4} \left(= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{529} = \frac{0.000,000,002,933,710,769}{529}\right) = 0.000,000,000,005,545,767;$

And $\frac{M}{m^5} \left(= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{529} = \frac{0.000,000,000,005,545,767}{529}\right) = 0.000,000,000,000,010,483;$

And $\frac{M}{m^6} \left(= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{529} = \frac{0.000,000,000,000,010,483}{529}\right) = 0.000,000,000,000,000,019;$

And $\frac{M}{m^7} \left(= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{529} = \frac{0.000,000,000,000,000,019}{529}\right) = 0.000,000,000,000,000,000;$

And consequently

$$\frac{M}{2m^2} \left(= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,001,551,932,997,320}{2}\right) = 0.000,000,775,966,498,660;$$

And $\frac{M}{3m^3} \left(= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,002,933,710,769}{3}\right) = 0.000,000,000,977,903,589;$

And $\frac{M}{4m^4} \left(= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,005,545,767}{4}\right) = 0.000,000,000,001,386,441;$

And $\frac{M}{5m^5} \left(= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,010,483}{5}\right) = 0.000,000,000,000,002,096;$

And $\frac{M}{6m^6} \left(= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,019}{6}\right) = 0.000,000,000,000,000,003.$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} +$ &c will be $=$

$$\begin{array}{r} 0;000,820,972,555,582,706, \\ +\ ;\ldots,\ldots,775,966,498,660, \\ +\ ;\ldots,\ldots,\ldots,977,903,589, \\ +\ ;\ldots,\ldots,\ldots,\ldots 1,386,441, \\ +\ ;\ldots,\ldots,\ldots,\ldots,\ldots 2,096, \\ +\ ;\ldots,\ldots,\ldots,\ldots,\ldots,\ldots 3, \\ \hline = 0.000,821,749,501,373,495. \end{array}$$

Therefore this number 0.000,821,749,501,373,495 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{529}$, or of 529 to 528. Q. E. I.

This value of the logarithm of the ratio of 529 to 528 is exact in all the figures except the last, which ought to be an 8 instead of a five; the more accurate value of this logarithm being 0.000,821,749,501,373,498,835. For, by Mr. Sharp's table, the logarithm of 23 is = 1.361,727,836,017,592,878,867; therefore the logarithm of 529 (which is the square of 23) will be = 2 × 1.361,727,836,017,592,878,867, or to 2.723,455,672,035,185,757,734. And the logarithm of 22 is 1.342,422, 680,822,206,235,963, and that of 24 is 1.380,211,241,711,606,022,936. There-

 fore

fore the logarithm of 528 (which is = 22 × 24) will be = 1.342,422,680,822, 206,235,963 + 1.380,211,241,711,606,022,936 = 2.722,633,922,533,812,258, 899; which if we fubtract from 2.723,455,672,035,185,757,734, (which is the logarithm of 529) the remainder, to wit, 0.000,821,749,501,373,498,835, will be the logarithm of the ratio of 529 to 528.

96. It appears therefore from thefe examples that Briggs's logarithms of the ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, 2401 to 2400, 169 to 168, 289 to 288, 361 to 360, and 529 to 528, are as follows, to wit:

$$\text{L. } \frac{10}{9} = 0.045,757,490,560,675,118;$$

$$\text{L. } \frac{11}{10} = 0.041,392,685,158,225,034;$$

$$\text{L. } \frac{81}{80} = 0.005,395,031,886,706,160;$$

$$\text{L. } \frac{121}{120} = 0.003,604,124,268,825,251;$$

$$\text{L. } \frac{2401}{2400} = 0.000,180,918,345,421,298;$$

$$\text{L. } \frac{169}{168} = 0.002,577,422,887,810,682;$$

$$\text{L. } \frac{289}{288} = 0.001,505,354,997,317,003;$$

$$\text{L. } \frac{361}{360} = 0.001,204,701,138,370,655;$$

$$\text{And L. } \frac{529}{528} = 0.000,821,749,501,373,495.$$

An Application of the foregoing Nine Logarithms of fmall Ratios in Briggs's Syftem of Logarithms to the Inveftigation of the Logarithms of the Ratios of the firft 23 *natural Numbers* (2, 3, 4, 5, &c, *to* 24 *inclufively) to* 1, *in the fame Syftem.*

97. From thefe original logarithms, which we have computed by means of the feries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} +$ *&c ad infinitum*, we may deduce the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, and of every following number to 1, as far as 24, in the following manner.

$$\text{L. } \frac{11}{9} \text{ is} = \text{L. } \frac{11}{10} + \text{L. } \frac{10}{9} = \left\{ \begin{array}{l} 0.041,392,685,158,225,034 \\ +\ 0.045,757,490,560,675,118 \end{array} \right\} = 0.087,150,175,718,900,152.$$

$$\text{L. } \frac{121}{81} \text{ is} = 2 \times \text{L. } \frac{11}{9} = 2 \times 0.087,150,175,718,900,152 = 0.174,300,351,437,800,304.$$

$$\text{L. } \frac{121}{80} \text{ is} = \text{L. } \frac{121}{81} + \text{L. } \frac{81}{80} = \left\{ \begin{array}{l} 0.174,300,351,437,800,304 \\ +\ 0.005,395,031,886,706,160 \end{array} \right\} = 0.179,695,383,324,506,464.$$

$$\text{L. } \frac{120}{80} \text{ is} = \text{L. } \frac{121}{80} - \text{L. } \frac{121}{120} = \left\{ \begin{array}{l} 0.179,695,383,324,506,464 \\ -\ 0.003,604,124,268,825,251 \end{array} \right\} = 0.176,091,259,055,681,213.$$

$$\text{L. } \frac{3}{2} = \text{L. } \frac{120}{80} = 0.176,091,259,055,681,213.$$

$$L.\ \frac{9}{4} = 2 \times L.\ \frac{3}{2} = 2 \times 0.176,091,259,055,681,213 = 0.352,182,518,111,362,426.$$

$$L.\ \frac{81}{16} = 2 \times L.\ \frac{9}{4} = 2 \times 0.352,182,518,111,362,426 = 0.704,365,036,222,724,852.$$

$$L.\ \frac{80}{16} \text{ is} = L.\ \frac{81}{16} - L.\ \frac{81}{80} = \left\{\begin{array}{r} 0.704,365,036,222,724,852 \\ -\ 0.005,395,031,886,706,160 \end{array}\right\} = 0.698,970,004,336,018,692.$$

$$L.\ \frac{5}{1} = L.\ \frac{80}{16} = 0.698,970,004,336,018,692. \qquad \text{Q. E. I.}$$

This number 0.698,970,004,336,018,692 is exact in all the figures except the three laſt, which ought to be 804 inſtead of 692; the more accurate value of the logarithm of 5, in Briggs's ſyſtem, being, according to Mr. Sharp's computation, 0.698,970,004,336,018,804,786.

$$L.\ \frac{10}{5} \text{ is} = L.\ \frac{10}{1} - L.\ \frac{5}{1} = \left\{\begin{array}{r} 1.000,000,000,000,000,000 \\ -\ 0.698,970,004,336,018,692 \end{array}\right\} = 0.301,029,995,663,981,308.$$

$$L.\ \frac{2}{1} \text{ is} = L.\ \frac{10}{5} = 0.301,029,995,663,981,308. \qquad \text{Q. E. I.}$$

This number 0.301,029,995,663,981,308 is exact in all but the three laſt figures; the more accurate value of the logarithm of 2, according to Mr. Sharp's computation, being 0.301,029,995,663,981,195,213.

$$L.\ \frac{4}{1} \text{ is} = 2 \times L.\ \frac{2}{1} = 2 \times 0.301,029,995,663,981,308 = 0.602,059,991,327,962,616.$$

Q. E. I.

$$L.\ \frac{8}{1} \text{ is} = 3 \times L.\ \frac{2}{1} = 3 \times 0.301,029,995,663,981,308 = 0.903,089,986,991,943,924.$$

Q. E. I.

$$L.\ \frac{16}{1} \text{ is} = 2 \times L.\ \frac{4}{1} = 2 \times 0.602,059,991,327,962,616 = 1.204,119,982,655,925,232.$$

Q. E. I.

$$L.\ \frac{20}{1} \text{ is} = L.\ \frac{20}{10} = L.\ \frac{10}{1} = L.\ \frac{2}{1} + L.\ \frac{10}{1} = \left\{\begin{array}{r} 0.301,029,995,663,981,308 \\ +\ 1.000,000,000,000,000,000 \end{array}\right\} = 1.301,029,995,663,981,308.$$

Q. E. I.

$$L.\ \frac{9}{1} \text{ is} = L.\ \frac{10}{1} - L.\ \frac{10}{9} = \left\{\begin{array}{r} 1.000,000,000,000,000,000 \\ -\ 0.045,757,490,560,675,118 \end{array}\right\} = 0.954,242,509,439,324,882.$$

Q. E. I.

$$L.\ \frac{3}{1} \text{ is} = \frac{1}{2} \times L.\ \frac{9}{1} = \frac{0.954,242,509,439,324,882}{2} = 0.477,121,254,719,662,441. \qquad \text{Q. E. I.}$$

$$L.\ \frac{6}{1} \text{ is} = L.\ \frac{6}{3} + L.\ \frac{3}{1} = L.\ \frac{2}{1} + L.\ \frac{3}{1} = \left\{\begin{array}{r} 0.301,029,995,663,981,308 \\ +\ 0.477,121,254,719,662,441 \end{array}\right\} = 0.778,151,250,383,643,749.$$

Q. E. I.

$$L.\ \frac{18}{1} \text{ is} = L.\ \frac{18}{9} + L.\ \frac{9}{1} = L.\ \frac{2}{1} + L.\ \frac{9}{1} = \left\{\begin{array}{r} 0.301,029,995,663,981,308 \\ +\ 0.954,242,509,439,324,882 \end{array}\right\} = 1.255,272,505,103,306,190.$$

Q. E. I.

$$L.\ \frac{15}{1} \text{ is} = L.\ \frac{15}{5} + L.\ \frac{5}{1} = L.\ \frac{3}{1} + L.\ \frac{5}{1} = \left\{\begin{array}{r} 0.477,121,254,719,662,441 \\ +\ 0.698,970,004,336,018,692 \end{array}\right\} = 1.176,091,259,055,681,133.$$

Q. E. I.

$$L.\frac{12}{1}\text{ is}=L.\frac{12}{6}+L.\frac{6}{1}=L.\frac{2}{1}+L.\frac{6}{1}=\left\{\begin{array}{r}0.301,029,995,663,981,308\\+0.778,151,250,383,643,749\end{array}\right\}=1.079,181,246,047,625,057.$$

Q. E. I.

$$L.\frac{24}{1}\text{ is}=L.\frac{24}{12}+L.\frac{12}{1}=L.\frac{2}{1}+L.\frac{12}{1}=\left\{\begin{array}{r}0.301,029,995,663,981,308\\+1.079,181,246,047,625,057\end{array}\right\}=1.380,211,241,711,606,365.$$

Q. E. I.

$$L.\frac{2400}{1}\text{ is}=L.\frac{2400}{100}+L.\frac{100}{1}=L.\frac{24}{1}+2\times L.\frac{10}{1}=L.\frac{24}{1}+2=\left\{\begin{array}{r}1.380,211,241,711,606,365\\+2.000,000,000,000,000,000\end{array}\right\}=3.380,211,241,711,606,365.$$

$$L.\frac{2401}{1}\text{ is}=L.\frac{2401}{2400}+L.\frac{2400}{1}=\left\{\begin{array}{r}0.000,180,918,345,421,298\\+3.380,211,241,711,606,365\end{array}\right\}=3.380,392,160,057,027,663.$$

$$L.\frac{7}{1}\text{ is}=\frac{1}{4}\times L.\frac{2401}{1}=\frac{3.380,392,160,057,027,663}{4}=0.845,098,040,014,256,915.$$ Q. E. I.

$$L.\frac{14}{1}\text{ is}=L.\frac{14}{7}+L.\frac{7}{1}=L.\frac{2}{1}+L.\frac{7}{1}=\left\{\begin{array}{r}0.301,029,995,663,981,308\\+0.845,098,040,014,256,915\end{array}\right\}=1.146,128,035,678,238,223.$$

Q. E. I.

$$L.\frac{21}{1}\text{ is}=L.\frac{21}{7}+L.\frac{7}{1}=L.\frac{3}{1}+L.\frac{7}{1}=\left\{\begin{array}{r}0.477,121,254,719,662,441\\+0.845,098,040,014,256,915\end{array}\right\}=1.322,219,294,733,919,356.$$

Q. E. I.

$$L.\frac{11}{1}\text{ is}=L.\frac{11}{10}+L.\frac{10}{1}=\left\{\begin{array}{r}0.041,392,685,158,225,034\\+1.000,000,000,000,000,000\end{array}\right\}=1.041,392,685,158,225,034.$$ Q. E. I.

$$L.\frac{22}{1}\text{ is}=L.\frac{22}{11}+L.\frac{11}{1}=L.\frac{2}{1}+L.\frac{11}{1}=\left\{\begin{array}{r}0.301,029,995,663,981,308\\+1.041,392,685,158,225,034\end{array}\right\}=1.342,422,680,822,206,342.$$

Q. E. I.

$$L.\frac{168}{1}\text{ is}=L.\frac{168}{14}+L.\frac{14}{1}=L.\frac{12}{1}+L.\frac{14}{1}=\left\{\begin{array}{r}1.079,181,246,047,625,057\\+1.146,128,035,678,238,223\end{array}\right\}=2.225,309,281,725,863,280.$$

$$L.\frac{169}{1}\text{ is}=L.\frac{169}{168}+L.\frac{168}{1}=\left\{\begin{array}{r}0.002,577,422,887,810,682\\+2.225,309,281,725,863,280\end{array}\right\}=2.227,886,704,613,673,962.$$

$$L.\frac{13}{1}\text{ is}=\frac{1}{2}\times L.\frac{169}{1}=\frac{1}{2}\times 2.227,886,704,613,673,962=1.113,943,352,306,836,981.$$

Q. E. I.

$$L.\frac{288}{1}\text{ is}=L.\frac{288}{18}+L.\frac{18}{1}=L.\frac{16}{1}+L.\frac{18}{1}=\left\{\begin{array}{r}1.204,119,982,655,925,232\\+1.255,272,505,103,306,190\end{array}\right\}=2.459,392,487,759,231,422.$$

$$L.\frac{289}{1}\text{ is}=L.\frac{289}{288}+L.\frac{288}{1}=\left\{\begin{array}{r}0.001,505,354,997,317,003\\+2.459,392,487,759,231,422\end{array}\right\}=2.460,897,842,756,548,425.$$

$$L.\frac{17}{1}\text{ is}=\frac{1}{2}\times L.\frac{289}{1}=\frac{2.460,897,842,756,548,425}{2}=1.230,448,921,378,274,212.$$ Q. E. I.

$$L.\frac{360}{1}\text{ is}=L.\frac{360}{20}+L.\frac{20}{1}=L.\frac{18}{1}+L.\frac{20}{1}=\left\{\begin{array}{r}1.255,272,505,103,306,190\\+1.301,029,995,663,981,308\end{array}\right\}=2.556,302,500,767,287,498$$

$$L.\frac{361}{1}\text{ is}=L.\frac{361}{360}+L.\frac{360}{1}=\left\{\begin{array}{r}0.001,204,701,138,370,655\\+2.556,302,500,767,287,498\end{array}\right\}=2.557,507,201,905,658,153.$$

$$L.\frac{19}{1}\text{ is}=\frac{1}{2}\times L.\frac{361}{1}=\frac{2.557,507,201,905,658,153}{2}=1.278,753,600,952,829,076.$$ Q. E. I.

$$L.\frac{528}{1}\text{ is} = L.\frac{528}{24} + L.\frac{24}{1} = L.\frac{22}{1} + L.\frac{24}{1} = \left\{\begin{array}{l} \ 1.342{,}422{,}680{,}822{,}206{,}342 \\ +\ 1.380{,}211{,}241{,}711{,}606{,}365 \end{array}\right\} = 2.722{,}633{,}922{,}533{,}812{,}707.$$

$$L.\frac{529}{1}\text{ is} = L.\frac{529}{528} + L.\frac{528}{1} = \left\{\begin{array}{l} \ 0.000{,}821{,}749{,}501{,}373{,}495 \\ +\ 2.722{,}633{,}922{,}533{,}812{,}707 \end{array}\right\} = 2.723{,}455{,}672{,}035{,}186{,}202.$$

$$L.\frac{23}{1}\text{ is} = \frac{1}{2} \times L.\frac{529}{1} = \frac{2.723{,}455{,}672{,}035{,}186{,}202}{2} = 1.361{,}727{,}836{,}017{,}593{,}101. \quad \text{Q. E. I.}$$

98. Thefe logarithms, if ranged in their proper order, will be as follows :

$L.\frac{2}{1} = 0.301{,}029{,}995{,}663{,}981{,}308$;

$L.\frac{3}{1} = 0.477{,}121{,}254{,}719{,}662{,}441$;

$L.\frac{4}{1} = 0.602{,}059{,}991{,}327{,}962{,}616$;

$L.\frac{5}{1} = 0.698{,}970{,}004{,}336{,}018{,}692$;

$L.\frac{6}{1} = 0.778{,}151{,}250{,}383{,}643{,}749$;

$L.\frac{7}{1} = 0.845{,}098{,}040{,}014{,}256{,}915$;

$L.\frac{8}{1} = 0.903{,}089{,}986{,}991{,}943{,}924$;

$L.\frac{9}{1} = 0.954{,}242{,}509{,}439{,}324{,}882$;

$L.\frac{10}{1} = 1.000{,}000{,}000{,}000{,}000{,}000$;

$L.\frac{11}{1} = 1.041{,}392{,}685{,}158{,}225{,}034$;

$L.\frac{12}{1} = 1.079{,}181{,}246{,}047{,}625{,}057$;

$L.\frac{13}{1} = 1.113{,}943{,}352{,}306{,}836{,}981$;

$L.\frac{14}{1} = 1.146{,}128{,}035{,}678{,}238{,}223$;

$L.\frac{15}{1} = 1.176{,}091{,}259{,}055{,}681{,}133$;

$L.\frac{16}{1} = 1.204{,}119{,}982{,}655{,}925{,}232$;

$L.\frac{17}{1} = 1.230{,}448{,}921{,}378{,}274{,}212$;

$L.\frac{18}{1} = 1.255{,}272{,}505{,}103{,}306{,}190$;

$L.\frac{19}{1} = 1.278{,}753{,}600{,}952{,}829{,}076$;

$$L.\ \frac{20}{1} = 1.301,029,995,663,981,308\ ;$$

$$L.\ \frac{21}{1} = 1.322,219,294,733,919,356\ ;$$

$$L.\ \frac{22}{1} = 1.342,422,680,822,206,342\ ;$$

$$L.\ \frac{23}{1} = 1.361,727,836,017,593,101\ ;$$

$$\text{And } L.\ \frac{24}{1} = 1.380,211,241,711,606,365.$$

Thefe values of the foregoing logarithms are exact in, at leaft, the firft fourteen places of decimal fractions, and much the greater part of them are exact alfo in the fifteenth place; their more accurate values, according to Mr. Sharp's computation, being as follows:

$$L.\ \frac{2}{1} = 0.301,029,995,663,981,195,213\ ;$$

$$L.\ \frac{3}{1} = 0.477,121,254,719,662,437,295\ ;$$

$$L.\ \frac{4}{1} = 0.602,059,991,327,962,390,427\ ;$$

$$L.\ \frac{5}{1} = 0.698,970,004,336,018,804,786\ ;$$

$$L.\ \frac{6}{1} = 0.778,151,250,383,643,632,508\ ;$$

$$L.\ \frac{7}{1} = 0.845,098,040,014,256,830,712\ ;$$

$$L.\ \frac{8}{1} = 0.903,089,986,991,943,585,641\ ;$$

$$L.\ \frac{9}{1} = 0.954,242,509,439,324,874,590\ ;$$

$$L.\ \frac{10}{1} = 1.000,000,000,000,000,000,000\ ;$$

$$L.\ \frac{11}{1} = 1.041,392,685,158,225,040,750\ ;$$

$$L.\ \frac{12}{1} = 1.079,181,246,047,624,827,722\ ;$$

$$L.\ \frac{13}{1} = 1.113,943,352,306,836,769,206\ ;$$

$$L.\ \frac{14}{1} = 1.146,128,035,678,238,025,925\ ;$$

$$L.\ \frac{15}{1} = 1.176,091,259,055,681,242,081\ ;$$

$$L.\ \frac{16}{1} = 1.204,119,982,655,924,780,854\ ;$$

$$L. \frac{17}{1} = 1.230,448,921,378,273,928,540;$$

$$L. \frac{18}{1} = 1.255,272,505,103,306,069,803;$$

$$L. \frac{19}{1} = 1.278,753,600,952,828,961,536;$$

$$L. \frac{20}{1} = 1.301,029,995,663,981,195,213;$$

$$L. \frac{21}{1} = 1.322,219,294,733,919,268,007;$$

$$L. \frac{22}{1} = 1.342,422,680,822,206,235,963;$$

$$L. \frac{23}{1} = 1.361,727,836,017,592,878,867;$$

$$L. \frac{24}{1} = 1.380,211,241,711,606,022,936.$$

99. If we were to purſue this matter further, and to compute the logarithms of the ratios of numbers greater than 24 to 1, by means of the aforeſaid ſerieſes of Mercator and Wallis, the labour of the computation would continually grow leſs and leſs as the numbers increaſed. For, as to the logarithms of the ratios of all compoſite numbers (or numbers formed by the multiplication of other numbers) to 1, they, it is evident, might be found by merely adding together the logarithms of the ratios of the factors of which they are compoſed to 1. Thus, for example, we ſhould have $L. \frac{25}{1}$ ($= L. \frac{25}{5} + L. \frac{5}{1} = L. \frac{5}{1} + L. \frac{5}{1}$) $= 2\,L. \frac{5}{1}$; and $L. \frac{26}{1}$ ($= L. \frac{26}{13} + L. \frac{13}{1}$) $= L. \frac{2}{1} + L. \frac{13}{1}$; and $L. \frac{27}{1}$ ($= L. \frac{27}{9} + L. \frac{9}{3} + L. \frac{3}{1} = L. \frac{3}{1} + L. \frac{3}{1} + L. \frac{3}{1}$) $= 3\,L. \frac{3}{1}$; and $L. \frac{28}{1}$ ($= L. \frac{28}{14} + L. \frac{14}{1}$) $= L. \frac{2}{1} + L. \frac{14}{1}$. Therefore the only logarithms for the computation of which it would be neceſſary to have recourſe to one of the foregoing infinite ſerieſes, would be thoſe of the ratios of the following prime numbers, ſuch as 29, 31, 37, *&c*, to 1. And, in computing theſe logarithms by means of the ſaid ſerieſes, it is evident that the labour of the computation muſt be continually leſs and leſs as the number becomes greater and greater, becauſe the terms of the ſeries will converge with a greater degree of ſwiftneſs as the number to which it relates increaſes. This will appear more clearly by reviewing the manner in which we found the logarithms of the ratios of the prime numbers 13, 17, 19, and 23, to 1, which was as follows.

A Review of the foregoing Manner of computing Logarithms.

100. We did not (as we might have done) compute by the ſerieſes the logarithms of the ratios of 13 to 12, of 17 to 16, of 19 to 18, and of 23 to 22, but, in

in order to obtain ſerieſes that would converge more ſwiftly, we ſquared theſe four prime numbers 13, 17, 19, and 23, whereby we obtained the numbers 169, 289, 361, and 529, and then we computed, by means of the ſerieſes, the logarithms of the ratios of 169 to 168, of 289 to 288, of 361 to 360, and of 529 to 528, and added the logarithms ſo found to the logarithms of the ratios of 168 to 1, of 288 to 1, of 360 to 1, and of 528 to 1, which were derived from the logarithms already known by adding together the logarithms of the ratios of the factors of thoſe numbers, to wit, the factors 12 and 14, or 13 — 1 and 13 + 1, 16 and 18, or 17 — 1 and 17 + 1, 18 and 20, or 19 — 1 and 19 + 1, and 22 and 24, or 23 — 1, and 23 + 1, to 1; and when we had thus found the logarithms of the ratios of 169, or $\overline{13}|^2$, and 289, or $\overline{17}|^2$, and 361, or $\overline{19}|^2$, and 529, or $\overline{23}|^2$, to 1, we took the halves of thoſe logarithms, and thereby obtained the logarithms of the ratios of thoſe prime numbers 13, 17, 19, and 23 themſelves, to 1.

Now, if this method be purſued with reſpect to any greater prime number, after the logarithms of the ratios of all the leſſer prime numbers to 1 have been computed, we muſt proceed as follows.

Let the prime number, of the ratio of which to 1 we want to find the logarithm, be called P.

Then, in the firſt place, it is evident that, ſince P is a prime number, and conſequently an odd number, P — 1 and P + 1 muſt, both of them, be even numbers, or diviſible by 2.

Secondly, ſince the logarithms of the ratios of all the prime numbers that are leſs than P to 1, are ſuppoſed to be already known, the logarithms of the ratios of the prime numbers that are factors of the compoſite numbers P — 1 and P + 1 to 1, muſt be already known; and conſequently the logarithms of the ratios of the ſaid compoſite numbers P — 1 and P + 1 to 1, may be derived from the logarithms already known, by mere addition.

Thirdly, the number PP — 1 is equal to $\overline{P-1}| \times \overline{P+1}|$, and conſequently the logarithm of the ratio of PP — 1 to 1 will be equal to the ſum of the logarithms of the ratios of P — 1 to 1, and of P + 1 to 1.

We muſt therefore find, by addition of the logarithms already known, the logarithms of the ratios of P — 1 to 1 and of P + 1 to 1, and add them together into one ſum; whereby we ſhall obtain the logarithm of the ratio of PP — 1 to 1. We muſt then find, by one of the foregoing ſerieſes, the logarithm of the ratio of PP to PP — 1, and add it to the logarithm of the ratio of PP — 1 to 1. The ſum will be the logarithm of the ratio of PP to 1; and half this logarithm will be the logarithm of the ratio of the prime number P itſelf to 1. Q. E. I.

Now, in computing by one of the foregoing ſerieſes the logarithm of the ratio of PP to PP — 1, it is evident that the greater the number PP is, and conſequently the greater the number P is, the greater will be the ſwiftneſs with which the terms of the ſaid logarithmick ſeries will converge. There will therefore be leſs trouble in computing the logarithm of the ratios of 29, 31, 37, 41, 43, and every following prime number, to 1, than there was in computing the logarithm of the ratio of 23 to 1; which yet, we have ſeen, was not very great, ſince only ſix terms of the

infinite ſeries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} +$ &c were ſufficient to enable us to find it exact to fifteen places of figures. In computing the logarithms of the ratios of 29, 31, and 37, to 1, by the method juſt deſcribed, or by firſt raiſing the ſquares of theſe prime numbers, to wit, 841, 961, and 1369, and finding by the ſaid infinite ſeries the logarithms of the ratios of thoſe ſquares to the next leſſer numbers 840, 960, and 1368, only five terms of the ſaid ſeries will be ſufficient to give the logarithms ſought to the ſame degree of exactneſs; and for the logarithms of the ratios of prime numbers greater than 100 to 1, only four terms of the ſaid ſeries will be ſufficient for the ſame purpoſe.

Of the great Expedition of the foregoing Series in computing the Logarithms of the Ratios of prime Numbers greater than 1100 *to* 1.

101. Mr. Abraham Sharp has already computed the logarithms of the ratios of all prime numbers under 1100 to 1 to no leſs than 61 places of figures; and they are publiſhed with Sherwin's Tables of Logarithms, in the third edition of that uſeful book, publiſhed in the year 1741, in pages 36, 37, 38, and 39. Theſe may therefore be made uſe of in finding the logarithm of the ratio of any prime number greater than 1100 to 1; as, for example, in finding the logarithm of the ratio of 1103 to 1. Now the ſquare of 1103 is 1,216,609; by which, if we divide M, or 0.434,294,481,903,251,830, the quotient will be = 0.000,000, 356,971, &c; that is, according to the notation uſed above, m will be = 1,216, 609, and $\frac{M}{m} =$ 0.000,000,356,971, &c. Therefore $\frac{M}{m^2}$ will be $= \frac{M}{m} \times \frac{1}{m} =$ 0.000,000,356,971, &c, $\times \frac{1}{1,206,609} = \frac{0.000,000,356,971,\ \&c}{1,216,609}$, the quotient of which diviſion will appear only in the 13th place of decimal fractions; and $\frac{M}{m^3}$ will be $= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{1,216,609}$, the quotient of which diviſion will appear only in the 19th place of decimal fractions. Therefore, if the logarithm of the ratio of 1,216,609 to 1, be ſought only to 18 places of decimal fractions, (which is the number of places to which the logarithms found in the foregoing nine examples are carried,) we need only compute the two firſt terms $\frac{M}{m}$ and $\frac{M}{2m^2}$ of the infinite ſeries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} +$ &c. And therefore to find the logarithm of the ratio of the prime number 1103 to 1, and, *à fortiori*, to find the logarithm of the ratio of any prime number greater than 1103 to 1, to 18 places of figures, it will be ſufficient to make uſe of only the two firſt terms of the ſeries $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} +$ &c. An eaſier method of computing ſuch a logarithm ſeems hardly to be expected, and needs not much to be deſired.

End of the Review of the foregoing Method of computing Logarithms.

102. I have now, I hope, ſufficiently explained the methods of applying theſe two infinite ſerieſes $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} +$ &c, and

$\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} +$ &c (which were invented by Mr. Mercator and Dr. Wallis) to the computation both of Napier's and Briggs's logarithms; by which their ufe and excellency have appeared moft manifeftly. I fhould therefore here put an end to thefe remarks, were it not that the abftrufenefs and fubtlety of the reafonings contained in the demonftrations of the two principal theorems in article 18 and article 66 are fo great, (notwithftanding the endeavours I have there ufed to facilitate them,) that I imagine a further attempt to ftate them clearly will not be unacceptable to the reader. Such an attempt I fhall therefore now proceed to make.

A further Attempt to illuftrate and facilitate the Demonftration of Theorem 1 *in Article* 18.

103. If there be two given ratios, that of E to F, and that of G to H, and a third ratio, to wit, that of X to Y, that is continually varying while its terms X and Y vary in their magnitude; and the faid varying ratio of X to Y continually approaches to each of the given ratios of E to F and of G to H, as to its limit, fo that it may be made to come as near as we pleafe to each of the faid given ratios, though it can never be abfolutely equal to either of them: I fay, that, upon this fuppofition, the ratio of G to H muft be equal to the ratio of E to F. This propofition I take to be fo evident as not to ftand in need of a demonftration. But, if it were thought to need a demonftration, it might, without much difficulty, be demonftrated *ex abfurdo* after the manner of the ancients, by fuppofing the ratios of E to F and of G to H not to be equal to each other, and then fhewing the contradictory confequences that would follow from fuch a fuppofition. But this, as it appears to me unneceffary, I fhall omit.

104. To this propofition we may bring the demonftration contained in article 18, by proceeding in the manner following.

Let k and q be any two quantities lefs than 1, whereof k is the greater; fuch, for example, as $\frac{9}{10}$ and $\frac{5}{10}$. We are to fhew, that the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1 in the fame proportion as the infinite feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c to the infinite feries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c, or that the ratio, or proportion, which the ratio of $1 + k$ to 1 bears to the ratio of $1 + q$ to 1, is equal to the ratio of the former of thefe two infinite feriefes to the latter. Now this may be done by fhewing that both thefe ratios (which are evidently fixt or given ratios) are the limits of a certain varying ratio, which we fhall now proceed to defcribe and confider.

Let us fuppofe the ratio of $1 + k$ to 1 to be firft bifected, or divided into two equal ratios, by extracting the fquare root of $1 + k$, or inferting a geometrical mean proportional between 1 and $1 + k$; and let the ratio of $1 + q$ to 1 be bifected in like manner, by extracting the fquare-root of $1 + q$, or inferting a geometrical mean proportional between 1 and $1 + q$. Let us then fuppofe the

the ſame two ratios of $1 + k$ to 1 and $1 + q$ to 1 to be quadriſected, or divided into four equal ratios, by the inſertion of three geometrical mean proportionals between 1 and $1 + k$, and three like mean proportionals between 1 and $1 + q$. And then let us ſuppoſe the ſame ratios to be divided into eight equal ratios by the inſertion of ſeven geometrical mean proportionals between their terms, and into ſixteen equal ratios, and into thirty-two equal ratios, and into a greater and greater number of equal ratios continually; every new number of ratios into which they are divided being double of the number in the diviſion next preceding.

Then it is evident, that, when theſe two ratios are divided into only two equal ratios by the inſertion of only one geometrical mean proportional between 1 and $1 + k$, and one geometrical mean proportional between 1 and $1 + q$, theſe geometrical mean proportionals will be $\sqrt{1+k}$ and $\sqrt{1+q}$, or $\overline{1+k}|^{\frac{1}{2}}$ and $\overline{1+q}|^{\frac{1}{2}}$; and when they are divided into four equal ratios by the inſertion of three geometrical mean proportionals between 1 and $1 + k$ and three geometrical mean proportionals between 1 and $1 + q$, the firſt, or leaſt, of the firſt ſet of mean proportionals will be $\sqrt[4]{1+k}$, or $\overline{1+k}|^{\frac{1}{4}}$, and the firſt, or leaſt, of the ſecond ſet of mean proportionals will be $\sqrt[4]{1+q}$, or $\overline{1+q}|^{\frac{1}{4}}$; and when they are divided into eight equal ratios by the inſertion of ſeven geometrical mean proportionals between 1 and $1 + k$ and ſeven ſuch mean proportionals between 1 and $1 + q$, the firſt, or leaſt, of the firſt ſet of mean proportionals will be $\sqrt[8]{1+k}$, or $\overline{1+k}|^{\frac{1}{8}}$, and the firſt, or leaſt, of the ſecond ſet of mean proportionals will be $\sqrt[8]{1+q}$, or $\overline{1+q}|^{\frac{1}{8}}$; and, in general, if n be the number of leſſer ratios into which each of the ſaid two ratios of $1 + k$ to 1 and $1 + q$ to 1 is divided by the inſertion of $n - 1$ geometrical mean proportionals, the firſt, or leaſt, of the firſt ſet of ſuch mean proportionals will be $\sqrt[n]{1+k}$, or $\overline{1+k}|^{\frac{1}{n}}$, and the firſt, or leaſt, of the ſecond ſet of ſuch mean proportionals will be $\sqrt[n]{1+q}$, or $\overline{1+q}|^{\frac{1}{n}}$.

Now during all theſe ſucceſſive diviſions of the ratios of $1 + k$ to 1 and of $1 + q$ to 1 into two, four, eight, ſixteen, and the following greater numbers of equal parts, the proportion of the ratio of the firſt, or leaſt, of the firſt ſet of mean proportionals (which are inſerted between 1 and $1 + k$) to 1 to the ratio of the firſt, or leaſt, of the ſecond ſet of mean proportionals (which are inſerted between 1 and $1 + q$) to 1, will always be the ſame, whatever be the number of parts, or leſſer ratios, into which the ſaid ratios of $1 + k$ to 1 and $1 + q$ to 1 be ſuppoſed to be divided, and will be the ſame with the proportion of the ratio of $1 + k$ to 1 to the ratio of $1 + q$ to 1. Thus, if theſe ratios are divided into only two parts, or leſſer ratios, by the inſertion of the mean proportionals $\overline{1+k}|^{\frac{1}{2}}$ and $\overline{1+q}|^{\frac{1}{2}}$, the proportion of the ratio of $\overline{1+k}|^{\frac{1}{2}}$ to 1 to the ratio of $\overline{1+q}|^{\frac{1}{2}}$ to 1, will be the ſame with the proportion of the ratio of $1 + k$ itſelf to 1 to the ratio of $1 + q$ itſelf to 1. For the two former ratios, to wit, the ratios of $\overline{1+k}|^{\frac{1}{2}}$ to 1 and of $\overline{1+q}|^{\frac{1}{2}}$ to 1, are the halves of the two latter ratios, or the ratios of $1 + k$ to 1 and $1 + q$ to 1; and therefore the proportion of the two former ratios muſt be the ſame as that of the

the two latter. And, in like manner, if the ratios of $1 + k$ to 1 and $1 + q$ to 1 are divided into four equal parts, or lesser ratios, by the insertion of three geometrical mean proportionals between 1 and $1 + k$ and of three like mean proportionals between 1 and $1 + q$, the ratio of $\overline{1+k}|^{\frac{1}{4}}$ (the first, or least, mean proportional of the first set) to 1 will be to the ratio of $\overline{1+q}|^{\frac{1}{4}}$ (the first, or least, mean proportional of the second set) to 1 in the same proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1; because the ratio of $\overline{1+k}|^{\frac{1}{4}}$ to 1 is a fourth part of the ratio of $1 + k$ to 1, and the ratio of $\overline{1+q}|^{\frac{1}{4}}$ to 1 is a fourth part of the ratio of $1 + q$ to 1, and the like parts of ratios (as well as of all other quantities) must be to each other in the same proportion as the whole quantities themselves. And, for the same reason, if each of the two ratios of $1 + k$ to 1 and of $1 + q$ to 1 be divided into any other and greater number of parts, or lesser ratios, denoted by n, by the insertion of $n - 1$ mean proportionals between 1 and $1 + k$ and the same number of mean proportionals between 1 and $1 + q$, the ratio of $\overline{1+k}|^{\frac{1}{n}}$ (the first, or least, of the first set of mean proportionals) to 1 will be to the ratio of $\overline{1+q}|^{\frac{1}{n}}$ (the first, or least, of the second set of mean proportionals) to 1 in the same proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1. So that, while the number n increases successively from 2 to 4, 8, 16, 32, 64, *&c, ad infinitum*, the proportion of the ratio of $\overline{1+k}|^{\frac{1}{n}}$ to 1 to the ratio of $\overline{1+q}|^{\frac{1}{n}}$ to 1 remains always the same, and is equal to the proportion of the ratio of $1 + k$ to 1 to the ratio of $1 + q$ to 1.

But, though the proportion of the ratio of $\overline{1+k}|^{\frac{1}{n}}$ to 1 to the ratio of $\overline{1+q}|^{\frac{1}{n}}$ to 1 continues always the same during all the increase of the number n from 2 to 4, 8, 16, 32, 64, *&c, ad infinitum*, the proportion of $\overline{1+k}|^{\frac{1}{n}} - 1$, or the excess of the first, or least, of the first set of mean proportionals above 1, to $\overline{1+q}|^{\frac{1}{n}} - 1$, or the excess of the first, or least, of the second set of mean proportionals above 1, does not continue always the same, but is continually varying. Thus, for example, if k be $= \frac{9}{10}$ and q be $= \frac{5}{10}$, and consequently $1 + k$ be $= 1 + \frac{9}{10}$, or 1.9, and $1 + q$ be $= 1 + \frac{5}{10}$, or 1.5, and n be supposed to be successively equal to 2, 4, and 8, we shall have $\overline{1+k}|^{\frac{1}{n}}$ successively equal to $\overline{1.9}|^{\frac{1}{2}}$, $\overline{1.9}|^{\frac{1}{4}}$, and $\overline{1.9}|^{\frac{1}{8}}$, that is, to 1.378,404, 1.174,054, and 1.083,538, and $\overline{1+q}|^{\frac{1}{n}}$ successively equal to $\overline{1.5}|^{\frac{1}{2}}$, $\overline{1.5}|^{\frac{1}{4}}$, and $\overline{1.5}|^{\frac{1}{8}}$, that is, to 1.224,744, 1.106,681, and 1.051,989. Therefore $\overline{1+k}|^{\frac{1}{n}} - 1$ will be successively equal to 0.378,404, 0.174,054, and 0.083,538, and $\overline{1+q}|^{\frac{1}{n}} - 1$ will be successively equal to 0.224,744, 0.106,681, and 0.051,989. Now, if we compare together the ratios of $\frac{9}{10}$ to $\frac{5}{10}$, or 0.900,000 to 0.500,000, or k and q, or $\overline{1+k}|^{1} - 1$ and $\overline{1+q}|^{1} - 1$, and of 0.378,404 to 0.224,744, or $\overline{1+k}|^{\frac{1}{2}} - 1$ to $\overline{1+q}|^{\frac{1}{2}} - 1$, and of 0.174,054 to 0.106,681, or $\overline{1+k}|^{\frac{1}{4}} - 1$ to $\overline{1+q}|^{\frac{1}{4}} - 1$, and of 0.083,538 to 0.051,989, or $\overline{1+k}|^{\frac{1}{8}} - 1$ to $\overline{1+q}|^{\frac{1}{8}} - 1$, we shall find that the first of these ratios is greater than the second, and the second than the

the third, and the third than the fourth; or that the ratio of $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$ decreaſes continually while the number n increaſes. For, in the firſt place, the ratio of 0.900,000 to 0.500,000 is equal to that of 0.378,404 to 0.210,224, and therefore is greater than that of 0.378,404 to 0.224,744; and, in the ſecond place, the ratio of 0.378,404 to 0.224,744 is equal to the ratio of 0.174,054 to 0.103,375, and therefore is greater than that of 0.174,054 to 0.106,681; and, laſtly, the ratio of 0.174,054 to 0.106,681 is equal to the ratio of 0.083,538 to 0.051,202, and therefore is greater than that of 0.083,538 to 0.051,989. Or, if we make 1 the common antecedent of theſe four ratios (by which their decreaſe while the number n increaſes will become more apparent), the firſt of theſe ratios, to wit, that of 0.900,000 to 0.500,000, will be equal to that of 1 to 0.555,555; the ſecond, to wit, that of 0.378,404 to 0.224,744, will be equal to that of 1 to 0.593,926; the third, to wit, that of 0.174,054 to 0.106,681, will be equal to that of 1 to 0.612,918; and the fourth, to wit, that of 0.083,538 to 0.051,989, will be equal to that of 1 to 0.622,339; ſo that the ſaid four ratios are equal to the four ratios following, to wit,

That of 1 to 0.555,555,
That of 1 to 0.593,926,
That of 1 to 0.612,918,
And That of 1 to 0.622,339;

of which it is evident, the firſt is greater than the ſecond, the ſecond than the third, and the third than the fourth.

It appears therefore that the ratio of $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$ decreaſes continually while the number n increaſes.

But, though this ratio decreaſes continually while the number n increaſes, it does not decreaſe *ad infinitum* ſo as to become at laſt equal to nothing, or to a ratio of equality (which is a ratio of no magnitude at all), but approaches only to a certain ratio of majority as its limit. And this ratio of majority, to which it ſo approaches as its limit, and to which it may, by increaſing the number n, be made to come as near as we pleaſe, is ſhewn in lemma 1, article 11, to be the ratio, or proportion, of the ratio of $\overline{1+k}|^{\frac{1}{n}}$ to 1 to the ratio of $\overline{1+q}|^{\frac{1}{n}}$ to 1, or the proportion of the n^{th} part of the ratio of $1+q$ to 1 to the n^{th} part of the ratio of $1+q$ to 1, and conſequently is equal to the proportion of the whole ratio of $1+k$ to 1, to the whole ratio of $1+q$ to 1; that is, the fixt ratio, or proportion, of the ratio of $1+k$ to 1 to the ratio of $1+q$ to 1, is the limit of the varying ratio of $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$.

But, by lemma 2, coroll. 2, it appears, that, by increaſing the number n, the ratio of the quantity $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\frac{1}{n} \times$ the infinite ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c may be made to come as near as we pleaſe to a ratio of equality; and the ratio of the quantity $\overline{1+q}|^{\frac{1}{n}} - 1$ to $\frac{1}{n} \times$ the infinite ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c may likewiſe be made to come as near as we pleaſe to a ratio of equality. Now, if theſe two ratios were abſolutely ratios of equality, it

it would follow that the ratio of $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$ would be equal to the ratio of $\frac{1}{n} \times$ the infinite ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c* to $\frac{1}{n}$ $\times$ the infinite ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c*, and conſequently to the ratio of n times $\frac{1}{n} \times$ the infinite ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6}$ $+$ *&c* to n times $\frac{1}{n} \times$ the infinite ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c*, or to the ratio of the whole ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* to the whole ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c ad infinitum.* Therefore, while the number n increaſes, the ratio of $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$ continually approaches to the ratio of the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6}$ $+$ *&c ad infinitum* to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c ad infinitum*, and may be made to come as near to it as we pleaſe; or, in other words, the fixt ratio of the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c ad infinitum* is the limit of the varying ratio of $\overline{1+k}|^{\frac{1}{n}} - 1$ to $\overline{1+q}|^{\frac{1}{n}} - 1$.

But we before ſhewed that the fixt ratio, or proportion, of the ratio of $1 + k$ to 1 to the ratio of $1 + q$ to 1 was alſo the limit of the varying ratio of $\overline{1+k}|^{\frac{1}{n}} -$ 1 to $\overline{1+q}|^{\frac{1}{n}} - 1$. It follows therefore that the fixt ratio, or proportion, of the ratio of $1 + k$ to 1 to the ratio of $1 + q$ to 1 muſt be equal to the fixt ratio of the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* to the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c ad infinitum*; or, in other words, the two ſerieſes $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* and $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ *&c ad infinitum* are proportional to, or meaſures of, the two ratios of $1 + k$ to 1 and $1 + q$ to 1. Q. E. D.

The foregoing Demonſtration expreſſed in fewer Words.

105. This reaſoning may be expreſſed more conciſely by adopting the notation uſed in article 103; which may be done as follows.

Let the ratio of $1 + k$ to 1 be denoted by the letter E, and the ratio of $1 + q$ to 1 be denoted by the letter F; and let the ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5}$ $- \frac{k^6}{6} +$ *&c ad infinitum* be denoted by the letter G, and the ſeries $q - \frac{q^2}{2} + \frac{q^3}{3}$

$-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ *&c ad infinitum* be denoted by the letter H. And let the quantities $\overline{1+k}|^{\frac{1}{n}}-1$ and $\overline{1+q}|^{\frac{1}{n}}-1$ (which vary continually in their magnitude while the number n increases, and of which the ratio varies at the same time,) be denoted, respectively, by the letters X and Y.

Then will the ratio of $\overline{1+k}|^{\frac{1}{n}}$ to 1 be $=\frac{E}{n}$, and the ratio of $\overline{1+q}|^{\frac{1}{n}}$ to 1 will be $=\frac{F}{n}$, and $\frac{1}{n}\times$ the series $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ *&c ad infinitum* will be $=\frac{G}{n}$, and $\frac{1}{n}\times$ the series $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ *&c ad infinitum* will be $=\frac{H}{n}$.

Now it is shewn in lemma 1, that the ratio of $\frac{E}{n}$ to $\frac{F}{n}$ (which is a fixt, or given, ratio, and is equal to the ratio of E to F) is the limit of the varying ratio of X to Y. And consequently the ratio of E to F (which is equal to the ratio of $\frac{E}{n}$ to $\frac{F}{n}$,) will be the limit of the varying ratio of X to Y.

And it is shewn in lemma 2, coroll. 2, that the ratio of X to $\frac{G}{n}$ approaches continually, while the number n increases, to a ratio of equality as its limit; and that the ratio of Y to $\frac{H}{n}$ approaches continually at the same time to a ratio of equality as its limit. But if X were absolutely equal to $\frac{G}{n}$, and Y were absolutely equal to $\frac{H}{n}$, the ratio of X to Y would be equal to the ratio of $\frac{G}{n}$ to $\frac{H}{n}$. Therefore, while the number n increases, and the ratio of X to $\frac{G}{n}$ approaches to a ratio of equality as its limit, and the ratio of Y to $\frac{H}{n}$ approaches also to a ratio of equality as its limit, the ratio of X to Y will approach continually to the ratio of $\frac{G}{n}$ to $\frac{H}{n}$, as its limit. But the ratio of $\frac{G}{n}$ to $\frac{H}{n}$ is equal to the ratio of G to H. Therefore the ratio of X to Y will approach to the ratio of G to H as its limit; or the fixt ratio of G to H will be the limit of the varying ratio of X to Y.

But it was before shewn, that the fixt ratio of E to F is also the limit of the varying ratio of X to Y.

It follows therefore that the fixt ratio of G to H must be equal to the fixt ratio of E to F, or that G will be to H in the same proportion as E to F; that is, the series $k-\frac{k^2}{2}+\frac{k^3}{3}-\frac{k^4}{4}+\frac{k^5}{5}-\frac{k^6}{6}+$ *&c ad infinitum* will be to the series $q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}-\frac{q^6}{6}+$ *&c ad infinitum* in the same proportion as the ratio of $1+k$ to 1 to the ratio of $1+q$ to 1. Q. E. D.

106. The

106. The reafonings ufed in demonftrating the other theorem in article 66, (which relates to the feries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} +$ *&c ad infinitum*, invented by Dr. Wallis,) might be illuftrated in the fame manner as thofe ufed in demonftrating the firft theorem, concerning Mercator's feries. But this, I apprehend, would prove tedious to my readers, after the very ample difcuffion of the fubject in the cafe of the faid firft theorem. I fhall therefore here put an end to thefe remarks.

AN

APPENDIX

TO THE

FOREGOING REMARKS

ON THE

Two logarithmick Seriefes of Mr. Nicholas Mercator and Dr. John Wallis:

CONTAINING

Inveftigations of two other infinite Seriefes, which were publifhed by Dr. Edmund Halley, and which are related to, and derived from, the two former; and by which we are enabled, when the value of either of the two former Seriefes is given, to difcover the Ratio to which it belongs, or of which it is the Logarithm.

ARTICLE I.

IN the foregoing remarks it has been fhewn, that if k be any quantity not greater than 1, and q be any quantity lefs than k, the infinite feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c will be to the infinite feries $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} +$ &c in the fame proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1, or that the faid two feriefes are meafures, or logarithms, of the faid two ratios: and it has alfo been fhewn, that if k be of any magnitude

 lefs

lefs than 1, and q be lefs than k, the infinite feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ will be to the infinite feries $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$ in the fame proportion as the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$, or that the faid two feriefes are meafures, or logarithms, of the faid two ratios. Now Dr. Edmund Halley, in his ingenious tract on logarithms, publifhed in the Philofophical Tranfactions, No. 216, and republifhed with Sherwin's Mathematical Tables, has given us two additional infinite feriefes relating to this fubject; by the former of which we may, from the value of the feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum*, derive the value of its firft term k, and confequently that of $1 + k$, and thereby difcover the ratio of $1 + k$ to 1, of which the faid feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum* is the logarithm; and by the latter of which we may, from the value of the feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ *ad infinitum*, derive the value of its firft term k, and confequently that of $1 - k$, and thereby difcover the ratio of 1 to $1 - k$, of which the faid feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ *ad infinitum* is the logarithm. The former of thefe feriefes, given us by Dr. Halley, is $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$ *ad infinitum*, in which L reprefents the value of the whole infinite feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^5}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$; and the latter of them is $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \&c$ *ad infinitum*, in which L reprefents the value of the whole infinite feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$. The former of them had been firft invented by Sir Ifaac Newton about the year 1669, but was not publifhed till many years after; and the latter was invented by the learned Mr. Leibnitz before the year 1676, but not publifhed till many years after. But Dr. Halley, in his tract on this fubject, does not mention either Newton or Leibnitz as the inventor of either of thefe feriefes; and therefore it feems probable that he did not know that thofe great men had invented them: and, in this cafe, he muft alfo be confidered as an inventor of them, though not as the firft inventor. The former of thefe feriefes is mentioned by Newton in his firft letter to Mr. Oldenburg, dated June 13, 1676; and the latter feries is mentioned by Mr. Leibnitz in his letter to the fame perfon, dated Auguft 27, 1676. See the *Commercium Epiftolicum*, pages 139 and 150.

Thefe two feriefes confifting of the powers of L may not improperly be called *anti-logarithmick feriefes*. The former of them, to wit, $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$ *ad infinitum*, may be derived from Mercator's feries

$k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum* by the help of Sir Iſaac Newton's binomial theorem, in the firſt and ſimpleſt caſe of it, or that in which the index of the power to which any binomial quantity, as $1 + x$, is to be raiſed, is a poſitive whole number. And the latter of them, to wit, $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, may be derived in the like manner from Dr. Wallis's ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, by the help of Sir Iſaac Newton's theorem for finding the powers of a reſidual quantity, ſuch as $1 - x$, in the firſt, or ſimpleſt, caſe of it, or that in which the index of the power to which ſuch reſidual quantity is to be raiſed, is a poſitive whole number. The manner in which the former of theſe anti-logarithmick ſerieſes, to wit, $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, may be derived from Mercator's logarithmick ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum*, may be explained as follows.

Of the Anti-logarithmick Infinite Series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ + &c.

2. We are therefore now to ſhew, that, if k be any quantity not greater than 1, and L be equal to the infinite ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c*, or to the logarithm of the ratio of $1 + k$ to 1, the firſt term k of the ſaid ſeries will be equal to the infinite ſeries $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ + *&c*; the terms of which will always decreaſe with a conſiderable degree of ſwiftneſs, becauſe both the co-efficients of the powers of L, to wit, $1, \frac{1}{2}, \frac{1}{2.3}, \frac{1}{2.3.4}, \frac{1}{2.3.4.5}, \frac{1}{2.3.4.5.6}$, *&c*, decreaſe with a very great, and an increaſing, degree of ſwiftneſs, and the powers of L themſelves, to wit, $L, L^2, L^3, L^4, L^5, L^6$, *&c*, will always decreaſe in ſome degree, becauſe the greateſt poſſible magnitude of L is leſs than 1, being equal to the decimal fraction 0.693,147,180,559,945,304, *&c*, or to the value of the infinite ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c* when k is of its greateſt poſſible magnitude, or is equal to 1, that is, to the value of the infinite ſeries $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ *&c ad infinitum*, or to the logarithm of the ratio of $1 + 1$, or 2, to 1, which has been ſhewn, in article 35 of the foregoing remarks, to be = 0.693,147,180,559,945,304,

 &c

&c. And, when L is the logarithm of a much smaller ratio than that of 2 to 1, as, for example, of the ratio of 10 to 9, or of the ratio of 11 to 10, (in the former of which cases it is equal to 0.105,360,515,657,826,302, *&c,* and in the latter to 0.095,310,179,804,324,858, *&c,*) the series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum* will converge with great swiftness.

An Investigation of the said Series.

3. In order to derive this *anti-logarithmick* series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum,* which is equal to the first term k of Mr. Mercator's logarithmick series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6}$ + *&c ad infinitum,* from the said latter series, it will be convenient to premise the following lemma.

LEMMA I.

If n denote any whole number whatsoever, the n^{th} power of the binomial quantity $1 + x$ will be equal to the following series of terms continued to $n + 1$ terms, to wit, $1 + \frac{n}{1} \times x + \frac{n}{1} \times \frac{n-1}{2} \times x^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times x^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times x^5 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times x^6$ + *&c,* or (if we put the capital letters A, B, C, D, E, F, G, *&c* for the first term 1, and the co-efficients of x, x^2, x^3, x^4, x^5, x^6, and the following powers of x, in the 2nd, 3rd, 4th, 5th, 6th, 7th, and other following terms of the series respectively,) to the series $1 + \frac{n}{1} A x + \frac{n-1}{2} B x^2 + \frac{n-2}{3} C x^3 + \frac{n-3}{4} D x^4 + \frac{n-4}{5} E x^5 + \frac{n-5}{6} F x^6$ + *&c* continued to $n + 1$ terms; in which the law of the continuation of the terms is very manifest, to wit, that the numerators n, $n-1$, $n-2$, $n-3$, $n-4$, $n-5$, *&c,* of the generating fractions $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, $\frac{n-4}{5}$, $\frac{n-5}{6}$, *&c* are formed from each other by the continual subtraction of an unit until the number n is exhausted, and the denominators 1, 2, 3, 4, 5, 6, *&c* of the same fractions are formed from each other by the continual addition of an unit.

4. This is the famous binomial theorem of Sir Isaac Newton, in its first and simplest case, or that in which the index of the power to which the binomial quantity

quantity is to be raifed, is a pofitive whole number. Its truth is univerfally admitted by mathematicians; and fome of them have given demonftrations of it; and, amongft the reft, Mr. James Bernoüilli, (a very clear and fatisfactory writer on mathematical fubjects,) in his excellent treatife on the doctrine of chances, intitled *Ars Conjectandi.*—See the faid treatife, book 2nd.——On the prefent occafion I fhall take this propofition for granted.

5. Coroll. 1. If x be taken equal to the n^{th} part of any given quantity called b, fo that $1 + x$ fhall be equal to $1 + \frac{b}{n}$, it will follow that $\overline{1 + x}|^n$, or $\overline{1 + \frac{b}{n}}|^n$, will be equal to the feries $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} +$ &c continued to $n + 1$ terms, or to the feries $1 + \frac{n}{1} \text{A} \times \frac{b}{n} + \frac{n-1}{2} \text{B} \times \frac{b^2}{n^2} + \frac{n-2}{3} \text{C} \times \frac{b^3}{n^3} + \frac{n-3}{4} \text{D} \times \frac{b^4}{n^4} + \frac{n-4}{5} \text{E} \times \frac{b^5}{n^5} + \frac{n-5}{6} \text{F} \times \frac{b^6}{n^6} +$ &c continued to $n + 1$ terms.

6. Coroll. 2. Whatever be the magnitude of n, the quantity $\overline{1 + \frac{b}{n}}|^n$ muft be lefs than the feries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} +$ &c continued to $n + 1$ terms.

For the feries $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} +$ &c continued to $n + 1$ terms, is evidently lefs than the feries $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{n-0}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{n-0}{4} \times \frac{n-0}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{n-0}{4} \times \frac{n-0}{5} \times \frac{n-0}{6} \times \frac{b^6}{n^6} +$ &c continued to the fame number of terms, or than the feries $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n^2}{2} \times \frac{b^2}{n^2} + \frac{n^3}{2.3} \times \frac{b^3}{n^3} + \frac{n^4}{2.3.4} \times \frac{b^4}{n^4} + \frac{n^5}{2.3.4.5} \times \frac{b^5}{n^5} + \frac{n^6}{2.3.4.5.6} \times \frac{b^6}{n^6} +$ &c continued to the fame number of terms, or than the feries $1 + \frac{nb}{n} + \frac{n^2 b^2}{2n^2} + \frac{n^3 b^3}{2.3.n^3} + \frac{n^4 b^4}{2.3.4.n^4} + \frac{n^5 b^5}{2.3.4.5.n^5} + \frac{n^6 b^6}{2.3.4.5.6.n^6} +$ &c continued to the fame number of terms, or than the feries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} +$

$\frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c* continued to the ſame number of terms. But by coroll. 1, the ſeries $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6}$ + *&c* continued to $n + 1$ terms is equal to $\overline{1 + \frac{b}{n}}\Big|^{n}$. Therefore $\overline{1 + \frac{b}{n}}\Big|^{n}$ muſt be leſs than the ſeries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n + 1$ terms. Q. E. D.

7. Coroll. 3. But though $\overline{1 + \frac{b}{n}}\Big|^{n}$ be always leſs than the ſeries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n + 1$ terms, yet the difference between them will continually decreaſe while the index n increaſes; and the ſaid index n may be taken of ſo great a magnitude that the ſaid difference ſhall become leſs than any aſſigned quantity, how ſmall ſoever.

For the ſeries $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6}$ + *&c* continued to $n + 1$ terms (to which the quantity $\overline{1 + \frac{b}{n}}\Big|^{n}$ is equal) is equal to the ſeries $1 + \frac{n}{n} \times b + \frac{n^2 - n}{n^2} \times \frac{b^2}{2} + \frac{n^3 - 3n^2 + 2n}{n^3} \times \frac{b^3}{2.3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4} \times \frac{b^4}{2.3.4} + \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{n^5} \times \frac{b^5}{2.3.4.5} + \frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{n^6} \times \frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n + 1$ terms, and conſequently to the ſeries $1 + b + \overline{1 - \frac{1}{n}} \times \frac{b^2}{2} + \overline{1 - \frac{3}{n} + \frac{2}{n^2}} \times \frac{b^3}{2.3} + \overline{1 - \frac{6}{n} + \frac{11}{n^2} - \frac{6}{n^3}} \times \frac{b^4}{2.3.4} + \overline{1 - \frac{10}{n} + \frac{35}{n^2} - \frac{50}{n^3} + \frac{24}{n^4}} \times \frac{b^5}{2.3.4.5} + \overline{1 - \frac{15}{n} + \frac{85}{n^2} - \frac{225}{n^3} + \frac{274}{n^4} - \frac{120}{n^5}} \times \frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n + 1$ terms, or to the ſeries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n + 1$ terms, — the ſeries $\frac{1}{n} \times \frac{b^2}{2} + \overline{\frac{3}{n} - \frac{2}{n^2}} \times \frac{b^3}{2.3} + \overline{\frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3}} \times \frac{b^4}{2.3.4} + \overline{\frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4}} \times \frac{b^5}{2.3.4.5} + \overline{\frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5}} \times \frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n - 1$ terms; the difference of which from the ſeries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} +$

$\frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + &c continued to $n+1$ terms, is the series $\frac{1}{n} \times \frac{b^2}{2} + \overline{\frac{3}{n} - \frac{2}{n^2}} \times \frac{b^3}{2.3} + \overline{\frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3}} \times \frac{b^4}{2.3.4} + \overline{\frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4}} \times \frac{b^5}{2.3.4.5} + \overline{\frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5}} \times \frac{b^6}{2.3.4.5.6}$ + &c continued to $n-1$ terms. Therefore this last series is equal to the difference whereby the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + &c continued to $n+1$ terms, exceeds the quantity $\overline{1 + \frac{b}{n}}|^n$. Now it is evident, that the said series $\frac{1}{n} \times \frac{b^2}{2} + \overline{\frac{3}{n} - \frac{2}{n^2}} \times \frac{b^3}{2.3} + \overline{\frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3}} \times \frac{b^4}{2.3.4} + \overline{\frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4}} \times \frac{b^5}{2.3.4.5} + \overline{\frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5}} \times \frac{b^6}{2.3.4.5.6}$ + &c continued to $n-1$ terms, will continually decrease while n increases, because the powers of n constitute the denominators of the several members of the co-efficients of its terms; and n may be taken of so great a magnitude that the said series shall be less than any assigned quantity, how small soever. Therefore, while the index n increases, the difference whereby the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + &c continued to $n+1$ terms, exceeds the quantity $\overline{1 + \frac{b}{n}}|^n$, will decrease continually *ad infinitum*, or the index n may be taken of so great a magnitude that the said difference shall be less than any assigned quantity, how small soever.

Q. E. D.

8. Coroll. 4. It follows, from the foregoing corollary, that the infinite series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + &c *ad infinitum* is the limit of the magnitude of $\overline{1 + \frac{b}{n}}|^n$, or the quantity to which $\overline{1 + \frac{b}{n}}|^n$ may, by increasing continually the index n, be made to come as near as we please.

For, by increasing n continually, the excess of the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + &c continued to $n+1$ terms, above the quantity $\overline{1 + \frac{b}{n}}|^n$ may be made to decrease till it is less than any assigned quantity, how small soever, as we have seen in the foregoing corollary. And, by increasing n continually, it is evident that the excess of the infinite series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + &c *ad infinitum* above the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + &c continued to $n+1$ terms, may be diminished as far as we please, or made to become less than any assigned

 quantity,

quantity, how ſmall ſoever. It follows therefore, that, by increaſing the index n continually, the ſum of theſe two exceſſes, or the exceſs of the infinite ſeries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c ad infinitum* above the quantity $\overline{1 + \frac{b}{n}}\Big|^{n}$, may be diminiſhed as far as we pleaſe, or made to become leſs than any aſſigned quantity, how ſmall ſoever. Or, in other words, the infinite ſeries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c ad infinitum* is the limit of the magnitude of the quantity $\overline{1 + \frac{b}{n}}\Big|^{n}$. Q. E. D.

9. Coroll. 5. If we make uſe of the language of infinites, (which, though not a correct way of ſpeaking, is often very convenient on account of its brevity, and has therefore been ſometimes adopted even by the moſt careful and elegant writers on mathematical ſubjects; and, amongſt the reſt, by the great Mr. Huygens himſelf, who deſerves that character above all other writers;) the ſubſtance of the laſt, or 4th, corollary may be expreſſed in the manner following.

When the index n of the power to which the binomial quantity $1 + \frac{b}{n}$ is to be raiſed, becomes infinite, the ſeries $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6}$ + *&c* continued to $n + 1$ terms, (to which $\overline{1 + \frac{b}{n}}\Big|^{n}$ is equal) or $1 + \frac{n}{n} \times b + \frac{nn - n}{nn} \times \frac{b^2}{2} + \frac{n^3 - 3n^2 + 2n}{n^3} \times \frac{b^3}{2.3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4} \times \frac{b^4}{2.3.4} + \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{n^5} \times \frac{b^5}{2.3.4.5} + \frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{n^6} \times \frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n + 1$ terms, becomes infinite in the number of its terms, and equal to the ſeries $1 + \frac{n}{n} \times b + \frac{nn}{nn} \times \frac{b^2}{2} + \frac{n^3}{n^3} \times \frac{b^3}{2.3} + \frac{n^4}{n^4} \times \frac{b^4}{2.3.4} + \frac{n^5}{n^5} \times \frac{b^5}{2.3.4.5} + \frac{n^6}{n^6} \times \frac{b^6}{2.3.4.5.6}$ + *&c ad infinitum*, or $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c ad infinitum*, becauſe all the members of the ſeveral numerators $nn - n$, $n^3 - 3n^2 + 2n$, $n^4 - 6n^3 + 11n^2 - 6n$, $n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$, and $n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n$, *&c*, after the firſt members n^2, n^3, n^4, n^5, n^6, *&c*, will vaniſh, or become infinitely ſmall, in reſpect of the ſaid firſt members, which contain a higher power of n than the ſaid following members; and conſequently thoſe whole numerators will be equal to their ſaid firſt members, n^2, n^3, n^4, n^5, n^6, *&c*, reſpectively. Therefore the quantity $\overline{1 + \frac{b}{n}}\Big|^{n}$ (which in all caſes is equal to the ſaid ſeries $1 +$

$\frac{n}{n} \times b + \frac{nn - n}{nn} \times \frac{b^2}{2} + \frac{n^3 - 3n^2 + 2n}{n^3} \times \frac{b^3}{2.3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4} \times \frac{b^4}{2.3.4} +$ $\frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{n^5} \times \frac{b^5}{2.3.4.5} + \frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{n^6} \times$ $\frac{b^6}{2.3.4.5.6}$ + *&c* continued to $n + 1$ terms,) will, in this cafe of the infinite magnitude of n, be equal to the infinite feries $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ + *&c ad infinitum.* Q. E. D.

10. Thefe things being premifed, the inveftigation of the firft anti-logarithmick feries $\text{L} + \frac{\text{L}^2}{2} + \frac{\text{L}^3}{2.3} + \frac{\text{L}^4}{2.3.4} + \frac{\text{L}^5}{2.3.4.5} + \frac{\text{L}^6}{2.3.4.5.6}$ + *&c ad infinitum*, or the derivation of it from Mr. Mercator's logarithmick feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6}$ + *&c ad infinitum*, may be performed in the manner following.

PROBLEM I.

Let k be any quantity not greater than 1, and L be equal to the infinite feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6}$ + *&c ad infinitum*; which is the logarithm of the ratio of $1 + k$ to 1 in Napier's fyftem of logarithms, and which confequently can never be greater than the logarithm of the ratio of $1 + 1$, or 2, to 1 in that fyftem, or than the decimal fraction 0.693,147,180,559,945,304, *&c*. And let L, or the value of this whole feries, be fuppofed to be known, but k, or the value of its firft term alone, and confequently the feparate values of all its other terms (which contain the powers of k,) to be unknown. It is required to find from L, or the value of the whole feries, the value of its firft term k.

SOLUTION.

Let us fuppofe the letter n to denote fome very great number, fuch as a nonillion, or the ninth power of a million.

Then it will follow, from lemma 2, coroll. 2, art. 17, of the foregoing remarks on Mercator's and Wallis's feriefes, that $\overline{1 + k}^{\frac{1}{n}} - 1$ will be very nearly equal to $\frac{1}{n} \times$ the infinite feries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^5}{6}$ + *&c ad infinitum*, that is, (becaufe L is fuppofed to be equal to the faid feries,) very nearly equal to $\frac{1}{n} \times \text{L}$, or $\frac{\text{L}}{n}$. Therefore, if we add 1 to both fides of the equation, we fhall have $\overline{1 + k}^{\frac{1}{n}}$ very nearly equal to $1 + \frac{\text{L}}{n}$, and (raifing both

 fides

sides of this last equation to the n^{th} power,) $1 + k$ very nearly equal to $\overline{1 + \frac{L}{n}}|^n$. And n may be taken of so great a magnitude that the ratio of $1 + k$ to $\overline{1 + \frac{L}{n}}|^n$ shall approach as near as we please to a ratio of equality.

But, because the index n is of an immensely-great magnitude, it follows, from the fourth corollary of the foregoing lemma, that $\overline{1 + \frac{L}{n}}|^n$ will be very nearly equal to the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, and that n may be taken of so great a magnitude that the difference between $\overline{1 + \frac{L}{n}}|^n$ and this series shall be less than any assigned quantity, how small soever. Therefore $1 + k$ will be very nearly equal to the said infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, and k will be very nearly equal to the infinite series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*. And n may be taken of so great a magnitude that the ratio of $1 + k$ to the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *ad infinitum* shall approach as near to a ratio of equality as we please. Therefore $1 + k$ must be not only very nearly, but accurately, equal to the said series, and consequently k will be not only very nearly, but accurately, equal to the infinite series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*.

Q. E. I.

11. This solution may be expressed with greater brevity in the language of infinites, as follows.

Let us suppose the letter n to denote a number infinitely great.

Then it will follow, from lemma 2, coroll. 2, art. 17, of the foregoing remarks, that $\overline{1 + k}|^{\frac{1}{n}} - 1$ will be equal to $\frac{1}{n} \times$ the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum*, that is, (because the said series is $= L$,) to $\frac{1}{n} \times L$, or to $\frac{L}{n}$. Therefore (adding 1 to both sides of the equation) we shall have $\overline{1 + k}|^{\frac{1}{n}} = 1 + \frac{L}{n}$, and consequently (raising both sides of this last equation to the n^{th} power,) $1 + k = \overline{1 + \frac{L}{n}}|^n$.

But, because the index n is an infinitely-great number, it follows, from coroll. 5 of

5 of the foregoing lemma, that $\overline{1 + \frac{L}{n}}\big|^n$ will be equal to the infinite feries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum.* Therefore $1 + k$ will be equal to the fame infinite feries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^5}{2.3.4.5.6} +$ *&c ad infinitum,* and confequently k will be equal to the infinite feries $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum.* Q. E. I.

12. Coroll. 1. Since k is equal to the infinite feries $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum,* it follows that the ratio of $1 + k$ to 1 is equal to the ratio of the infinite feries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* to 1; or the ratio correfponding to any given logarithm in Napier's fyftem denoted by L (that is not greater than 0.693,147,180, 559,945,304, *&c,* or the logarithm of the ratio of 2 to 1,) is the ratio of the infinite feries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* to 1.

An Example of the foregoing Method of difcovering the Ratio correfponding to a given Logarithm L *in Napier's Syftem, by Means of the Infinite Series* $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ &c ad infinitum.

13. Let the given logarithm, of which we are to find the ratio, be $\frac{1}{2}$, or 0.500,000,000.

Then will L^2 be $= \frac{1}{4}$, and L^3 be $= \frac{1}{8}$, and $L^4 = \frac{1}{16}$, and $L^5 = \frac{1}{32}$, and $L^6 = \frac{1}{64}$, and L^7, L^8, L^9, L^{10}, *&c* equal to the 7th, 8th, 9th, 10th, and other following powers of $\frac{1}{2}$, and confequently the feries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c* will be equal to $1 + \frac{1}{2} + \frac{1}{4.2} + \frac{1}{8.2.3} + \frac{1}{16.2.3.4} + \frac{1}{32.2.3.4.5} + \frac{1}{64.2.3.4.5.6} +$ *&c*; or if we denote thefe feveral terms by the capital letters A, B, C, D, E, F, *&c,* fo that the firft term 1 fhall be = A, and the fecond term $\frac{1}{2}$ fhall be B, and the third term $\frac{1}{4.2}$ fhall be = C, and the fourth, fifth, fixth, feventh, eighth, and other following terms, fhall be equal to D, E, F, G, H, *&c,* the

 feries

series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ &c will be equal to $1 + \frac{A}{2} + \frac{B}{2.2} + \frac{C}{2.3} + \frac{D}{2.4} + \frac{E}{2.5} + \frac{F}{2.6} + \frac{G}{2.7} + \frac{H}{2.8} + \frac{I}{2.9} + \frac{K}{2.10} +$ &c *ad infinitum*. These terms may be computed in the manner following.

$$A = 1.000,000,000;$$

$$B \left(= \frac{A}{2} = \frac{1.000,000,000}{2}\right) = 0.500,000,000;$$

$$C \left(= \frac{B}{2.2} = \frac{B}{4} = \frac{0.500,000,000}{4}\right) = 0.125,000,000;$$

$$D \left(= \frac{C}{2.3} = \frac{C}{6} = \frac{0.125,000,000}{6}\right) = 0.020,833,333;$$

$$E \left(= \frac{D}{2.4} = \frac{D}{8} = \frac{0.020,833,333}{8}\right) = 0.002,604,166;$$

$$F \left(= \frac{E}{2.5} = \frac{E}{10} = \frac{0.002,604,166}{10}\right) = 0.000,260,416;$$

$$G \left(= \frac{F}{2.6} = \frac{F}{12} = \frac{0.000,260,416}{12}\right) = 0.000,021,701;$$

$$H \left(= \frac{G}{2.7} = \frac{G}{14} = \frac{0.000,021,701}{14}\right) = 0.000,001,550;$$

$$I \left(= \frac{H}{2.8} = \frac{H}{16} = \frac{0.000,001,550}{16}\right) = 0.000,000,096;$$

$$K \left(= \frac{I}{2.9} = \frac{I}{18} = \frac{0.000,000,096}{18}\right) = 0.000,000,005;$$

$$\text{And } L \left(= \frac{K}{2.10} = \frac{K}{20} = \frac{0.000,000,005}{20}\right) = 0.000,000,000.$$

Therefore the series $A + B + C + D + E + F + G + H + I + K + L +$ &c, or $1 + \frac{A}{2} + \frac{B}{2.2} + \frac{C}{2.3} + \frac{D}{2.4} + \frac{E}{2.5} + \frac{F}{2.6} + \frac{G}{2.7} + \frac{H}{2.8} + \frac{I}{2.9} + \frac{K}{2.10} +$ &c will be equal to 1.000,000,000 + 0.500,000,000 + 0.125,000,000 + 0.020, 833,333 + 0.002,604,166 + 0.000,260,416 + 0.000,021,701 + 0.000,001, 550 + 0.000,000,096 + 0.000,000,005 + 0.000,000,000 + &c = 1.648, 721,267, &c; and consequently the ratio sought, or that of which the fraction $\frac{1}{2}$, or 0.500,000,000, is the logarithm in Napier's system, is the ratio of 1.648, 721,267, &c to 1. Q. E. I.

14. The ratio of the square of this number 1.648,721,267, &c to 1 is double of the ratio of this number itself to 1; and consequently the logarithm of the ratio of the square of this number to 1 will be double of the logarithm of the ratio of this number itself to 1, or will be double of $\frac{1}{2}$, or 0.500,000,000, or equal to 1. But the square of the number 1.648,621,267 is 2.718,281,816, &c. Therefore the ratio corresponding to the logarithm 1, or of which 1 is the logarithm in Napier's system, is that of 2.718,281,816, &c to 1.

SCHOLIUM.

15. In the ſolution of the foregoing problem the letter L was put for the value of the whole ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6}$ *&c ad infinitum*, in which k can never be greater than 1, and the value of which can never therefore be greater than that of the infinite ſeries $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ *&c ad infinitum*, or than the logarithm of the ratio of $1 + 1$, or 2, to 1, or than the decimal fraction 0.693,147,180,559,945,304, *&c.* And therefore it cannot be concluded from the reaſonings uſed in the ſaid ſolution, (which are grounded on the ſaid ſuppoſition, "that L is equal to the ſaid ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c*, and conſequently is ſubject to the ſaid reſtriction,") that, when L is greater than Napier's logarithm of the ratio of 2 to 1, or than the decimal fraction 0.693,147,180,559,945,304, *&c*, the ratio correſponding to it, or of which it is the logarithm in Napier's ſyſtem, is the ratio of the ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* to 1. Nevertheleſs this propoſition is true in theſe caſes as well as in the former, though it does not follow from the foregoing premiſſes. But it may be ſhewn from other premiſſes, that, whatever be the magnitude of L, even though it ſhould be much greater than the decimal fraction 0.693,147,180,559,945,304, *&c*, and even than 1, as, for example, equal to 10, or 100, or 1000, or any greater number, yet the ratio correſponding to the logarithm L, or of which L is the logarithm in Napier's ſyſtem, will ſtill be the ratio of the ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* to 1.

For though, when L is greater than 1, the powers of L, to wit, L, L^2, L^3, L^4, L^5, L^6, *&c*, which form the numerators of the terms of the ſaid ſeries, will be a ſeries of increaſing quantities, every one of which will be greater than that which immediately precedes it in the proportion of L to 1, yet the increaſe of the whole terms L, $\frac{L^2}{2}$, $\frac{L^3}{2.3}$, $\frac{L^4}{2.3.4}$, $\frac{L^5}{2.3.4.5}$, $\frac{L^6}{2.3.4.5.6}$, *&c ad infinitum*, ariſing from ſuch increaſe of their numerators L, L^2, L^3, L^4, L^5, L^6, *&c*, will, in ſome part or other of the ſeries, be more than counter-balanced by the decreaſe of them ariſing from the increaſe of their denominators 2, 2.3, 2.3.4, 2.3.4.5, 2.3.4.5.6, *&c*; and conſequently the ſaid whole terms (though they will have diverged, or increaſed, continually from the beginning of the ſeries to that term at which the ſaid over-balancing takes place) will afterwards converge, or decreaſe, continually, and that with an increaſing velocity, or faſter than in a continued geometrical proportion; and therefore the value, or ſum of all the terms, of the ſaid ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c* will (notwithſtanding the divergency, or increaſe of its firſt terms,) be only of a finite magnitude.

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This convergency of the terms of this series, in some part or other of it, even when L is much greater than 1, may be easily deduced from an attentive consideration of the manner in which the denominators of its terms increase: and, further, it may be demonstrated from other premisses, that, whatever be the magnitude of L, the ratio corresponding to it, or of which it is the logarithm in Napier's system, will be the ratio of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* to 1. See below, in articles 39, 40, and 41. But this is a speculation of more curiosity than use, on account of the great labour of computing a sufficient number of the terms of this series, when L is much greater than 1, to obtain its value to any considerable degree of exactness. It may not, however, be amiss to give the reader an instance of the truth of this proposition in one remarkable case of it, that lies out of the limits of the supposition made in the foregoing problem, by computing the value of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, when L is equal to 1, and thereby determining (in case this proposition holds true in the extent we have just now assigned to it,) the ratio corresponding to 1, or of which 1 is the logarithm in Napier's system; which we shall find to be that of the number 2.718,281,816, *&c* to 1, as it has been already shewn to be in article 14. The computation of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* in this case, in which it becomes equal to the series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} +$ *&c ad infinitum*, may be performed as follows.

The Computation of the Series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ + &c ad infinitum, *when* L *is* = 1.

16. Let the first term 1 of the series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} +$ *&c ad infinitum*, to which the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c* is in this case equal, be called A, and the second term 1 be called B, and the third term $\frac{1}{2}$ be called C, and the fourth term $\frac{1}{2.3}$ be called D, and the fifth term $\frac{1}{2.3.4}$ be called E, and the sixth, seventh, eighth, ninth, and other following terms, or $\frac{1}{2.3.4.5}$, $\frac{1}{2.3.4.5.6}$, $\frac{1}{2.3.4.5.6.7}$, $\frac{1}{2.3.4.5.6.7.8}$, *&c*, be denoted by F, G, H, I, and the following capital letters of the alphabet. And the foregoing series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} + \frac{1}{2.3.4.5.6.7}$

$+ \frac{1}{2.3.4.5.6.7.8} +$ &c will become equal to the feries $1 + \frac{A}{1} + \frac{B}{2} + \frac{C}{3} + \frac{D}{4} + \frac{E}{5} + \frac{F}{6} + \frac{G}{7} + \frac{H}{8} +$ &c. Thefe terms, when reduced into decimal fractions, will be as follows.

$$A = 1.000,000,000,000,000,000,000,000;$$

$$B = \frac{A}{1} = 1.000,000,000,000,000,000,000,000;$$

$$C = \frac{B}{2} = 0.500,000,000,000,000,000,000,000;$$

$$D = \frac{C}{3} = 0.166,666,666,666,666,666,666,666;$$

$$E = \frac{D}{4} = 0.041,666,666,666,666,666,666,666;$$

$$F = \frac{E}{5} = 0.008,333,333,333,333,333,333,333;$$

$$G = \frac{F}{6} = 0.001,388,888,888,888,888,888,888;$$

$$H = \frac{G}{7} = 0.000,198,412,698,412,698,412,698;$$

$$I = \frac{H}{8} = 0.000,024,801,587,301,587,301,587;$$

$$K = \frac{I}{9} = 0.000,002,755,731,922,398,589,065;$$

$$L = \frac{K}{10} = 0.000,000,275,573,192,239,858,906;$$

$$M = \frac{L}{11} = 0.000,000,025,052,108,385,441,718;$$

$$N = \frac{M}{12} = 0.000,000,002,087,675,698,786,809;$$

$$O = \frac{N}{13} = 0.000,000,000,160,590,438,368,216;$$

$$P = \frac{O}{14} = 0.000,000,000,011,470,745,597,729;$$

$$Q = \frac{P}{15} = 0.000,000,000,000,764,716,373,181;$$

$$R = \frac{Q}{16} = 0.000,000,000,000,047,794,773,323;$$

$$S = \frac{R}{17} = 0.000,000,000,000,002,811,457,254;$$

$$T = \frac{S}{18} = 0.000,000,000,000,000,156,192,069;$$

$$V = \frac{T}{19} = 0.000,000,000,000,000,008,220,635;$$

$$\text{W} = \frac{\text{V}}{20} = 0.000{,}000{,}000{,}000{,}000{,}000{,}411{,}031\,;$$

$$\text{X} = \frac{\text{W}}{21} = 0.000{,}000{,}000{,}000{,}000{,}000{,}019{,}572\,;$$

$$\text{Y} = \frac{\text{X}}{22} = 0.000{,}000{,}000{,}000{,}000{,}000{,}000{,}889\,;$$

$$\text{Z} = \frac{\text{Y}}{23} = 0.000{,}000{,}000{,}000{,}000{,}000{,}000{,}038\,;$$

$$\text{A} = \frac{\text{Z}}{24} = 0.000{,}000{,}000{,}000{,}000{,}000{,}000{,}001.$$

$$2.718{,}281{,}828{,}459{,}045{,}235{,}360{,}274.$$

Therefore, when L is $=$ 1, the ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c* is $= 2.718{,}281{,}828{,}459{,}045{,}235{,}360{,}274$, *&c*; and the ratio correſponding to 1, or of which 1 is the logarithm in Napier's ſyſtem, is the ratio of 2.718,281,828,459,045,235,360,274, *&c* to 1; which is the ſame with the ratio aſſigned above in article 14 for the ratio correſponding to the ſame logarithm 1, to wit, the ratio of 2.718,281,816, *&c* to 1, except that it is expreſſed with more exactneſs, or to a greater number of figures.

17. Coroll. 2. If B be the logarithm of the ſame ratio in Briggs's ſyſtem of logarithms of which L is the logarithm in Napier's ſyſtem, B will be leſs than L in the proportion of Briggs's logarithm of the ratio of 10 to 1 to Napier's logarithm of the ſame ratio, that is, in the proportion of 1 to 2.302,585,092, 994,045,668, *&c*; or L will be $=$ 2.302,585,092,994,045,668, *&c* $\times$ B. Therefore, if B be any logarithm, in Briggs's ſyſtem, of which we wiſh to find the correſpondent ratio, we muſt firſt multiply B by the number 2.302, 585,092,994,045,668, *&c*, whereby we ſhall obtain the value of L, or of Napier's logarithm of the ſame ratio; and then we muſt compute as many terms of the ſeries $1 + L + \frac{L^2}{2} = \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* as may be neceſſary to expreſs the ratio we are ſeeking to the propoſed degree of exactneſs. For the proportion of the ſaid ſeries, or of the ſum of ſo many terms of it as we ſhall have computed, to 1 will be the ratio ſought.

An Example of this Method of diſcovering the Ratio correſponding to a given Logarithm in Briggs's Syſtem of Logarithms.

18. Let it be required to find the ratio correſponding to the logarithm $\frac{1}{100}$, or 0.01, in Briggs's ſyſtem of logarithms.

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The logarithm of this ratio in Napier's fyftem is $= 2.302,585,092,994,045,668$, *&c* $\times 0.01 = 0.023,025,850,929,940,456,68$, *&c.* Call this logarithm L. Then will L^2 be equal to $\overline{0.023,025,850,929,940,456,68, \&c}|^2$, or (neglecting the laft eleven figures, as inconfiderable,) equal to $\overline{0.023,025,850}|^2 = 0.000,530,189$; and L^3 will be $= 0.000,012,208$; and L^4 will be $= 0.000,000,281$; and L^5 will be $= 0.000,000,006$. Therefore $\frac{L^2}{2}$ will be $= \frac{0.000,530,189}{2} = 0.000,265,094$; and $\frac{L^3}{2.3}$ will be $= \frac{0.000,012,208}{6} = 0.000,002,034$; and $\frac{L^4}{2.3.4}$ will be $= \frac{0.000,000,281}{24} = 0.000,000,011$; and $\frac{L^5}{2.3.4.5}$ will be $= \frac{0.000,000,006}{120} = 0.000,000,000$; and confequently the feries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5}$ + *&c* will be $= 1,000,000,000 + 0.023,025,850 + 0.000,265,094 + 0.000,002,034 + 0.000,000,011 + 0.000,000,000$ + *&c* $= 1.023,292,989$, *&c.* Therefore the ratio correfponding to L, or 0.023,025,850, or of which 0.023,025,850 is the logarithm in Napier's fyftem of logarithms, or of which 0.01, or $\frac{1}{100}$, is the logarithm in Briggs's fyftem of logarithms, is the ratio of 1.023,292,989, *&c* to 1. Q. E. I.

19. This number 1.023,292,989, *&c* is the hundredth root of 10.

For the ratio of the 100th root of 10 to 1 is the 100th part of the ratio of 10 to 1; and confequently the logarithm of the ratio of the hundredth root of 10 to 1 is the hundredth part of the logarithm of the ratio of 10 to 1. But in Briggs's fyftem of logarithms 1 is the logarithm of the ratio of 10 to 1. Therefore $\frac{1}{100}$, or 0.01, is the logarithm of the ratio of the hundredth root of 10 to 1. But it has been fhewn, that 0.01 is the logarithm of the ratio of 1.023,292,989, *&c* to 1 in Briggs's fyftem of logarithms. Therefore the ratio of the hundredth root of 10 to 1 is equal to the ratio of 1.023,292,989, *&c* to 1, they having both the fame logarithm. Therefore (by El. 5, 9) the hundredth root of 10 muft be equal to 1.023,292,989, *&c.* Q. E. D.

CONCERNING THE

SECOND ANTI-LOGARITHMICK SERIES

PUBLISHED BY DR. HALLEY;

TO WIT,

The Series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum.*

20. WE come now to consider the second anti-logarithmick series invented by Dr. Halley, to wit, the series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, in which L denotes the value of the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, invented by Dr. Wallis, or the logarithm of the ratio of 1 to $1 - k$. Now, in order to derive the former of these serieses from the latter, it will be convenient to premise the following lemma.

LEMMA II.

21. If x be any quantity less than 1, and n denote any whole number whatsoever, the n^{th} power of the residual quantity $1 - x$, or of the excess of 1 above x, will be equal to the following series of terms, to wit, $1 - \frac{n}{1} \times x + \frac{n}{1} \times$

$\frac{n-1}{2} \times x^2 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times x^4 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times x^5 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times x^6 -$ &c, continued to $n + 1$ terms, or (if we put the capital letters A, B, C, D, E, F, G, &c for the first term 1, and the several co-efficients of x, x^2, x^3, x^4, x^5, x^6, and the following powers of x respectively) to the series $1 - \frac{n}{1} \text{A} x + \frac{n-1}{2} \text{B} x^2 - \frac{n-2}{3} \text{C} x^3 + \frac{n-3}{4} \text{D} x^4 - \frac{n-4}{5} \text{E} x^5 + \frac{n-5}{6} \text{F} x^6 -$ &c; in which the law of continuation of the terms is very manifest, to wit, that the terms are alternately positive and negative, or that the 3rd, 5th, 7th, and all the following odd terms, are to be added to the first term 1, and the 2nd, 4th, 6th, and other following even terms, are to be subtracted from it; and, with respect to the co-efficients of x, x^2, x^3, x^4, x^5, x^6, &c, that they are generated from the first term 1 by the continual multiplication of the generating fractions $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, $\frac{n-4}{5}$, $\frac{n-5}{6}$, &c, of which the numerators n, $n-1$, $n-2$, $n-3$, $n-4$, $n-5$, &c, are formed from each other by the continual subtraction of an unit, and the denominators 1, 2, 3, 4, 5, 6, &c, are formed from each other by the continual addition of an unit.

22. This proposition may be demonstrated in the same manner as the celebrated binomial theorem, of which it is usually considered as a branch. On the present occasion I shall take it for granted, as I did the binomial theorem itself in article 3.

23. Coroll. 1. If x be equal to the n^{th} part of any given quantity called b, so that $1 - x$ shall be equal to $1 - \frac{b}{n}$, it will follow that $\overline{1-x}|^n$, or $\overline{1 - \frac{b}{n}}|^n$, will be equal to the series $1 - \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} -$ &c, continued to $n + 1$ terms, or to the series $1 - \frac{n}{1} \text{A} \times \frac{b}{n} + \frac{n-1}{2} \text{B} \times \frac{b^2}{n^2} - \frac{n-2}{3} \text{C} \times \frac{b^3}{n^3} + \frac{n-3}{4} \text{D} \times \frac{b^4}{n^4} - \frac{n-4}{5} \text{E} \times \frac{b^5}{n^5} + \frac{n-5}{6} \text{F} \times \frac{b^6}{n^6} -$ &c, continued to $n + 1$ terms.

24. Coroll. 2. Whatever be the magnitude of n, the quantity $\overline{1 - \frac{b}{n}}|^n$ will be greater than the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} -$ &c, continued to $n + 1$ terms.

For,

For, by the foregoing corollary, we have $\overline{1-\frac{b}{n}}^{\,n} = 1 - \frac{n}{1}\times\frac{b}{n} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{b^2}{n^2} - \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{b^3}{n^3} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{b^4}{n^4} - \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}\times\frac{b^5}{n^5} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}\times\frac{n-5}{6}\times\frac{b^6}{n^6} -$ &c, continued to $n+1$ terms; which is $= 1 - b + \frac{n^2-n}{n^2}\times\frac{b^2}{2} - \overline{\frac{n^3-3n^2+2n}{n^3}}\times\frac{b^3}{2.3} + \frac{n^4-6n^3+11n^2-6n}{n^4}\times\frac{b^4}{2.3.4} - \overline{\frac{n^5-10n^4+35n^3-50n^2+24n}{n^5}}\times\frac{b^5}{2.3.4.5} + \frac{n^6-15n^5+85n^4-225n^3+274n^2-120n}{n^6}\times\frac{b^6}{2.3.4.5.6} -$ &c, continued to $n+1$ terms, or to $1 - b + \overline{1-\frac{1}{n}}\times\frac{b^2}{2} - \overline{1-\frac{3}{n}+\frac{2}{n^2}}\times\frac{b^3}{2.3} + \overline{1-\frac{6}{n}+\frac{11}{n^2}-\frac{6}{n^3}}\times\frac{b^4}{2.3.4} - \overline{1-\frac{10}{n}+\frac{35}{n^2}-\frac{50}{n^3}+\frac{24}{n^4}}\times\frac{b^5}{2.3.4.5} + \overline{1-\frac{15}{n}+\frac{85}{n^2}-\frac{225}{n^3}+\frac{274}{n^4}-\frac{120}{n^5}}\times\frac{b^6}{2.3.4.5.6} +$ &c, continued to $n+1$ terms, or to the series $1-b+\frac{b^2}{2}-\frac{b^3}{2.3}+\frac{b^4}{2.3.4}-\frac{b^5}{2.3.4.5}+\frac{b^6}{2.3.4.5.6}$ — &c, continued to $n+1$ terms, — the series $\frac{1}{n}\times\frac{b^2}{2} - \overline{\frac{3}{n}-\frac{2}{n^2}}\times\frac{b^3}{2.3} + \overline{\frac{6}{n}-\frac{11}{n^2}+\frac{6}{n^3}}\times\frac{b^4}{2.3.4} - \overline{\frac{10}{n}-\frac{35}{n^2}+\frac{50}{n^3}-\frac{24}{n^4}}\times\frac{b^5}{2.3.4.5} + \overline{\frac{15}{n}-\frac{85}{n^2}+\frac{225}{n^3}-\frac{274}{n^4}+\frac{120}{n^5}}\times\frac{b^6}{2.3.4.5.6}$ — &c, continued to $n-1$ terms. Therefore $\overline{1-\frac{b}{n}}^{\,n}$ is greater than the series $1-b+\frac{b^2}{2}-\frac{b^3}{2.3}+\frac{b^4}{2.3.4}-\frac{b^5}{2.3.4.5}+\frac{b^6}{2.3.4.5.6}$ — &c, continued to $n+1$ terms, by an excess, or difference, equal to the series $\frac{1}{n}\times\frac{b^2}{2} - \overline{\frac{3}{n}-\frac{2}{n^2}}\times\frac{b^3}{2.3} + \overline{\frac{6}{n}-\frac{11}{n^2}+\frac{6}{n^3}}\times\frac{b^4}{2.3.4} - \overline{\frac{10}{n}-\frac{35}{n^2}+\frac{50}{n^3}-\frac{24}{n^4}}\times\frac{b^5}{2.3.4.5} + \overline{\frac{15}{n}-\frac{85}{n^2}+\frac{225}{n^3}-\frac{274}{n^4}+\frac{120}{n^5}}\times\frac{b^6}{2.3.4.5.6}$ — &c, continued to $n-1$ terms. Q. E. D.

25. Coroll. 3. But, though the quantity $\overline{1-\frac{b}{n}}^{\,n}$ be always greater than the ſeries $1-b+\frac{b^2}{2}-\frac{b^3}{2.3}+\frac{b^4}{2.3.4}-\frac{b^5}{2.3.4.5}+\frac{b^6}{2.3.4.5.6}$ — &c, continued to $n+1$ terms, yet the difference between them will continually decreaſe while the index n increaſes; and the ſaid index n may be taken of ſo great a magnitude that the ſaid difference ſhall become leſs than any aſſigned quantity, how ſmall ſoever.

For it has been ſhewn in the preceding corollary, that this difference is equal to the ſeries $\frac{1}{n}\times\frac{b^2}{2} - \overline{\frac{3}{n}-\frac{2}{n^2}}\times\frac{b^3}{2.3} + \overline{\frac{6}{n}-\frac{11}{n^2}+\frac{6}{n^3}}\times\frac{b^4}{2.3.4}$ —

$\overline{\frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4}} \times \frac{b^5}{2.3.4.5} + \overline{\frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5}} \times \frac{b^6}{2.3.4.5.6}$ — &c, continued to $n - 1$ terms. Now it is evident, that this series will continually decrease while n increases; because the powers of n constitute the denominators of the several members of the co-efficients of its terms; and n may be taken of so great a magnitude that the said series shall be less than any assigned quantity, how small soever. Therefore, while the index n increases, the difference whereby the quantity $\overline{1 - \frac{b}{n}}|^n$ exceeds the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ — &c, continued to $n + 1$ terms, will decrease continually *ad infinitum*, or the index n may be taken of so great a magnitude that the said difference shall be less than any assigned quantity, how small soever.

Q. E. D.

26. Coroll. 4. It follows, from the foregoing corollary, that the infinite series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ — &c *ad infinitum*, is the limit of the magnitude of $\overline{1 - \frac{b}{n}}|^n$, or the quantity to which $\overline{1 - \frac{b}{n}}|^n$ may, by increasing continually the index n, be made to come as near as we please.

For, by increasing the index n continually, the excess of the quantity $\overline{1 - \frac{b}{n}}|^n$ above the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ — &c, continued to $n + 1$ terms, may be made to decrease till it is less than any assigned quantity, how small soever, as we have seen in the foregoing corollary. And, by increasing the said index n continually, it is evident that the difference between the said series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ — &c, continued to $n + 1$ terms, and the same series, continued *ad infinitum*, may be diminished as far as we please, or made to become less than any assigned quantity, how small soever. It follows therefore, that by increasing the index n continually, the difference of the quantity $\overline{1 - \frac{b}{n}}|^n$ and the infinite series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ — &c *ad infinitum* (which will be equal to the sum of the said excess of $\overline{1 - \frac{b}{n}}|^n$ above the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6}$ — &c, continued to $n + 1$ terms, and of the excess of the said finite series above the same series continued *ad infinitum*, in case the said finite series is greater than the same series continued *ad infinitum*; but otherwise will be equal to the difference between the said excess of $\overline{1 - \frac{b}{n}}|^n$ above the said finite series and the excess of the said infinite series above the said finite series,) may be diminished as far as we please, or made to become less than any assigned quantity, how small soever; or, in other words, the infinite series $1 -$

$b +$

$b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} -$ *&c ad infinitum*, is the limit of the magnitude of the quantity $\overline{1 - \frac{b}{n}}|^{n}$. Q. E. D.

27. Coroll. 5. If we make use of the language of infinites, the substance of the last, or 4th corollary may be expressed in the manner following.

When the index n of the power to which the residual quantity $1 - \frac{b}{n}$ is to be raised, becomes infinite, the series $1 - \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} - \&c$, continued to $n + 1$ terms (to which the quantity $\overline{1 - \frac{b}{n}}|^{n}$ is equal,) or its equal, the series $1 - \frac{n}{n} \times b + \frac{nn-n}{nn} \times \frac{b^2}{2} - \overline{\frac{n^3 - 3n^2 + 2n}{n^3}} \times \frac{b^3}{2.3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4} \times \frac{b^4}{2.3.4} - \overline{\frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{n^5}} \times \frac{b^5}{2.3.4.5} + \frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{n^6} \times \frac{b^6}{2.3.4.5.6} - \&c$, continued to $n + 1$ terms, will become infinite in the number of its terms, and will be equal to the series $1 - \overline{\frac{n}{n}} \times b + \frac{n^2}{n^2} \times \frac{b^2}{2} - \overline{\frac{n^3}{n^3}} \times \frac{b^3}{2.3} + \frac{n^4}{n^4} \times \frac{b^4}{2.3.4} - \overline{\frac{n^5}{n^5}} \times \frac{b^5}{2.3.4.5} + \frac{n^6}{n^6} \times \frac{b^6}{2.3.4.5.6} -$ *&c ad infinitum*, or to the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} -$ *&c ad infinitum*; because all the members of the several numerators $nn - n$, $n^3 - 3n^2 + 2n$, $n^4 - 6n^3 + 11n^2 - 6n$, $n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$, and $n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n$, *&c*, after the first members n^2, n^3, n^4, n^5, n^6, *&c*, will vanish, or become infinitely small, in respect of the said first members, which contain a higher power of n than the said following members, and consequently those whole numerators will be equal to their said first members n^2, n^3, n^4, n^5, n^6, *&c*, respectively. Therefore the quantity $\overline{1 - \frac{b}{n}}|^{n}$ (which is in all cases equal to the said series $1 - \overline{\frac{n}{n}} \times b + \frac{n^2 - n}{n^2} \times \frac{b^2}{2} - \overline{\frac{n^3 - 3n^2 + 2n}{n^3}} \times \frac{b^3}{2.3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4} \times \frac{b^4}{2.3.4} - \overline{\frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{n^5}} \times \frac{b^5}{2.3.4.5} + \frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{n^6} \times \frac{b^6}{2.3.4.5.6} - \&c$, continued to $n + 1$ terms,) will, in this case of the infinite magnitude of n, be equal to the infinite series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} -$ *&c ad infinitum*. Q. E. D.

28. These

28. Thefe things being premifed, the inveftigation of the fecond anti-logarithmick feries $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, or the derivation of it from Dr. Wallis's logarithmick feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, may be performed in the manner following.

PROBLEM II.

Let k be any quantity lefs than 1, and let L be equal to the infinite feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*; which is the logarithm of the ratio of 1 to $1 - k$ in Napier's fyftem of logarithms. And let L, or the value of this whole feries, be fuppofed to be known, but k, or the value of its firft term alone, and confequently the feparate values of all its other terms (which contain the powers of k), to be unknown. It is required to find from L, or the value of the whole feries, the value of its firft term k.

SOLUTION.

Let us fuppofe the letter n to denote fome very great number, fuch as a nonillion, or the ninth power of a million.

Then it will follow, from lemma 3, coroll. 2, art. 65, of the foregoing remarks on Mercator's and Wallis's feriefes, that $1 - \overline{1 - k}|^{\frac{1}{n}}$ will be very nearly equal to $\frac{1}{n} \times$ the infinite feries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, that is, (becaufe L is fuppofed to be equal to the faid feries,) very nearly equal to $\frac{1}{n} \times L$, or to $\frac{L}{n}$. Therefore, if we add $\overline{1 - k}|^{\frac{1}{n}}$ to both fides, we fhall have 1 very nearly $= \frac{L}{n} + \overline{1 - k}|^{\frac{1}{n}}$, and (fubtracting $\frac{L}{n}$ from both fides) $\overline{1 - k}|^{\frac{1}{n}}$ very nearly $= 1 - \frac{L}{n}$, and (raifing both fides of this laft equation to the n^{th} power) $1 - k$ very nearly equal $\overline{1 - \frac{L}{n}}|^{n}$. And n may be taken of fo great a magnitude that the ratio of $1 - k$ to $\overline{1 - \frac{L}{n}}|^{n}$ fhall approach as near as we pleafe to a ratio of equality.

But, becaufe the index n is of an immenfely-great magnitude, it follows, from the fourth corollary of the foregoing lemma, that $\overline{1 - \frac{L}{n}}|^{n}$ will be very nearly equal to the infinite feries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum*, and that n may be taken of fo great a magnitude that the difference

ference between $\overline{1 - \frac{L}{n}}\Big|^{n}$ and the said series shall be less than any assigned quantity, how small soever. Therefore $1 - k$ will be very nearly equal to the said infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c *ad infinitum*, and consequently (adding k to both sides) 1 will be very nearly equal to $k + 1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c *ad infinitum*, and (adding the series $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} +$ &c *ad infinitum* to both sides) $1 + L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} +$ &c *ad infinitum*, will be very nearly equal to $k + 1 + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} +$ &c *ad infinitum*, and (subtracting 1 from both sides) $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} +$ &c *ad infinitum*, will be very nearly equal to $k + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} +$ &c *ad infinitum*, and, lastly, (subtracting the infinite series $\frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.2.4.5.6} +$ &c *ad infinitum* from both sides,) k will be very nearly equal to the infinite series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} +$ &c *ad infinitum*. And since, by increasing the magnitude of n continually, the quantity $1 - k$ may be made to approach as near as we please to an equality with the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — &c *ad infinitum*, it follows, that it is not only very nearly, but accurately, equal to the said infinite series, and consequently that k is not only very nearly, but accurately, equal to the infinite series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} +$ &c *ad infinitum*. Q. E. I.

11. This solution may be expressed with greater brevity in the language of infinites, as follows.

Let us suppose the letter n to denote a number infinitely great.

Then it will follow, from lemma 3, coroll. 2, art. 65, of the foregoing remarks on Mercator's and Wallis's serieses, that $1 - \overline{1 - k}\Big|^{\frac{1}{n}}$ will be equal to $\frac{1}{n} \times$ the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ &c *ad infinitum*, that is, (because the said series is $= L$,) to $\frac{1}{n} \times L$, or to $\frac{L}{n}$. Therefore (adding $\overline{1 - k}\Big|^{\frac{1}{n}}$ to both sides) we shall have $1 = \frac{L}{n} + \overline{1 - k}\Big|^{\frac{1}{n}}$, and (subtracting $\frac{L}{n}$

2 from

from both fides) $1 - \frac{L}{n} = \overline{1-k}|^{\frac{1}{n}}$, or $\overline{1-k}|^{\frac{1}{n}} = 1 - \frac{L}{n}$, and confequently (raifing both fides of this laft equation to the n^{th} power,) $1 - k = \overline{1 - \frac{L}{n}}|^{n}$.

But, becaufe the index n is an infinitely-great number, it follows, from coroll. 5 of the foregoing lemma, that $\overline{1 - \frac{L}{n}}|^{n}$ will be equal to the infinite feries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum.* Therefore $1 - k$ will be equal to the fame infinite feries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum;* and confequently (adding k to both fides of the equation) 1 will be $= k + 1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum;* and (adding $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5}$ + *&c ad infinitum* to both fides,) $1 + L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5}$ + *&c ad infinitum* will be $= k + 1 + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum;* and (fubtracting 1 from both fides) $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5}$ + *&c ad infinitum* will be $= k + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum;* and, laftly, (fubtracting the infinite feries $\frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum* from both fides,) k will be = the infinite feries $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum.* Q. E. I.

30. Coroll. 1. Since k is equal to the infinite feries $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum,* it follows that the ratio of 1 to $1 - k$ will be equal to the ratio of 1 to 1 — the infinite feries $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum,* or of 1 to the infinite feries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum;* or the ratio correfponding to any given logarithm in Napier's fyftem denoted by L is the ratio of 1 to the infinite feries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum.*

An Example of the foregoing Method of discovering the Ratio corresponding to a given Logarithm L *in Napier's System, by Means of the Infinite Series* $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c ad infinitum.

31. Let the given logarithm, of which we are to find the ratio, be $\frac{1}{2}$, or 0.500,000,000.

Then will L^2 be $= \frac{1}{4}$, $L^3 = \frac{1}{8}$, $L^4 = \frac{1}{16}$, $L^5 = \frac{1}{32}$, $L^6 = \frac{1}{64}$, and L^7, L^8, L^9, L^{10}, *&c* will be equal to the 7th, 8th, 9th, 10th, and other following powers of $\frac{1}{2}$, as in the example given above in article 13. Therefore the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum* will in this case be equal to 1.000,000,000 — 0.500,000,000 + 0.125,000,000 — 0.020,833,333 + 0.002,604,166 — 0.000,260,416 + 0.000,021,701 — 0.000,001,550 + 0.000,000,096 — 0.000,000,005 + 0.000,000,000 — *&c* = 1.127,625,963 — 0.521,095,304 = 0.606,530,659. And consequently the ratio sought, or that of which $\frac{1}{2}$, or 0.500,000,000, is the logarithm in Napier's system, is that of 1 to 0.606,530,659. Q. E. I.

32. This ratio of 1 to 0.606,530,659 is the same with that which was found above in article 13 for the ratio corresponding to the same logarithm $\frac{1}{2}$, or 0.500, 000,000, to wit, the ratio of 1.648,721,267 to 1.

For 1.648,721,267 is to 1 as 1 is to $\frac{1.000,000,000,000,000,000,0}{1.648,721,267} = 0.606,530,661$; which exceeds 0.606,530,659 by only 0.000,000,002, or $\frac{2}{1.000,000,000}$, or $\frac{1}{500,000,000}$, or one five-hundred-millionth part of an unit.

33. This equality of the two ratios which are found to be correspondent to the same logarithm $\frac{1}{2}$, or 0.500,000,000, by means of the two serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum*, which were derived from the two logarithmick serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum*, and $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, invented by Mr. Mercator and Dr. Wallis, affords an additional proof of the truth of what is asserted in theorem 3, article 83, of the foregoing remarks on those two serieses,

feriefes, to wit, that the logarithms exhibited by the faid two feriefes of Mercator and Wallis belong to the fame fyftem.

Another Example of the foregoing Method of difcovering the Ratio correfponding to a given Logarithm L *in Napier's Syftem by Means of the Infinite Series* $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — &c ad infinitum.

34. Let the given logarithm L be equal to 1. It is required to find the ratio of which it is the logarithm by means of the feries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum.*

Now, when L is $= 1$, we fhall have $L^2 = 1$, and $L^3 = 1$, and $L^4 = 1$, and every following power of $L = 1$; and confequently the feries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum,* will in this cafe become equal to the feries $1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6}$ — *&c ad infinitum,* or $\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6}$ — *&c ad infinitum,* the feveral terms of which feries have been already found, in article 16, to be as follows.

$$C = \frac{B}{2} = 0.500,000,000,000,000,000,000,000;$$

$$D = \frac{C}{3} = 0.166,666,666,666,666,666,666,666;$$

$$E = \frac{D}{4} = 0.041,666,666,666,666,666,666,666;$$

$$F = \frac{E}{5} = 0.008,333,333,333,333,333,333,333;$$

$$G = \frac{F}{6} = 0.001,388,888,888,888,888,888,888;$$

$$H = \frac{G}{7} = 0.000,198,412,698,412,698,412,698;$$

$$I = \frac{H}{8} = 0.000,024,801,587,301,587,301,587;$$

$$K = \frac{I}{9} = 0.000,002,755,731,922,398,589,065;$$

$$L = \frac{K}{10} = 0.000,000,275,573,192,239,858,906;$$

$$M = \frac{L}{11} = 0.000,000,025,052,108,385,441,718;$$

$$N = \frac{M}{12} = 0.000,000,002,087,675,698,786,809;$$

$$O = \frac{N}{13} = 0.000,000,000,160,590,438,368,216;$$

$$P = \frac{O}{14} = 0.000,000,000,011,470,745,597,729;$$

$$Q = \frac{P}{15} = 0.000,000,000,000,764,716,373,181;$$

$$R = \frac{Q}{16} = 0.000,000,000,000,047,794,773,323;$$

$$S = \frac{R}{17} = 0.000,000,000,000,002,811,457,254;$$

$$T = \frac{S}{18} = 0.000,000,000,000,000,156,192,069;$$

$$V = \frac{T}{19} = 0.000,000,000,000,000,008,220,635;$$

$$W = \frac{V}{20} = 0.000,000,000,000,000,000,411,031;$$

$$X = \frac{W}{21} = 0.000,000,000,000,000,000,019,572;$$

$$Y = \frac{X}{22} = 0.000,000,000,000,000,000,000,889;$$

$$Z = \frac{Y}{23} = 0.000,000,000,000,000,000,000,038;$$

$$A' = \frac{Z}{24} = 0.000,000,000,000,000,000,000,001.$$

Therefore $\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} -$ *&c ad infinitum*, or $\frac{1}{2} - \frac{C}{3} + \frac{D}{4} - \frac{E}{5} + \frac{F}{6} - \frac{G}{7} + \frac{H}{8} - \frac{I}{9} + \frac{K}{10} - \frac{L}{11} + \frac{M}{12} - \frac{N}{13} + \frac{O}{14} - \frac{P}{15} + \frac{Q}{16} - \frac{R}{17} + \frac{S}{18} - \frac{T}{19} + \frac{V}{20} - \frac{W}{21} + \frac{X}{22} - \frac{Y}{23} + \frac{Z}{24} -$ *&c ad infinitum*, or $\frac{1}{2} + \frac{D}{4} + \frac{F}{6} + \frac{H}{8} + \frac{K}{10} + \frac{M}{12} + \frac{O}{14} + \frac{Q}{16} + \frac{S}{18} + \frac{V}{20} + \frac{X}{22} + \frac{Z}{24} +$ *&c* $- \frac{C}{3} - \frac{E}{5} - \frac{G}{7} - \frac{I}{9} - \frac{L}{11} - \frac{N}{13} - \frac{P}{15} - \frac{R}{17} - \frac{T}{19} - \frac{W}{21} - \frac{Y}{23} -$ *&c*, will be =

0;500,000,000,000,000,000,000,000,
\+ ;. 41,666,666,666,666,666,666,666,
\+ ;.. 1,388,888,888,888,888,888,888,
\+ ;...,. 24,801,587,301,587,301,587,
\+ ;...,...,275,573,192,239,858,906,
\+ ;...,...,.. 2,087,675,698,786,809,
\+ ;...,...,...,. 11,470,745,597,729,
\+ ;...,...,...,...,. 47,794,773,323,
\+ ;...,...,...,...,...,156,192,069,
\+ ;...,...,...,...,...,...,411,031,
\+ ;...,...,...,...,...,...,...,889,
\+ ;...,...,...,...,...,...,...,.. 1,
\+ *&c.*

$-0;166,666,666,666,666,666,666,666,$
$- \;;..8,333,333,333,333,333,333,333,$
$- \;;...,198,412,698,412,698,412,698,$
$- \;;...,..2,755,731,922,398,589,065,$
$- \;;...,...,.25,052,108,385,441,718,$
$- \;;...,...,...,160,590,438,368,216,$
$- \;;...,...,...,...,764,716,373,181,$
$- \;;...,...,...,...,..2,811,457,254,$
$- \;;...,...,...,...,...,..8,220,635,$
$- \;;...,...,...,...,...,...,.19,572,$
$- \;;...,...,...,...,...,...,...,.38,$
$-$ &c $=$

0.543,080,634,815,243,778,477,898, &c $-$ 0.175,201,193,643,801,456,882, 376, &c $=$ 0.367,879,441,171,442,321,595,522, &c. Therefore the ratio sought, or that of which 1 is the logarithm in Napier's system, is that of 1 to the decimal fraction 0.367,879,441,171,442,321,595,522, &c. Q. E. I.

35. This ratio is equal to the ratio of 2.718,281,828,459,045,235,360,274, &c to 1; which was found above, in article 16, to be the ratio of which 1 was the logarithm in the system of logarithms exhibited by Mercator's series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ &c *ad infinitum*. For 2.718,281,828,459, 045,235,360,274 is to 1 as 1 is to $\frac{1.000,000,000,000,000,000,000,000,000,000, \&c}{2.718,281,828,459,045,235,360,274} =$ 0.367,879,441,171,442,321,595,525; which agrees with the foregoing number 0.367,879,441,171,442,321,595,522, in all the figures but the last.

The operation of this division of 1.000,000,000,000,000,000,000,000,000,000, &c by the long number 2.718,281,828,459,045,235,360,274, is as follows.

In order to avoid mistakes in performing so long and laborious a division, it will be expedient, before we enter upon it, to find all the products that can arise by multiplying the divisor 2.718,281,828,459,045,235,360,274 into the eight first numbers 2, 3, 4, 5, 6, 7, 8, and 9, by the more simple operation of addition, in which it is hardly possible that we should make any slips. This may be done in the following manner, the said divisor 2.718,281,828,459,045,235,360,274 being (for brevity's sake) denoted by the capital letter A.

$$\begin{array}{rr}
 & 2.718,281,828,459,045,235,360,274 = \text{A}. \\
 & \underline{2.718,281,828,459,045,235,360,274} \\
2\,\text{A} = & 5.436,563,656,918,090,470,720,548 \\
 & \underline{2.718,281,828,459,045,235,360,274} \\
3\,\text{A} = & 8.154,845,485,377,135,706,080,822 \\
 & \underline{2.718,281,828,459,045,235,360,274} \\
4\,\text{A} = & 10.873,127,313,836,180,941,441,096 \\
 & \underline{2.718,281,828,459,045,235,360,274} \\
5\,\text{A} = & 13.591,409,142,295,226,176,801,370 \\
 & 2.718,281,828,459,045,235,360,274
\end{array}$$

6 A = 16.309,690,970,754,271,412,161,644
2.718,281,828,459,045,235,360,274

7 A = 19.027,972,799,213,316,647,521,918
2.718,281,828,459,045,235,360,274

8 A = 21.746,254,627,672,361,882,882,192
2.718,281,828,459,045,235,360,274

9 A = 24.464,536,456,131,407,118,242,466

These several products being thus previously obtained in a manner hardly liable to error, the operation of dividing 1, or 1.000,000,000,000,000,000,000,000,000, 000,000, *&c*, by the long number 2.718,281,828,459,045,235,360,274, may be performed as follows.

The Division of 1, *or* 1.000,000,000,000,000,000,000,000,000,000,000, &c, *by* 2.718,281,828,459,045,235,360,274.

The divisor.	The quotient.
2.718,281,828,459,045,235,360,274)	(0.367,879,441,171,442,321,595,525

The dividend.

1.000,000,000,000,000,000,000,000,000,000,000, *&c.*
815484548537713570608082 2

.1845154514622864293919178 0
1630969097075427141216164 4

.2141854175474371527030136 0
1902797279921331664752191 8

.2390568955530398622779442 0
2174625462767236188288219 2

.2159434927631624344912228 0
1902797279921331664752191 8

.2566376477102926801600362 0
2446453645613140711824246 6

.1199228314897860897761154 0
1087312731383618094144109 6

.1119155835142428036170444 0
1087312731383618094144109 6

..3184310375880994202633440
2718281828459045235360274

.4660285474219489672731660
2718281828459045235360274

.1942003645760444437371386 0
1902797279921331664752191 8

..3920636583911277261919420
2718281828459045235360274

.1202354755452232026559146 0
1087312731383618094144109 6

1.1504202406861393241503640

.11504202406861393241503640

1087312731383618094l441096

..6310750930252123000625440

5436563656918090470720548

.8741872733340325299048920

8154845485377135706080822

.5870272479631895929680980

5436563656918090470720548

.4337088227138054589604320

2718281828459045235360274

1618806398679009354244046

1359140914229522617680137

.2596654844494867365639090

2446453645613140711824246

.1502011988817266538148434

1359140914229522617680137

.1428710745877439204682970

1359140914229522617680137

..6956983164791658700283300

5436563656918090470720548

1520419507873568229562752

1359140914229522617680137

.1612785936440456118826150

36. This equality of these two ratios, of which 1 is the logarithm in the two systems of logarithms exhibited by the two serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum*, and $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, invented by Mr. Mercator and Dr. Wallis, affords another proof of the truth of what is asserted in theorem 3, article 83, of the foregoing remarks on those two serieses, to wit, that the logarithms exhibited by both serieses belong to one and the same system.

A general Demonstration of the Equality between the Ratio of the Infinite Series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ &c *to* 1, *and the Ratio of* 1 *to the Infinite Series* $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c.

37. But the equality of the two ratios that are found, by means of the two infinite serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum*, to belong to the same logarithm L, or the equality of the ratio of 1 to the infinite series

ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c ad infinitum* to the ratio of the infinite ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ + *&c ad infinitum* to 1, will appear in the cleareſt manner by dividing 1 by the infinite ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ — *&c.* For it will be found that the quotient of this diviſion will be the other ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6}$ + *&c*; from whence it follows, that the ratio of 1 to the former ſeries, (which is the diviſor of this operation of diviſion,) will be equal to the ratio of the latter ſeries (which is the quotient of the ſame diviſion) to 1. This diviſion may be performed in the manner following.

The diviſor.

$$1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c.$$

The quotient.

$$1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c.$$

The dividend.

$$1 \quad * \quad * \quad * \quad * \quad * \quad * \quad *$$

$$1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c.$$

$$* + L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \&c.$$

$$+ L - L^2 + \frac{L^3}{2} - \frac{L^4}{2.3} + \frac{L^5}{2.3.4} - \frac{L^6}{2.3.4.5} + \&c.$$

$$* + \frac{L^2}{2} - \frac{2L^3}{2.3} + \frac{3L^4}{2.3.4} - \frac{4L^5}{2.3.4.5} + \frac{5L^6}{2.3.4.5.6} - \&c.$$

$$+ \frac{L^2}{2} - \frac{L^3}{2} + \frac{L^4}{2.2} - \frac{L^5}{2.3.2} + \frac{L^6}{2.3.4.2} - \&c.$$

$$* + \frac{L^3}{2.3} - \frac{3L^4}{2.3.4} + \frac{6L^5}{2.3.4.5} - \frac{10L^6}{2.3.4.5.6} + \&c.$$

$$+ \frac{L^3}{2.3} - \frac{L^4}{2.3} + \frac{L^5}{2.3.2} - \frac{L^6}{2.3.2.3} + \&c.$$

$$* + \frac{L^4}{2.3.4} - \frac{4L^5}{2.3.4.5} + \frac{10L^6}{2.3.4.5.6} - \&c.$$

$$+ \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4} + \frac{L^6}{2.3.4.2} - \&c.$$

$$* + \frac{L^5}{2.3.4.5} - \frac{5L^6}{2.3.4.5.6} + \&c.$$

$$+ \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5} + \&c.$$

$$* + \frac{L^6}{2.3.4.5.6} - \&c.$$

$$+ \frac{L^6}{2.3.4.5.6} - \&c.$$

$$* \quad + \&c.$$

38. This is a very remarkable property of theſe two ſeriefes, "that the diviſion of unity by one of them ſhould produce the other, or that the quotient of ſuch diviſion ſhould be a ſeries conſiſting of exactly the ſame terms as the diviſor, only with different ſigns prefixed to the terms that fill the even places of the ſeries, or that involve the odd powers of L." I do not recollect any other ſuch inſtance of ſimilarity between the diviſor and the quotient of a diviſion in all the operations of this kind that I have ſeen in Algebra.

39. From this ſimilarity between the two ſeriefes $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum*, and $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, it follows univerſally, that, whatever be the magnitude of L, the ratio of the ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum* (which is the quotient of the foregoing diviſion of unity by the other ſeries) to 1 will be equal to the ratio of 1 to the ſaid other ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum*. Therefore any quantity denoted by L will be the logarithm of the ſame ratio in the ſyſtem of logarithms exhibited by Mr. Mercator's ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum*, from which the ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c* was derived, as in the ſyſtem of logarithms exhibited by Dr. Wallis's ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, from which the ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c* was derived; or, in other words, the logarithms exhibited by Mr Mercator's infinite ſeries $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$ *&c ad infinitum* belong to the ſame ſyſtem as the logarithms exhibited by Dr. Wallis's infinite ſeries $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} +$ *&c ad infinitum*, agreeably to theorem 3, article 83, of the foregoing remarks on thoſe ſeriefes of Mercator and Wallis.

A SCHOLIUM.

40. When L is leſs than 1, the numerators of the terms of the two ſeriefes $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} +$ *&c ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} -$ *&c ad infinitum*, will converge, or decreaſe, continually; and conſequently the whole terms will converge, or decreaſe, with conſiderable ſwiftneſs, becauſe the denominators of the terms, to wit, 2, 2.3, 2.3.4, *&c*, increaſe continually, and thereby

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contribute to the decreafe of the whole terms. Of this decreafe of the terms of thefe feriefes we have had examples above, in articles 13 and 31, in which L was fuppofed to be $= \frac{1}{2}$, or 0.500,000,000.

And when L is equal to 1, and confequently the numerators of the terms of thefe two feriefes, to wit, 1, L, L^2, L^3, L^4, *&c*, are all equal to 1, the whole terms of them, to wit, 1, L, $\frac{L^2}{2}$, $\frac{L^3}{2.3}$, $\frac{L^4}{2.3.4}$, *&c*, or 1, 1, $\frac{1}{2}$, $\frac{1}{2.3}$, $\frac{1}{2.3.4}$ *&c* after the firft term 1, will ftill converge, or decreafe, and with a confiderable degree of fwiftnefs, on account of the continual increafe of their denominators 2, 2.3, 2.3.4, *&c*. Of the decreafe of the terms of thefe feriefes in this cafe we have had examples above in articles 16 and 34, in which we have computed the values of thefe feriefes exact to no fewer than 23 places of figures.

But, when L is greater than 1, the numerators of the terms of thefe two feriefes, to wit, 1, L, L^2, L^3, L^4, *&c*, will continually diverge, or increafe, in the proportion of L to 1, and thereby will tend to make the whole terms increafe. And the fecond term L will be greater than the firft term 1; and if L be confiderably greater than 1, as, for inftance, if it be equal to 10, there will be feveral of the terms in the beginning of the feries, befides the fecond term, which will be greater than the terms that immediately precede them: or the feriefes will, for a while, be diverging feriefes. Yet fuch is the tendency of the terms of thefe feriefes to converge, or decreafe, arifing from the continual increafe of their denominators 2, 2.3, 2.3.4, 2.3.4.5, 2.3.4.5.6, 2.3.4.5.6.7, 2.3.4.5.6.7.8, *&c*, that, whatever be the magnitude of L, and the confequent increafe, or divergency, of the numerators 1, L, L^2, L^3, L^4, L^5, L^6, L^7, L^8, *&c*, the diminution of the terms arifing from the increafe of their denominators will, in fome part or other of the faid feriefes, more than counterbalance the faid divergency, or increafe, of them arifing from the increafe of their numerators, and make the whole terms, after the term in which fuch over-balancing firft takes place, continually converge, or decreafe; and this with an increafing velocity, or more fwiftly than in a continued geometrical proportion.

For, fince the denominators of the terms of thefe feriefes, to wit, 2, 2.3, 2.3.4, 2.3.4.5, 2.3.4.5.6, 2.3.4.5.6.7, 2.3.4.5.6.7.8, 2.3.4.5.6.7.8.9, 2.3.4.5.6.7.8.9.10, *&c*, increafe by the continual multiplication of the natural numbers 3, 4, 5, 6, 7, 8, 9, 10, *&c ad infinitum*, it follows, that, however great we may fuppofe L to be, we fhall, in fome part or other of the feries, come to a term of which the denominator will be derived from the denominator of the next preceding term, by the multiplication of a number ftill greater than L; and this term will be lefs than the faid next preceding term, becaufe its denominator will be greater than the denominator of the faid next preceding term in a greater proportion than that in which its numerator will exceed the numerator of the faid preceding term, which is that of L to 1. Thus, for example, if L is equal to 1000, it is evident that the thoufand and firft term of either of thefe feriefes will have for its numerator the 1000th power of L, or L^{1000}, and for its denominator the product 2.3.4.5.6.7.8.9.10, *&c*, continued to 1000; and the thoufand and fecond term will have for its numerator the 1001ft power of L, or L^{1001}, and for its denominator

nator the product 2.3.4.5.6.7.8.9.10, *&c*, continued to 1001. Therefore the 1002nd term will be equal to the 1001ft term multiplied into the fraction $\frac{L}{1001}$, or $\frac{1000}{1001}$, and confequently will be lefs than the 1001ft term. And, in like manner, the 1003'd term will be derived from the 1002nd term, by multiplying it into $\frac{L}{1002}$, or $\frac{1000}{1002}$; and the 1004th term will be derived from the 1003rd term, by multiplying it into $\frac{L}{1003}$, or $\frac{1000}{1003}$; and every following term will be derived from the term immediately preceding it, by multiplying it into a fraction that is lefs than $\frac{1000}{1003}$, and alfo than the multiplying, or generating, fraction next before it. And confequently, after the 1001ft term, the feveral terms of the feries will continually decreafe, and that with an increafing velocity, or more fwiftly than in a continued geometrical proportion. Q. E. D.

41. And hence it follows, that the two feriefes $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum*, will always be of finite magnitudes, whatever be the magnitude of L; agreeably to what was afferted above in the fcholium in article 15. For, when L is of a very great magnitude, as, for example, when it is equal to 1000, the firft part of each of thefe feriefes, though it will confift of terms that continually diverge, or increafe, yet will be of a finite magnitude, becaufe the number of its terms is finite; and the fecond part of the feries, from the part where it begins to converge, will converge with an increafing degree of velocity, or more fwiftly than in a continued geometrical proportion, and therefore will be of a finite magnitude, notwithftanding the infinite number of its terms, becaufe every infinite feries of terms that decreafe in a continued geometrical proportion is of a finite magnitude. Therefore the fum of the firft and fecond parts, or of the diverging and converging parts, of each of thefe feriefes, that is, the whole of each of thefe feriefes, will be of a finite magnitude.

Q. E. D.

42. But thefe feriefes can never, as I conceive, be applied to any ufeful purpofe when L is greater than 1; nor, indeed, will they be of much ufe in the computation of numbers from the logarithms that belong to them, even when L is lefs than 1, but nearly equal to it. But when L is much lefs than 1, as, for example, when it is equal to $\frac{1}{10}$, or any leffer number, thefe feriefes will converge with fufficient fwiftnefs to be of great ufe in making fuch computations.

43 In the foregoing corollary to problem 2, we have fhewn how to find the ratio correfponding to a given logarithm of Napier's fyftem: it remains that we fhew how, when a logarithm of Briggs's fyftem is given, we may find the ratio which correfponds to it. This will be the fubject of a fecond corollary.

Coroll. 2. If B be the logarithm of any ratio in Briggs's ſyſtem of logarithms, and L be the logarithm of the ſame ratio in Napier's ſyſtem, L will be greater than B in the ſame proportion in which Napier's logarithm of the ratio of 10 to 1 is greater than Briggs's logarithm of the ratio of 10 to 1, that is, in the proportion of 2.302,585,092,994,045,668, *&c* to 1; or L will be = 2.302,585,092,994, 045,668, *&c* × B.

Therefore, if B be any logarithm in Briggs's ſyſtem, and we are required to find the ratio correſponding to it, or of which it is the logarithm, by means of the foregoing ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum*, we muſt firſt multiply B by the number 2.302,585,092,994,045,668, *&c*; whereby we ſhall obtain the value of L, or of Napier's logarithm of the ſame ratio: and then we muſt compute as many terms of the ſaid ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ *&c ad infinitum* as may be neceſſary to expreſs the ratio we are ſeeking to the propoſed degree of exactneſs. For the proportion of 1 to the ſaid ſeries, or to ſo many terms of it as we ſhall have computed, will be the ratio ſought.

An Example of this Method of diſcovering the Ratio correſponding to a given Logarithm in Briggs's Syſtem of Logarithms.

44. Let it be required to find the ratio correſponding to the logarithm $\frac{1}{100}$, or 0.01, in Briggs's ſyſtem of logarithms.

The logarithm of this ratio in Napier's ſyſtem of logarithms is = 2.302,585, 092,994,045,668, *&c* × 0.01 = 0.023,025,850,929,940,456,68, *&c*, or (neglecting the laſt eleven figures, as inconſiderable) 0.023,025,850. Call this logarithm L.

Then will L^2 be $= \overline{0.023,025,850}|^2 = 0.000,530,189$, and L^3 will be = 0.000, 012,208, and L^4 will be = 0.000,000,281, and L^5 will be = 0.000,000,006. Therefore $\frac{L^2}{2}$ will be $= \frac{0.000,530,189}{2} = 0.000,265,094$, and $\frac{L^3}{2.3}$ will be $= \frac{0.000,012,208}{6} = 0.000,002,034$, and $\frac{L^4}{2.3.4}$ will be $= \frac{0.000,000,281}{24} = 0.000,000,011$; and $\frac{L^5}{2.3.4.5}$ will be $= \frac{0.000,000,006}{120} = 0.000,000,000$, and conſequently the ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} +$ *&c* will be = 1.000,000, 000 − 0.023,025,850 + 0.000,265,094 − 0.000,002,034 + 0.000,000,011 − 0.000,000,000 + *&c* = 1.000,265,105 − 0.023,027,884 = 0.977,237, 221. Therefore the ratio correſponding to L, or 0.023,025,850, or of which 0.023,025,850 is the logarithm in Napier's ſyſtem of logarithms, or of which 0.01, or $\frac{1}{100}$, is the logarithm in Briggs's ſyſtem of logarithms, is the ratio of 1 to 0.977,237,221.

Q. E. I.

45. This

45. This ratio of 1 to 0.977,237,221, which is here found by means of the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ to be the ratio of which 0.01, or $\frac{1}{100}$, is the logarithm in Briggs's syſtem, is equal to the ratio of 1.023,292,989, &c to 1, which was found above in article 18, by means of the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$, to be the ratio corresponding to the ſame logarithm. For 1.023,292,989 is to 1 as 1 is to $\frac{1.000,000,000,000,\ \&c}{1.023,292,989} = 0.977,237,224$; which agrees with the number 0.977,237,221 in the firſt eight figures, and exceeds it by only 0.000,000,003, or $\frac{3}{1000,000,000}$, or three thouſand-millionth parts of an unit.

Another Example of this Method of diſcovering the Ratio correſponding to a given Logarithm in Briggs's Syſtem of Logarithms.

46. Suppoſe it were required to find the 365th root of $1 + \frac{1}{20}$, or $1 + \frac{5}{100}$, or 1.05, the logarithm of the ratio of 1.05 to 1 in Briggs's ſyſtem being given.

The logarithm of the ratio of 1.05 to 1 in Briggs's ſyſtem is $=$ log. $\frac{105}{100} =$ log. $\frac{21}{20} =$ log. $\frac{21}{1}$ — log. $\frac{20}{1} =$ (according to the computations in the foregoing remarks, article 98) 1.322,219,294,733,919,356 — 1.301,029,995,663,981, 308 = 0.021,189,299,069,938,048. Therefore the logarithm of the ratio of $\sqrt[365]{1.05}$, or $\overline{1.05}|^{\frac{1}{365}}$, to 1, will be $= \frac{0.021,189,299,069,938,048}{365} = 0.000,058,$ 052,874,164,213. This is Briggs's logarithm of the ratio of $\overline{1.05}|^{\frac{1}{365}}$ to 1, from which we are now to diſcover that ratio itſelf, or the value of $\overline{1.05}|^{\frac{1}{365}}$.

Now, ſince Briggs's logarithm of this ratio is 0.000,058,052,874,164,213, or (neglecting the laſt nine figures, as inconſiderable,) 0.000,058,052, it follows that Napier's logarithm of the ſame ratio will be = 2.302,585,092, &c × 0.000, 058,052 = 0.000,133,669, &c; that is, L will be = 0.000,133,669, &c. We ſhall therefore have L^2 (= $\overline{0.000,133,669}|^2$) = 0.000,000,017, and L^3 (= $L^2 \times L$ = 0.000,000,017 × 0.000,133,669) = 0.000,000,000; and conſequently $\frac{L^2}{2}$ (= $\frac{0.000,000,017}{2}$) = 0.000,000,008, and $L - \frac{L^2}{2}$ (= 0.000,133,669 — 0.000,000,008) = 0.000,133,661, and $1 - L + \frac{L^2}{2}$ = 1.000,000,000 — 0.000, 133,661 = 0.999,866,339; that is, the ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ will be = 0.999,866,339. Therefore the ratio

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correſponding to L, or 0.000,133,669, &c, or of which 0.000,133,669, &c is the logarithm in Napier's ſyſtem of logarithms, or of which 0.000,058,052, &c is the logarithm in Briggs's ſyſtem of logarithms, is the ratio of 1 to 0.999,866, 339; and conſequently $\overline{1.05}|^{\frac{1}{365}}$ is to 1 as 1 is to 0.999,866,339, and $\overline{1.05}|^{\frac{1}{365}}$ is $= \frac{1.000,000,000}{0.999,866,339} =$ 1.000,133,678, or the 365th root of 1.05 is 1.000,133,678.

Q. E. I.

47. The ratio correſponding to L is equal to that of the ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ &c to 1 as well as to the ratio of 1 to the ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c. Now the ſeries $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ &c, is, in this caſe, equal to 1 + 0.000,133,669 + 0.000,000,008 + &c = 1.000,133,677, &c. Therefore the ratio correſponding to L, or 0.000,133,669, &c, or of which 0.000,133,669, &c is the logarithm in Napier's ſyſtem of logarithms, or of which 0.000,058,052, &c is the logarithm in Briggs's ſyſtem of logarithms, is the ratio of 1.000,133,677, &c to 1; and conſequently the ratio of $\overline{1.05}|^{\frac{1}{365}}$ to 1 is equal to the ratio of 1.000,133,677, &c to 1, and therefore $\overline{1.05}|^{\frac{1}{365}}$ is = 1.000,133,677, &c.

Q. E. I.

This value of $\overline{1.05}|^{\frac{1}{365}}$ agrees with the former value of it, to wit, 1.000,133, 678, (which was obtained by means of the ſeries $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} -$ &c,) in all the figures but the laſt, which differs from the laſt figure of the ſaid former value only by an unit.

48. This number 1.000,133,677, which is equal to the 365th root of 1.05, is the quantity to which one pound ſterling, or any other original ſum of money denoted by 1, would increaſe by being lent out at intereſt for the ſpace of only one day, if the terms of the loan were ſuch that the money was to be paid back to the lender every night, with the intereſt due upon it, and then immediately to be lent out again, together with the intereſt due upon it added to it, and yet that the whole intereſt that ſhould be gained upon it in the courſe of a whole year, or 365 days, by theſe 365 repeated loans, ſhould amount only to one twentieth part, or five hundredth parts, of one pound, or the other original ſum, whatever it was, which was denoted by 1. It is a very little leſs than $1 + \frac{.05}{365}$, or 1 + 0.000, 136,986, or 1.000,136,986, to which the ſum 1 would increaſe in the ſpace of one day, if the intereſt to be paid for it for one day was the 365th part of $\frac{5}{100}$, or of the intereſt paid for it for a whole year, when lent for a year at a time, at the intereſt of 5 *per cent. per annum.*

2 49. Before

49. Before I put an end to this appendix, I think it neceffary to declare, that the inveftigations of the two anti-logarithmick infinite feriefes $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} +$ *&c*, and $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} +$ *&c*, which have been given above in the preceding lemmas and problems, were fuggefted to me by reading Dr. Halley's difcourfe on logarithms above-mentioned, and that I therefore fuppofe they are in fubftance the fame with his inveftigations of the fame feriefes contained, or, rather, hinted at, in the faid difcourfe. But this I am not quite fure of, becaufe, from the extreme concifenefs with which he has there treated the fubject, I find myfelf unable to underftand him fo thoroughly as I could wifh. However, whether the inveftigations which I have here given of thefe feriefes are the fame with his or not, I hope they will be found fufficient to eftablifh his conclufions, and that they have been explained in fuch a manner as to give my readers as little trouble as poffible in underftanding them.

THE END OF VOL. I.